Sensitive instances of the Constraint Satisfaction Problem

Libor Barto
Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovska 83, 186 00 Praha 8, Czech Republic
barto@karlin.mff.cuni.cz

Marcin Kozik
Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
marcin.kozik@uj.edu.pl

Johnson Tan
Department of Mathematics, University of Illinois, Urbana-Champaign, Urbana, IL USA, 61801
jgtan2@illinois.edu

Matt Valeriote
Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1
matt@math.mcmaster.ca

Abstract

We investigate the impact of modifying the constraining relations of a Constraint Satisfaction Problem (CSP) instance, with a fixed template, on the set of solutions of the instance. More precisely we investigate sensitive instances: an instance of the CSP is called sensitive, if removing any tuple from any constraining relation invalidates some solution of the instance. Equivalently, one could require that every tuple from any one of its constraints extends to a solution of the instance.

Clearly, any non-trivial template has instances which are not sensitive. Therefore we follow the direction proposed (in the context of strict width) by Feder and Vardi in [13] and require that only the instances produced by a local consistency checking algorithm are sensitive. In the language of the algebraic approach to the CSP we show that a finite idempotent algebra $\mathbf{A}$ has a $k+2$ variable near unanimity term operation if and only if any instance that results from running the $(k, k+1)$-consistency algorithm on an instance over $\mathbf{A}^2$ is sensitive.

A version of our result, without idempotency but with the sensitivity condition holding in a variety of algebras, settles a question posed by G. Bergman about systems of projections of algebras that arise from some subalgebra of a finite product of algebras.

Our results hold for infinite (albeit in the case of $\mathbf{A}$ idempotent) algebras as well and exhibit a surprising similarity to the strict width k condition proposed by Feder and Vardi. Both conditions can be characterized by the existence of a near unanimity operation, but the arities of the operations differ by 1.

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1 Introduction

One important algorithmic approach to deciding if a given instance of the Constraint Satisfaction Problem (CSP) has a solution is to first consider whether it has a consistent set of local solutions. Clearly, the absence of local solutions will rule out having any (global) solutions, but in general having local solutions does not guarantee the presence of a solution. A major thrust of the recent research on the CSP has focused on coming up with suitable notions of local consistency and then characterizing those CSPs for which local consistency implies outright consistency or some stronger property. A good source for background material is the survey article [7].

Early results of Feder and Vardi [13] and also Jeavons, Cooper, and Cohen [15] establish that when a template (i.e., a relational structure) $\mathbb{A}$ has a special type of polymorphism, called a near unanimity operation, then not only will an instance of the CSP over $\mathbb{A}$ that has a suitably consistent set of local solutions have a solution, but that any partial solution of it can always be extended to a solution. The notion of local consistency that we investigate in this paper is related to that considered by these researchers but that, as we shall see, is weaker.

The following operations are central to our investigation.

▶ **Definition 1.** *An operation $n(x_1, \ldots, x_{k+1})$ on a set A of arity $k + 1$ is called a* near unanimity operation *on A if it satisfies the equalities*

$$n(b, a, a, \ldots, a) = n(a, b, a, \ldots, a) = \cdots = n(a, a, \ldots, a, b) = a$$

for all a, $b \in A$.

Near unanimity operations have played an important role in the development of universal algebra and first appeared in the 1970's in the work of Baker and Pixley [1] and Huhn [14]. More recently they have been used in the study of the CSP [13, 15] and related questions [2, 12]. The main results of this paper can be expressed in terms of the CSP and also in algebraic terms and we start by presenting them from both perspectives. In the concluding section, Section 6, a translation of parts of our results into a relational language is provided, along with some open problems.

1.1 CSP viewpoint

In their seminal paper, Feder and Vardi [13] introduced the notion of bounded width for the class of CSP instances over a finite template $\mathbb{A}$. Their definition of bounded width was presented in terms of the logic programming language DATALOG but there is an equivalent formulation using local consistency algorithms, also given in [13]. Given a CSP instance $\mathcal{I}$ and $k < l$, the (k, l)-consistency algorithm will produce a new instance having all k variable constraints that can be inferred by considering l variables at a time of $\mathcal{I}$. This algorithm rejects $\mathcal{I}$ if it produces an empty constraint. The class of CSP instances over a finite template $\mathbb{A}$ will have width (k, l) if the (k, l)-consistency algorithm rejects all instances from the class that do not have solutions, i.e., the (k, l)-consistency algorithm can be used to decide if a given instance from the class has a solution or not. The class has bounded width if it has width (k,l) for some $k < l$.

A lot of effort, in the framework of the algebraic approach to the CSP, has gone in to analyzing various properties of instances that are the outputs of these types of local consistency algorithms. On one end of the spectrum of the research is a rather wide class of templates of bounded width [5] and on the other a very restrictive class of templates having bounded strict width [13].

To be more precise, we now formally introduce instances of the CSP.

▶ **Definition 2.** *An* instance $\mathcal{I}$ *of the* CSP *is a pair* $(V, \mathcal{C})$ *where* V *is a finite set of variables, and* $\mathcal{C}$ *is a set of constraints of the form* $((x_1, \ldots, x_n), R)$ *where all* x_i *are in* V *and* R *is an* n*-ary relation over (possibly infinite) sets* A_i *associated to each variable* x_i*.*

A solution *of* $\mathcal{I}$ *is an evaluation* f *of variables such that, for every* $((x_1, \ldots, x_n), R) \in \mathcal{C}$ *we have* $(f(x_1), \ldots, f(x_n)) \in R$*;* a partial solution *is a partial function satisfying the same condition.*

The CSP over a relational structure $\mathbb{A}$*, written* $\mathsf{CSP}(\mathbb{A})$*, is the class of* CSP *instances whose constraint relations are from* $\mathbb{A}$*.*

▶ **Example 3.** For $k > 1$, the template associated with the graph k-colouring problem is the relational structure $\mathbb{D}_{k\text{colour}}$ that has universe $\{0, 1, \ldots, k-1\}$ and a single relation $\neq_k = \{(x, y) \mid x, y < k \text{ and } x \neq y\}$. The template associated with the HORN-3-SAT problem is the relational structure $\mathbb{D}_{\text{horn}}$ that has universe $\{0, 1\}$ and two ternary relations R_0, R_1, where R_i contains all the triples but $(1, 1, i)$. It is known that $\mathsf{CSP}(\mathbb{D}_{\text{horn}})$ has width $(1, 2)$, that $\mathsf{CSP}(\mathbb{D}_{2\text{colour}})$ has width $(2, 3)$, and that for $k > 2$, $\mathsf{CSP}(\mathbb{D}_{k\text{colour}})$ does not have bounded width (see [7]).

Instances produced by the (k,l)-consistency algorithm have uniformity and consistency properties that we highlight.

▶ **Definition 4.** *The* CSP *instance* $\mathcal{I}$ *is* k-uniform *if all of its constraints are* k*-ary and every set of* k *variables is constrained by a single constraint.*

An instance is a (k,l)-instance *if it is* k*-uniform and for every choice of a set* W *of* l *variables no additional information about the constraints can be derived by restricting the instance to the variables in* W*.*

This last, very important, property can be rephrased in the following way: for every set $W \subseteq V$ of size l, every tuple in every constraint of $\mathcal{I}_{|W}$ participates in a solution to $\mathcal{I}_{|W}$ (where $\mathcal{I}_{|W}$ is obtained from $\mathcal{I}$ by removing all the variables outside of W and all the constraints that contain any such variables).

Consider the notion of strict width k introduced by Feder and Vardi [13, Section 6.1.2]. Let $\mathbb{A}$ be a template and let us assume, to avoid some technical subtleties, that every relation in $\mathbb{A}$ has arity at most k. The class $\mathsf{CSP}(\mathbb{A})$ has *strict width* (k,l) if whenever the (k,l)-consistency algorithm does not reject an instance $\mathcal{I}$ from the class then "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints." In other words, if $\mathcal{I}$ is the result of applying the (k,l)-consistency algorithm to an instance of $\mathsf{CSP}(\mathbb{A})$, then any partial solution of $\mathcal{I}$ can be extended to a solution. The template $\mathbb{A}$ is said to have *strict width* k if it has strict width (k,l) for some $l > k$.

A *polymorphism* of a template $\mathbb{A}$ is a function on A that preserves all of the relations of $\mathbb{A}$. Feder and Vardi prove the following.

▶ **Theorem 5** (see Theorem 25, [13]). *Let $k > 1$ and let $\mathbb{A}$ be a finite relational structure with relations of arity at most k. The class* $\mathsf{CSP}(\mathbb{A})$ *has strict width k if and only if it has strict width $(k, k+1)$ if and only if $\mathbb{A}$ has a $(k+1)$-ary near unanimity operation as a polymorphism.*

Using this Theorem we can conclude that $\mathsf{CSP}(\mathbb{D}_{2\text{colour}})$ from Example 3 has strict width 2 since the ternary majority operation preserves the relation $\neq_2$. In fact this operation preserves all binary relations over the set $\{0, 1\}$. On the other hand, $\mathsf{CSP}(\mathbb{D}_{\text{horn}})$ does not have strict width k for any $k \geq 3$.

Following the algebraic approach to the CSP we replace templates $\mathbb{A}$ with algebras $\mathbf{A}$.

▶ **Definition 6.** *An* algebra $\mathbf{A}$ *is a pair $(A, \mathcal{F})$ where A is a non-empty set, called the* universe of $\mathbf{A}$ *and $\mathcal{F} = (f_i \mid i \in I)$ is a set of finitary operations on A called the* set of basic operations of $\mathbf{A}$. *The function that assigns the arity of the operation f_i to i is called the* signature of $\mathbf{A}$. *If $t(x_1, \ldots, x_n)$ is a term in the signature of $\mathbf{A}$ then the interpretation of t by $\mathbf{A}$ as an operation on A is called a* term operation of $\mathbf{A}$ *and is denoted by $t^{\mathbf{A}}$.*

The CSP over $\mathbf{A}$, *written* $\mathsf{CSP}(\mathbf{A})$, *is the class of* CSP *instances whose constraint relations are amongst those relations over A that are preserved by the operations of $\mathbf{A}$ (i.e., they are subuniverses of powers of $\mathbf{A}$).*

A number of important questions about the CSP can be reduced to considering templates that have all of the singleton unary relations [7]; the algebraic counterpart to these types of templates are the *idempotent algebras*.

▶ **Definition 7.** *An operation $f : A^n \to A$ on a set A is* idempotent *if $f(a, a, \ldots, a) = a$ for all $a \in A$. An algebra $\mathbf{A}$ is* idempotent *if all of its basic operations are.*

It follows that if $\mathbf{A}$ is idempotent then every term operation of $\mathbf{A}$ is an idempotent operation. As demonstrated in Example 22, several of the results in this paper do not hold in the absence of idempotency.

The characterization of strict width in Theorem 5 has the following consequence in terms of algebras.

▶ **Corollary 8.** *Let $k > 1$ and let $\mathbb{A}$ be a finite relational structure with relations of arity at most k. Let $\mathbf{A}$ be the algebra with the same universe as $\mathbb{A}$ whose basic operations are exactly the polymorphisms of $\mathbb{A}$. The following are equivalent:*

1. *$\mathbf{A}$ has a near unanimity term operation of arity $k + 1$;*
2. *in every $(k, k+1)$-instance over $\mathbf{A}$, every partial solution extends to a solution.*

The implication “1 implies 2” in Corollary 8 remains valid for general algebras, not necessarily coming from finite relational structures with restricted arities of relations. However, the converse implication fails even if $\mathbf{A}$ is assumed to be finite and idempotent.

▶ **Example 9.** Consider the rather trivial algebra $\mathbf{A}$ that has universe $\{0, 1\}$ and no basic operations. If $\mathcal{I}$ is a $(2, 3)$-instance over $\mathbf{A}$ then since, as noted just after Theorem 5, every binary relation over $\{0, 1\}$ is invariant under the ternary majority operation on $\{0, 1\}$ it follows that every partial solution of $\mathcal{I}$ can be extended to a solution. Of course, $\mathbf{A}$ does not have a near unanimity term operation of any arity.

What this example demonstrates is that in general, for a fixed k, the k-ary constraint relations arising from an algebra do not capture that much of the structure of the algebra. Example 22 provides further evidence for this.

Our first theorem shows that for finite idempotent algebras $\mathbf{A}$, by considering a slightly bigger set of $(k, k+1)$-instances, over $\mathsf{CSP}(\mathbf{A}^2)$, rather than over $\mathsf{CSP}(\mathbf{A})$, we can detect the presence of a $(k+1)$-ary near unanimity term operation. Moreover, it is enough to consider only instances with $k+2$ variables. We note that every $(k, k+1)$-instance over $\mathbf{A}$ can be easily encoded as a $(k, k+1)$-instance over $\mathbf{A}^2$.

▶ **Theorem 10.** *Let $\mathbf{A}$ be a finite, idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k+1$;*
2. *in every $(k, k+1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k+1)$-instance over $\mathbf{A}^2$ **on** $k+2$ **variables**, every partial solution extends to a solution.*

In Theorem 20 we extend our result to infinite idempotent algebras by working with local near unanimity term operations.

Going back the original definition of strict width: "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints" we note that the requirement that the assignment should be greedy is rather restrictive. The main theorem of this paper investigates an arguably more natural concept where the assignment need not be greedy.

▶ **Definition 11.** *An instance of the* CSP *is called* sensitive*, if removing any tuple from any constraining relation invalidates some solution of the instance.*

In other words, an instance is sensitive if every tuple in every constraint of the instance extends to a solution. For $(k, k+1)$-instances, being sensitive is equivalent to the instance being a (k, n)-instance, where n is the number of variables present in the instance. We provide the following characterization.

▶ **Theorem 12.** *Let $\mathbf{A}$ be a finite, idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k+2$;*
2. *every $(k, k+1)$-instance over $\mathbf{A}^2$ is sensitive;*
3. *every $(k, k+1)$-instance over $\mathbf{A}^2$ **on** $k+2$ **variables** is sensitive.*

Exactly as in Theorem 10 we can consider infinite algebras at the cost of using local near unanimity term operations (see Theorem 21).

In conclusion we investigate a natural property of instances motivated by the definition of strict width and provide a characterization of this new condition in algebraic terms. A surprising conclusion is that the new concept is, in fact, very close to the strict width concept, i.e., for a fixed k one characterization is equivalent to a near unanimity operation of arity $k+1$ and the second of arity $k+2$.

1.2 Algebraic viewpoint

Our work has as an antecedent the papers of Baker and Pixley [1] and of Bergman [8] on algebras having near unanimity term operations. In these papers the authors considered subalgebras of products of algebras and systems of projections associated with them. Baker and Pixley showed that in the presence of a near unanimity term operation, such a subalgebra is closely tied with its projections onto small sets of coordinates.

▶ **Definition 13.** *A* variety of algebras *is a class of algebras of the same signature that is closed under taking homomorphic images, subalgebras, and direct products. For $\mathbf{A}$ an algebra, $\mathcal{V}(\mathbf{A})$ denotes the smallest variety that contains $\mathbf{A}$ and is called the* variety generated by $\mathbf{A}$. *A variety $\mathcal{V}$ has a near unanimity term of arity $k+1$ if there is some $(k+1)$-ary term in the signature of $\mathcal{V}$ whose interpretation in each member of $\mathcal{V}$ is a near unanimity operation.*

Here is one version of the Baker-Pixley Theorem:

▶ **Theorem 14** (see Theorem 2.1 from [1]). *Let* $\mathbf{A}$ *be an algebra and* $k > 1$*. The following are equivalent:*

1. $\mathbf{A}$ *has a* $(k+1)$*-ary near unanimity term operation;*
2. *for every* $r > k$ *and every* $\mathbf{A}_i \in \mathcal{V}(\mathbf{A})$*,* $1 \leq i \leq r$*, every subalgebra* $\mathbf{R}$ *of* $\prod_{i=1}^{r} \mathbf{A}_i$ *is* ***uniquely*** *determined by the projections of* R *on all products* $A_{i_1} \times \cdots \times A_{i_k}$ *for* $1 \leq i_1 < i_2 < \cdots < i_k \leq r$*;*
3. *the same as condition 2, with* r *set to* $k+1$*.*

In other words, an algebra has a $(k+1)$-ary near unanimity term operation if and only if every subalgebra of a product of algebras from $\mathcal{V}(\mathbf{A})$ is uniquely determined by its system of k-fold projections into its factor algebras. A natural question, extending the result above, was investigated by Bergman [8]: when does a given "system of k-fold projections" arise from a product algebra?

Note that such a system can be viewed as a k-uniform CSP instance: indeed, following the notation of Theorem 14, we can introduce a variable x_i for each $i \leq r$ and a constraint $((x_{i_1}, \ldots, x_{i_k}); \mathrm{proj}_{i_1,\ldots,i_k} R)$ for each $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. In this way the original relation R consists of solutions of the created instance (but in general will not contain all of them). In this particular instance, different variables can be evaluated in different algebras. Note that the instance is sensitive, if and only if it "arises from a product algebra" in the sense investigated by Bergman.

We will say that $\mathcal{I}$ is a CSP instance *over the variety* $\mathcal{V}$ *(denoted* $\mathcal{I} \in \mathsf{CSP}(\mathcal{V})$*)* if all the constraining relations of $\mathcal{I}$ are algebras in $\mathcal{V}$. In the language of the CSP, Bergman proved the following:

▶ **Theorem 15** ([8]). *If* $\mathcal{V}$ *is a variety that has a* $(k+1)$*-ary near unanimity term then every* $(k, k+1)$*-instance over* $\mathcal{V}$ *is sensitive.*

In commentary that Bergman provided on his proof of this theorem he noted that a stronger conclusion could be drawn from it and he proved the following theorem. We note that this theorem anticipates the results from [13] and [15] dealing with templates having near unanimity operations as polymorphisms.

▶ **Theorem 16** ([8]). *Let* $k > 1$ *and* $\mathcal{V}$ *be a variety. The following are equivalent:*

1. $\mathcal{V}$ *has a* $(k+1)$*-ary near unanimity term;*
2. *any partial solution of a* $(k, k+1)$*-instance over* $\mathcal{V}$ *extends to a solution.*

Theorem 15 provides a partial answer to the question that Bergman posed in [8], namely that in the presence of a $(k+1)$-ary near unanimity term, a necessary and sufficient condition for a k-fold system of algebras to arise from a product algebra is that the associated CSP instance is a $(k, k+1)$-instance.

In [8] Bergman asked whether the converse to Theorem 15 holds, namely, that if all $(k, k+1)$-instances over a variety are sensitive, must the variety have a $(k+1)$-ary near unanimity term? He provided examples that suggested that the answer is no, and we confirm this by proving that the condition is actually equivalent to the variety having a near unanimity term of arity $k+2$. The main result of this paper, viewed from the algebraic perspective (but stated in terms of the CSP), is the following:

▶ **Theorem 17.** *Let* $k > 1$*. A variety* $\mathcal{V}$ *has a* $(k+2)$*-ary near unanimity term if and only if each* $(k, k+1)$*-instance of the* CSP *over* $\mathcal{V}$ *is sensitive.*

The "if" direction of this theorem is proved in Section 3, while a sketch of a proof of the "only if" direction can be found in Section 5 (the complete reasoning is included in the full version of this paper). We note that a novel and significant feature of this result is that it does not assume any finiteness or idempotency of the algebras involved.

1.3 Structure of the paper

The paper is structured as follows. In the next section we introduce local near unanimity operations and state Theorem 10 and Theorem 12 in their full power. In Section 3 we collect the proofs that establish the existence of (local) near unanimity operations. Section 4 contains a proof of a new loop lemma, which can be of independent interest, and is necessary in the proof in Section 5. In Section 5 we provide a sketch of the proof showing that, in the presence of a near unanimity operation of arity $k+2$, the $(k, k+1)$-instances are sensitive. A complete proof of this fact, which is our main contribution, can be found in the full version of this paper. Finally, Section 6 contains conclusions.

2 Details of the CSP viewpoint

In order to state our results in their full strength, we need to define local near unanimity operations. This special concept of local near unanimity operations is required, when considering infinite algebras.

▶ **Definition 18.** *Let $k > 1$. An algebra $\mathbf{A}$ has* local near unanimity term operations of arity *$k+1$ if for every finite subset S of A there is some $(k+1)$-ary term operation n_S of $\mathbf{A}$ such that*

$$n_S(b, a, \dots, a, a) = n_S(a, b, a, \dots, a) = \dots = n_S(a, a, \dots, b, a) = n_S(a, a, \dots, a, b) = a.$$

for all $a, b \in S$.

It should be clear that, for finite algebras, having local near unanimity term operations of arity $k+1$ and having a near unanimity term operation of arity $k+1$ are equivalent, but for arbitrary algebras they are not. The following provides a characterization of when an idempotent algebra has local near unanimity term operations of some given arity; it will be used in the proofs of Theorems 20 and 21. It is similar to Theorem 14 and is proved in the full version of this paper.

▶ **Theorem 19.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ has local near unanimity term operations of arity $k+1$;*
2. *for every $r > k$, every subalgebra of $\mathbf{A}^r$ is uniquely determined by its projections onto all k-element subsets of coordinates;*
3. *every subalgebra of $\mathbf{A}^{k+1}$ is uniquely determined by its projections onto all k-element subsets of coordinates.*

We are ready to state Theorem 10 in its full strength:

▶ **Theorem 20.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*

1. *$\mathbf{A}$ (or equivalently $\mathbf{A}^2$) has local near unanimity term operations of arity $k+1$;*
2. *in every $(k, k+1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k+1)$-instance over $\mathbf{A}^2$ **on** $k+2$ **variables**, every partial solution extends to a solution.*

Proof. Obviously condition 2 implies condition 3. A proof of condition 3 implying condition 1 can be found in Section 3. The implication from 1 to 2 is covered by Theorem 16. ◀

Analogously, the main result of the paper, for idempotent algebras, and the full version of Theorem 12 states:

▶ **Theorem 21.** *Let* $\mathbf{A}$ *be an idempotent algebra and* $k > 1$*. The following are equivalent:*

1. $\mathbf{A}$ *(or equivalently* $\mathbf{A}^2$*) has local near unanimity term operations of arity* $k+2$*;*
2. *every* $(k, k+1)$*-instance over* $\mathbf{A}^2$ *is sensitive;*
3. *every* $(k, k+1)$*-instance over* $\mathbf{A}^2$ ***on*** $k+2$ ***variables*** *is sensitive.*

Proof. Obviously condition 2 implies condition 3. For a proof that condition 3 implies condition 1 see Section 3. A sketch of the proof of the remaining implication can be found in Section 5 (see the full version of this paper for a complete proof). ◀

The following examples show that in Theorems 19, 20, and 21 the assumption of idempotency is necessary.

▶ **Example 22.** For $n > 2$, let $\mathbf{S}_n$ be the algebra with domain $[n] = \{1, 2, \ldots, n\}$ and with basic operations consisting of all unary operations on $[n]$ and all non-surjective operations on $[n]$ of arbitrary arity. The collection of such operations forms a finitely generated clone, called the Słupecki clone. Relevant details of these algebras can be found in [16, Example 4.6] and [20]. It can be shown that for $m < n$, the subuniverses of $\mathbf{S}_n^m$ consist of all m-ary relations R_θ over $[n]$ determined by a partition θ of $[m]$ by

$$R_\theta = \{(a_1, \ldots, a_m) \mid a_i = a_j \text{ whenever } (i, j) \in \theta\}.$$

These rather simple relations are preserved by any operation on $[n]$, in particular by any majority operation or more generally, by any near unanimity operation.

It follows from Theorem 16 that if $k > 1$ and $\mathcal{I}$ is a $(k, k+1)$-instance of $\mathsf{CSP}(\mathbf{S}^2_{2k+1})$ then any partial solution of $\mathcal{I}$ extends to a solution. This also implies that $\mathcal{I}$ is sensitive. Furthermore any subalgebra of $\mathbf{S}^{k+1}_{k+2}$ is determined by it projections onto all k-element sets of coordinates. As noted in [16, Example 4.6], for $n > 2$, $\mathbf{S}_n$ does not have a near unanimity term operation of any arity, since the algebra $\mathbf{S}_n^n$ has a quotient that is a 2-element essentially unary algebra.

3 Constructing near unanimity operations

In this section we collect the proofs providing, under various assumptions, near unanimity or local near unanimity operations. That is: the proofs of "3 implies 1" in Theorems 20 and Theorem 21 as well as a proof of the "if" direction from Theorem 17.

In the following proposition we construct instances over $\mathbf{A}^2$ (for some algebra $\mathbf{A}$). By a minor abuse of notation, we allow in such instances two kinds of variables: variables x evaluated in A and variables y evaluated in A^2. The former kind should be formally considered as variables evaluated in A^2 where each constraint enforces that x is sent to $\{(b, b) \mid b \in A\}$.

Moreover, dealing with k-uniform instances, we understand the condition "every set of k variables is constrained by a single constraint" flexibly: in some cases we allow for more constraints with the same set of variables, as long as the relations are proper permutations so that every constraint imposes the same restriction.

▶ **Proposition 23.** *Let $k > 1$ and let $\mathbf{A}$ be an algebra such that, for every $(k, k+1)$-instance $\mathcal{I}$ over $\mathbf{A}^2$ on $k+2$ variables every partial solution of $\mathcal{I}$ extends to a solution. Then each subalgebra of $\mathbf{A}^{k+1}$ is determined by its k-ary projections.*

Proof. Let $\mathbf{R} \leq \mathbf{A}^{k+1}$ and we will show that it is determined by the system of projections $\mathrm{proj}_I(R)$ as I ranges over all k elements subsets of coordinates. Using $\mathbf{R}$ we define the following instance $\mathcal{I}$ of $\mathsf{CSP}(\mathbf{A}^2)$. The variables of $\mathcal{I}$ will be the set $\{x_1, x_2, \dots, x_{k+1}, y_{12}\}$ and the domain of each x_i is A, while the domain of y_{12} is A^2.

For $U \subseteq \{x_1, \dots, x_{k+1}\}$ of size k, let C_U be the constraint with scope U and constraint relation $R_U = \mathrm{proj}_U(R)$. For U a $(k-1)$-element subset of $\{x_1, \dots, x_{k+1}\}$, let $C_{U \cup \{y_{12}\}}$ be the constraint with scope $U \cup \{y_{12}\}$ and constraint relation $R_{U \cup \{y_{12}\}}$ that consists of all tuples $(b_v \mid v \in U \cup \{y_{12}\})$ such that there is some $(a_1, \dots, a_{k+1}) \in R$ with $b_v = a_i$ if $v = x_i$ and with $b_{y_{12}} = (a_1, a_2)$.

The instance $\mathcal{I}$ is k-uniform and we will show that it is sensitive. Indeed every tuple in every constraining relation originates in some tuple $\mathbf{b} \in R$. Setting $x_i \mapsto b_i$ and $y_{12} \mapsto (b_1, b_2)$ defines a solution that extends such a tuple.

In particular $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ with $k+2$ variables and so any partial solution of it can be extended to a solution. Let $\mathbf{b} \in A^{k+1}$ such that $\mathrm{proj}_I(\mathbf{b}) \in \mathrm{proj}_I(R)$ for all k element subsets I of $[k+1]$. Then $\mathbf{b}$ is a partial solution of $\mathcal{I}$ over the variables $\{x_1, \dots, x_{k+1}\}$ and thus there is some extension of it to the variable y_{12} that produces a solution of $\mathcal{I}$. But there is only one consistent way to extend $\mathbf{b}$ to y_{12} namely by setting y_{12} to the value (b_1, b_2). By considering the constraint with scope $\{x_3, \dots, x_{k+1}, y_{12}\}$ it follows that $\mathbf{b} \in R$, as required. ◀

Now we are ready to prove the first implication tackled in this section: 3 implies 1 in Theorem 20.

Proof of "3 implies 1" in Theorem 20. By Theorem 19 it suffices to show that each subalgebra of $\mathbf{A}^{k+1}$ is determined by its k-ary projections. Fortunately, Proposition 23 provides just that. ◀

We move on to proofs of "3 implies 1" in Theorem 21 and the "if" direction of Theorem 17. Similarly, as in the theorem just proved, we start with a proposition.

▶ **Proposition 24.** *Let $k > 1$ and let $\mathbf{A}$ be an algebra such that every $(k, k+1)$-instance $\mathcal{I}$ over $\mathbf{A}^2$ on $k+2$ variables is sensitive. Then each subalgebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections.*

Proof. We will show that if $\mathbf{R}$ is a subalgebra of $\mathbf{A}^{k+2}$ then $R = R^*$ where

$$R^* = \{a \in A^{k+2} \mid \mathrm{proj}_I(a) \in \mathrm{proj}_I(R) \text{ whenever } |I| = k+1\}.$$

In other words, we will show that the subalgebra $\mathbf{R}$ is determined by its projections into all $(k+1)$-element sets of coordinates.

We will use R and R^* from the previous paragraph to construct a $(k, k+2)$-instance $\mathcal{I} = (V, \mathcal{C})$ with $V = \{x_5, \dots, x_{k+2}, y_{12}, y_{34}, y_{13}, y_{24}\}$ where each x_i is evaluated in A while all the y's are evaluated in A^2.

The set of constraints is more complicated. There is a *special constraint* on a *special variable set* $((y_{12}, y_{34}, x_5, \dots, x_{k+2}), C)$ where

$$C = \{((a_1, a_2), (a_3, a_4), a_5, \dots, a_{k+2}) \mid (a_1, \dots, a_{k+2}) \in R^*\}.$$

The remaining constraints are defined using the relation R. For each set of variables $S = \{v_1, \dots, v_k\} \subseteq V$ (which is different than the set for the special constraint) we define a constraint $((v_1, \dots, v_k), D_S)$ with $(b_1, \dots, b_k) \in D_S$ if and only if there exists a tuple $(a_1, \dots, a_{k+2}) \in R$ such that:

- if v_i is x_j then $b_i = a_j$, and
- if v_i is y_{lm} then $b_i = (a_l, a_m)$.

Note that the instance $\mathcal{I}$ is k-uniform.

▷ Claim 25. $\mathcal{I}$ is a $(k, k+1)$-instance.

Let $S \subseteq V$ be a set of size k. If S is not the special variable set, then every tuple in the relation constraining S originates in some $(b_1, \dots, b_{k+2}) \in R$ and, as in Proposition 23, sending $x_i \mapsto b_i$ and $y_{lm} \mapsto (b_l, b_m)$ defines a solution that extends such a tuple. We immediately conclude, that the potential failure of the $(k, k+1)$ condition must involve the special constraint.

Thus $S = \{y_{12}, y_{34}, x_5, \dots, x_{k+2}\}$ and if $\mathbf{b}$ is a tuple from the special constraint C then there is some $(a_1, \dots, a_{k+2}) \in R^*$ with

$$\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \dots, a_{k+2}).$$

The extra variable that we want to extend the tuple $\mathbf{b}$ to is either y_{13} or y_{24}. Both cases are similar and we will only work through the details when it is y_{13}. In this case, assigning the value (a_1, a_3) to the variable y_{13} will produce an extension $\mathbf{b}'$ of $\mathbf{b}$ to a tuple over $S \cup \{y_{13}\}$ that is consistent with all constraints of $\mathcal{I}$ whose scopes are subsets of $\{y_{12}, y_{34}, x_5, \dots, x_{k+2}, y_{13}\}$.

To see this, consider a k element subset S' of $\{y_{12}, y_{34}, x_5, \dots, x_{k+2}, y_{13}\}$ that excludes some variable x_j. Then, by the definition of R^* there exists some tuple of the form $(a_1, a_2, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_{k+2}) \in R$. This tuple from R can be used to witness that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$ since the scope of this constraint does not include the variable x_j.

Suppose that S' is a k element subset of $\{y_{12}, y_{34}, x_5, \dots, x_{k+2}, y_{13}\}$ that excludes y_{12}. By the definition of R^* there is some tuple of the form $(a_1, a'_2, a_3, \dots, a_{k+2}) \in R$. Using this tuple it follows that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$. This is because neither of the variables y_{12} and y_{24} are in S' and so the value $a'_2 \in A_2$ does not matter. A similar argument works when S' is assumed to exclude y_{34} and the claim is proved.

Since $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ and it has $k+2$ variables then by assumption, $\mathcal{I}$ is sensitive. We can use this to show that $R^* \subseteq R$ to complete the proof of this proposition. Let $(a_1, \dots, a_{k+2}) \in R^*$ and consider the associated tuple $\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \dots, a_{k+2}) \in C$. Since $\mathcal{I}$ is sensitive then this k-tuple can be extended to a solution $\mathbf{b}'$ of $\mathcal{I}$. Using any constraints of $\mathcal{I}$ whose scopes include combinations of y_{12} or y_{34} with y_{13} or y_{24} it follows that the value of $\mathbf{b}'$ on the variables y_{13} and y_{24} are (a_1, a_3) and (a_2, a_4) respectively. Then considering the restriction of $\mathbf{b}'$ to $S = \{x_5, \dots, x_{k+2}, y_{13}, y_{24}\}$ it follows that $(a_1, \dots, a_{k+2}) \in R$ since this restriction lies in the constraint relation D_S. ◀

We are in a position to provide the two final proofs in this section.

Proof of "3 implies 1" in Theorem 21. By Theorem 19 it suffices to show that each subalgebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections. Fortunately Propositions 24 provides just that. ◀

Proof of the "if" direction in Theorem 17. For this direction we apply Proposition 24 to a special member of $\mathcal{V}$, namely the $\mathcal{V}$-free algebra freely generated by $\mathbf{x}$ and $\mathbf{y}$, which we will denote by $\mathbf{F}$. Up to isomorphism, this algebra is unique and its defining property is that $\mathbf{F} \in \mathcal{V}$ and for any algebra $\mathbf{A} \in \mathcal{V}$, any map $f : \{\mathbf{x}, \mathbf{y}\} \to A$ extends uniquely to a homomorphism from $\mathbf{F}$ to $\mathbf{A}$. Consequently, for any two terms $s(x, y)$ and $t(x, y)$ in the signature of $\mathcal{V}$ if $s^{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = t^{\mathbf{F}}(\mathbf{x}, \mathbf{y})$ then the equation $s(x, y) \approx t(x, y)$ holds in $\mathcal{V}$.

Let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^{k+2}$ generated by the tuples $(\mathbf{y}, \mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$, $(\mathbf{x}, \mathbf{y}, \mathbf{x}, \dots, \mathbf{x})$, $\dots$, $(\mathbf{x}, \dots, \mathbf{x}, \mathbf{y})$. By Proposition 24, the algebra $\mathbf{R}$ is determined by its $(k+1)$-ary projections and so the constant tuple $(\mathbf{x}, \dots, \mathbf{x})$ belongs to R. The term generating this tuple from the given generators of $\mathbf{R}$ defines the required $(k + 2)$-ary near unanimity operation. ◀

4 New loop lemmata

A *loop lemma* is a theorem stating that a binary relation satisfying certain structural and algebraic requirements necessarily contains a *loop* – a pair (a, a). In this section we provide two new loop lemmata, Theorem 31 and Theorem 32, which generalize an "infinite loop lemma" of Olšák [18] and may be of independent interest. Theorem 32 is a crucial tool for the proof presented in Section 5.

The algebraic assumptions in the new loop lemmata concern absorption, a concept that has proven to be useful in the algebraic theory of CSPs and in universal algebra [6]. We adjust the standard definition to our specific purposes. We begin with a very elementary definition.

▶ **Definition 26.** *Let R and S be sets. We call a tuple $(a_1, \dots, a_n)$ a* one-S-in-R tuple *if for exactly one i we have $a_i \in S$ and all the other a_i's are in R.*

Next we proceed to define a relaxation of the standard absorbing notion. We follow a standard notation, silently extending operations of an algebra to powers (by computing them coordinate-wise).

▶ **Definition 27.** *Let $\mathbf{A}$ be an algebra, $\mathbf{R} \leq \mathbf{A}^k$ and $S \subseteq A^k$. We say that R locally n-absorbs S if, for every finite set $\mathcal{C}$ of one-S-in-R tuples of length n, there is a term operation t of $\mathbf{A}$ such that $t(\mathbf{a^1}, \dots, \mathbf{a^n}) \in R$ whenever $(\mathbf{a^1}, \dots, \mathbf{a^n}) \in \mathcal{C}$. We will say that R* locally absorbs *S, if R locally n-absorbs S for some n.*

Absorption, even in this form, is stable under various constructions. The following lemma lists some of them and we leave it without a proof (the reasoning is identical to the one in e.g. Proposition 2 in [6]).

▶ **Lemma 28.** *Let $\mathbf{A}$ be an algebra and $\mathbf{R} \leq \mathbf{A}^2$ such that R locally n-absorbs S. Then R^{-1} locally n-absorbs S^{-1}; and $R \circ R$ locally n-absorbs $S \circ S$, and $R \circ R \circ R$ locally n-absorbs $S \circ S \circ S$ etc.*

Let us prove a first basic property of local absorption.

▶ **Lemma 29.** *Let $\mathbf{A}$ be an idempotent algebra and $\mathbf{R} \leq \mathbf{A}^2$ such that R locally n-absorbs S. Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ be directed walks in R, and let $(a_i, b_i) \in S$ for each i (see Figure 1). Then there exists a directed walk from a_1 to b_n of length n in R.*

Proof. We will show that there is a term operation t of the algebra $\mathbf{A}$ such that the following

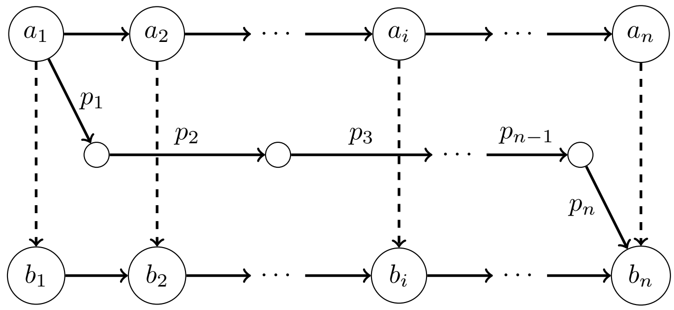

Figure 1 Solid arrows represent tuples from R and dashed arrows represent tuples from S.

$(n+1)$-tuple of elements of A is a walk of length n in R from a_1 to b_n.

$$\begin{aligned}
(a_1 =& t(a_1, a_1, a_1, \ldots, a_1),\\
& t(b_1, a_2, a_2, \ldots, a_2),\\
& t(b_2, b_2, a_3, \ldots, a_3),\\
& \vdots\\
& t(b_{n-1}, b_{n-1}, \ldots, b_{n-1}, a_n),\\
b_n =& t(b_n, b_n, b_n, \ldots, b_n)).
\end{aligned}$$

In order to choose a proper t we apply the definition of local absorption to the set of $(n+1)$ one-S-in-R tuples corresponding to the steps in the path. ◀

The loop lemma of Olšák concerns symmetric relations absorbing the equality relation $\{(a, a) \mid a \in A\}$, which is denoted $=_A$. The original result, stated in a slightly different language, does not cover the case of local absorption. However, a typographical modification of a proof mentioned in [18] shows that the theorem holds. For completeness sake, we present this proof in the full version of this paper.

▶ **Theorem 30** ([18]). *Let $\mathbf{A}$ be an **idempotent** algebra and $\mathbf{R} \leq \mathbf{A}^2$ be nonempty and symmetric. If R locally absorbs $=_A$, then R contains a loop.*

In order to apply this theorem in the case of sensitive instances, we need to generalize it. In the following two theorems we will gradually relax the requirement that R is symmetric. In the first step, we substitute it with a condition requiring a closed, directed walk in the graph (i.e., a sequence of possibly repeating vertices, with consecutive vertices connected by forward edges and the first and last vertex identical). Recall that R^{-1} is the inverse relation to R and let us denote by $R^{\circ l}$ the l-fold relational composition of R with itself.

▶ **Theorem 31.** *Let $\mathbf{A}$ be an **idempotent** algebra and $\mathbf{R} \leq \mathbf{A}^2$ contain a directed closed walk. If R locally absorbs $=_A$, then R contains a loop.*

Proof. Let n denote the arity of the absorbing operations. The proof is by induction on $l \geq 0$, where l is a number such that there exists a directed closed walk from a_1 to a_1 of length 2^l.

We start by verifying that such an l exists. Take a directed walk $(a_1, \ldots, a_{k-1}, a_k = a_1)$ in R. We may assume that its length k is at least n, since we can, if necessary, traverse the walk multiple times. An application of Lemma 29 to the relations $R, =_A$ and tuples $(a_1, \ldots, a_n)$, $(a_1, \ldots, a_n)$ gives us a directed walk from a_1 to a_n of length n. Appending this walk with the walk $(a_n, a_{n+1}, \ldots, a_k = a_1)$ yields a directed walk from a_1 to a_1 of length $k+1$. In this way, we can get a directed walk from a_1 to a_1 of any length greater than k.

Now we return to the inductive proof and start with the base of induction for $l = 0$ or $l = 1$. If $l = 0$, then we have found a loop. If $l = 1$ we have a closed walk of length 2, that is, a pair (a, b) which belongs to both R and R^{-1}. We set $R' = R \cap R^{-1}$ and observe that R' is nonempty and symmetric, and it is not hard to verify that R' locally absorbs $=_A$. Olšák's loop lemma, in the form of Theorem 30, gives us a loop in R.

Finally, we make the induction step from $l-1$ to l. Take a closed walk $(a_1, a_2, \ldots)$ of length 2^l and consider $R' = R^{\circ 2}$. Observe that R' contains a directed closed walk of length 2^{l-1} (namely $(a_1, a_3, \ldots)$), and that R' locally absorbs $=_A$ (by Lemma 28), so, by the inductive hypothesis, R' has a loop. In other words, R has a directed closed walk of length 2 and we are done by the case $l = 1$. ◀

Note that we cannot further relax the assumption on the graph by requiring that, for example, it has an infinite directed walk. Indeed the natural order of the rationals (taken for R) locally 2-absorbs the equality relation by the binary arithmetic mean operation $(a+b)/2$ (i.e., all the absorbing evaluations are realized by a single operation). The same relation locally 4-absorbs equality with the near unanimity operation $n(x, y, z, w)$ which, when applied to $a \leq b \leq c \leq d$, in any order, returns $(b+c)/2$.

Nevertheless, we can strengthen the algebraic assumption and still provide a loop; the following theorem is one of the key components in the proof sketch provided in Section 5 (albeit applied there with $l = 1$).

▶ **Theorem 32.** *Let* $\mathbf{A}$ *be an* ***idempotent*** *algebra and* $\mathbf{R} \leq \mathbf{A}^2$ *contain a directed walk of length* $n-1$. *If* R *locally* n*-absorbs* $=_A$ *and* $R^{\circ l}$ *locally* n*-absorbs* R^{-1} *for some* $l \in \mathbb{N}$ *then* R *contains a loop.*

Proof. By applying Lemma 29 similarly as in the proof of Theorem 31, we can get, from a directed walk of length $n-1$, a directed walk $(a_1, a_2, \ldots)$ of an arbitrary length. Moreover, by the same reasoning, for each i and j with $j \geq i + n - 1$, there is a directed walk from a_i to a_j of any length greater than or equal to $j - i$.

Consider the relations $R' = R^{\circ l n^2}$ and $S = (R^{-1})^{\circ n^2}$, and tuples

$$\mathbf{c} = (c_1, \ldots, c_n) := (a_{n^2}, a_{(n+1)n}, \ldots a_{(2n-1)n}), \text{ and}$$
$$\mathbf{d} = (d_1, \ldots, d_n) := (a_n, a_{2n} \ldots, a_{n^2})$$

By the previous paragraph and the definitions, both $\mathbf{c}$ and $\mathbf{d}$ are directed walks in R', and $(c_i, d_i) \in S$ for each i. Moreover, since $R^{\circ l}$ locally n-absorbs R^{-1}, Lemma 28 implies that R' locally absorbs S. We can thus apply Lemma 29 to the relations R', S and the tuples $\mathbf{c}, \mathbf{d}$ and obtain a directed walk from $c_1 = a_{n^2}$ to $d_{n-1} = a_{n^2}$ in R'. This closed walk in turn gives a closed directed walk in R and we are in a position to finish the proof by applying Theorem 31. ◀

5 Consistent instances are sensitive (sketch of a proof)

In this section we present the main ideas that are used to prove the "only if" direction in Theorem 17 and "1 implies 2" in Theorem 21. These ideas are shown in a very simplified situation, in particular, only the case that $k = 2$ and $\mathbf{A}$ is finite is considered. In the end of this section we briefly discuss the necessary adjustments in the general situation. A complete proof is given in the full version of this paper.

Consider a finite idempotent algebra $\mathbf{A}$ with a 4-ary near unanimity term operation and a $(2,3)$-instance $\mathcal{I} = (V, \mathcal{C})$ over $\mathbf{A}$. Each pair $\{x, y\}$ of variables is constrained by a unique constraint $((x, y), R_{xy})$ or $((y, x), R_{yx})$. For convenience we also define $R_{yx} = R_{yx}^{-1}$ (or $R_{xy} = R_{yx}^{-1}$ in the latter case) and R_{xx} to be the equality relation on A. Our aim is to show that every pair in every constraint relation extends to a solution. The overall structure of the proof is by induction on the number of variables of $\mathcal{I}$.

We fix a pair of variables $\{x_1, x_2\}$ and a pair $(a_1, a_2) \in R_{x_1x_2}$ that we want to extend. The strategy is to consider the instance $\mathcal{J}$ obtained by removing x_1 and x_2 from the set of variables and shrinking the constraint relations R_{uv} to R'_{uv} so that only the pairs consistent with the fixed choice remain, that is,

$$R'_{uv} = \{(b, c) \in R_{uv} \mid (a_1, b) \in R_{x_1u}, (a_2, b) \in R_{x_2u}, (a_1, c) \in R_{x_1v}, (a_2, c) \in R_{x_2v}\}.$$

We will show that $\mathcal{J}$ contains a nonempty $(2,3)$-subinstance, that is, an instance whose constraint relations are nonempty subsets of the original ones. The induction hypothesis then gives us a solution to $\mathcal{J}$ which, in turn, yields a solution to $\mathcal{I}$ that extends the fixed choice.

Having a nonempty $(2,3)$-subinstance can be characterized by the solvability of certain relaxed instances. The following concepts will be useful for working with relaxations of $\mathcal{I}$ and $\mathcal{J}$.

▶ **Definition 33.** *A* pattern *is a triple $\mathbb{P} = (W; \mathcal{F}, l)$, where $(W; \mathcal{F})$ is an undirected graph, and l is a mapping $l : W \to V$. The variable $l(i)$ is referred to as the* label *of i.*

A realization *(*strong realization*, respectively) of $\mathbb{P}$ is a mapping $\alpha : W \to A$, which* satisfies *every edge $\{w_1, w_2\} \in \mathcal{F}$, that is, $(\alpha(w_1), \alpha(w_2)) \in R_{l(w_1),l(w_2)}$ $((\alpha(w_1), \alpha(w_2)) \in R'_{l(w_1),l(w_2)}$, respectively). (Strong realization only makes sense if $l(W) \subseteq V \setminus \{x_1, x_2\}$.)*

*A pattern is (*strongly*)* realizable *if it has a (strong) realization.*

The most important patterns for our purposes are *2-trees*, these are patterns obtained from the empty pattern by gradually adding triangles (patterns whose underlying graph is the complete graph on 3 vertices) and merging them along a vertex or an edge to the already constructed pattern. Their significance stems from the following well known fact.

▶ **Lemma 34.** *An instance (over a finite domain) contains a nonempty (2,3)-subinstance if and only if every 2-tree is realizable in it.*

The "only if" direction of the lemma applied to the instance $\mathcal{I}$ implies that every 2-tree is realizable. The "if" direction applied to the instance $\mathcal{J}$ tells us that our aim boils down to proving that every 2-tree is strongly realizable. This is achieved by an induction on a suitable measure of complexity of the tree using several constructions. We will not go into full technical details here, we rather present several lemmata whose proofs contain essentially all the ideas that are necessary for the complete proof.

▶ **Lemma 35.** *Every edge (i.e., a pattern whose underlying graph is a single edge) is strongly realizable.*

Proof sketch. Let $\mathbb{Q}$ be the pattern formed by an undirected edge with vertices w^1 and w^2 labeled z_1 and z_2, respectively. Let $\mathbb{P}$ be the pattern obtained from $\mathbb{Q}$ by adding a set of four fresh vertices $W' = \{w_{11}, w_{12}, w_{21}, w_{22}\}$ labeled x_1, x_2, x_1, x_2, respectively, and adding the edges $\{w^i, w_{i1}\}$ and $\{w^i, w_{i2}\}$ for $i = 1, 2$, see Figure 2. Observe that the restriction of a

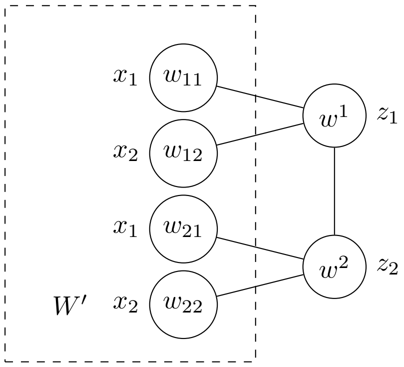

Figure 2 Pattern $\mathbb{P}$ in Lemma 35.

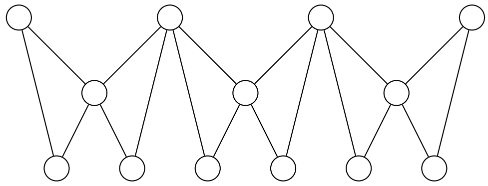

Figure 3 Path of three bow ties.

realization β of $\mathbb{P}$, such that $\beta(w_{ij}) = a_j$ for each $i, j \in \{1, 2\}$, to the set $\{w^1, w^2\}$ is a strong realization of $\mathbb{Q}$.

We consider the set T of restrictions of realizations of $\mathbb{P}$ to the set W'. Since constraint relations are subuniverses of $\mathbf{A}^2$, it follows that T is a subuniverse of $\mathbf{A}^4$.

$$T = \{(\beta(w_{11}), \beta(w_{12}), \beta(w_{21}), \beta(w_{22})) \mid \beta \text{ realizes } \mathbb{P}\} \leq \mathbf{A}^4$$

We need to prove that the tuple $\mathbf{a} = (a_1, a_2, a_1, a_2)$ is in T. By the Baker-Pixley theorem, Theorem 14, it is enough to show that for any 3-element set of coordinates, the relation T contains a tuple that agrees with $\mathbf{a}$ on this set. This is now our aim.

For simplicity, consider the set of the first three coordinates. We will build a realization β of $\mathbb{P}$ in three steps. After each step, β will satisfy all the edges where it is defined. First, since $(a_1, a_2) \in R_{x_1x_2}$ and $\mathcal{I}$ is a (2,3)-instance, we can find $b_1 \in A$ such that $(a_1, b_1) \in R_{x_1z_1}$ and $(a_2, b_1) \in R_{x_2z_1}$, and we set $\beta(w_{11}) = a_1$, $\beta(w_{12}) = a_2$, and $\beta(w^1) = b_1$. Second, we find $b_2 \in A$ such that $(a_1, b_2) \in R_{x_1z_2}$ and $(b_1, b_2) \in R_{z_1z_2}$ (here we use $(a_1, b_1) \in R_{x_1z_1}$ and that $\mathcal{I}$ is a (2,3)-instance), and set $\beta(w_{21}) = a_1$, $\beta(w^2) = b_2$. Third, using $(a_1, b_2) \in R_{x_1z_2}$ we find a_2' such that $(b_2, a_2') \in R_{z_2x_2}$ and set $\beta(w_{22}) = a_2'$. By construction, β is a realization of $\mathbb{P}$ and $(\beta(w_{11}), \beta(w_{12}), \beta(w_{21})) = (a_1, a_2, a_1)$, so our aim has been achieved. ◀

Using Lemma 35, one can go a step further and prove that every pattern built on a graph which is a triangle is strongly realizable. We are not going to prove this fact here.

▶ **Lemma 36.** *Every bow tie (a pattern whose underlying graph is formed by two triangles with a single common vertex) is strongly realizable.*

Proof sketch. Let $\mathbb{W}_1'$ and $\mathbb{W}_2'$ be two triangles (viewed as undirected graphs) with a single common vertex w. Let $\mathbb{Q}'$ be any pattern over $W_1' \cup W_2'$ with labelling l' sending $W_1' \cup W_2'$ to $V \setminus \{x_1, x_2\}$. Similarly as in the proof of Lemma 35 we form a pattern $\mathbb{Q}$ by adding to $\mathbb{Q}'$ ten additional vertices (five of them labeled x_1, the other five x_2) and edges so that the restriction of a realization α of $\mathbb{Q}$ to the set $W_1' \cup W_2'$ is a strong realization of $\mathbb{Q}'$ whenever the additional vertices have proper values (that is, value a_i for vertices labeled x_i).

We will gradually construct a realization α of $\mathbb{Q}$, which sends all the vertices labeled by x_1 to a_1, and all the vertices labeled by x_2 and adjacent to a vertex in W_1' to a_2. First use the discussion after Lemma 35 to find a strong realization of $\mathbb{Q}'$ restricted to W_1'. This defines α on W_1' and its adjacent vertices labeled by x_1 and x_2.

Next, we want to use Lemma 35 for assigning values to the two remaining vertices of W_2'. However, in order to accomplish that, we need to shift the perspective: the role of

x_1 is played by x_1, but the role of x_2 is played by $l'(w)$; and the role of (a_1, a_2) is played by $(a_1, \alpha(w))$. In this new context, we use Lemma 35 to find a strong realization of the edge-pattern formed by the two remaining vertices of W_2' (with a proper restriction of l'). This defines α on all the vertices of $\mathbb{Q}$, except for the two vertices adjacent to $W_2' \setminus \{w\}$ and labeled by x_2. Finally, similarly as in the third step in the proof of Lemma 35, we define α on the remaining two vertices (labeled x_2) to get a sought after realization of $\mathbb{Q}$.

Now α assigns proper values (a_1 or a_2) to all additional vertices, except those two coming from the non-central vertices of W_2' and labeled by x_2. We apply the 4-ary near unanimity term operation to the realization α and its 3 variants obtained by exchanging the roles of W_1' and W_2' and x_1 and x_2. The result of this application is a realization of $\mathbb{Q}$ which defines a strong realization of $\mathbb{Q}'$. ◀

In the same way it is possible to prove strong realizability of further patterns, such as those in the following corollary.

▶ **Corollary 37.** *Every "path of 3 bow ties" (i.e., a pattern whose underlying graph is as in Figure 3) is strongly realizable.*

The application of the loop lemma is illustrated by the final lemma in this section.

▶ **Lemma 38.** *Every diamond (i.e., a pattern whose underlying graph is formed by two triangles with a single common edge) is strongly realizable.*

Proof sketch. The idea is to merge two vertices in a bow tie using the loop lemma. Let $\mathbb{Q}'$ be a pattern over a graph which is a bow tie on two triangles W_1' and W_2' (just like in the proof of Lemma 36). Let $w_1 \in W_1' \setminus W_2'$ and $w_2 \in W_2' \setminus W_1'$ be such that $l(w_1) = l(w_2)$.

Let $\mathbb{Q}$ be obtained from $\mathbb{Q}'$ exactly as in the proof of Lemma 36 and notice that a proper realization α of $\mathbb{Q}$ with $\alpha(w_1) = \alpha(w_2)$ gives us a strong realization of a diamond. Let $\mathbb{Q}^3$ be the pattern obtained by taking the disjoint union of 3 copies of $\mathbb{Q}$ and identifying the vertex w_2 in the i-th copy with the vertex w_1 in the $(i+1)$-first copy, for each $i \in \{1, 2\}$ (Figure 3 shows $\mathbb{Q}^3$ without the additional vertices).

Denote by T the set of all the realizations β of $\mathbb{Q}$ and denote by $S \subseteq T$ the set of those $\beta \in T$ that are proper. By a straightforward argument, both T and S are subuniverses of $\prod_{w \in Q} \mathbf{A}$. Using the near unanimity term operation of arity 4, S clearly 4-absorbs T.

The plan is to apply Theorem 32 to the binary relation $\mathrm{proj}_{w_1,w_2} S \subseteq A \times A$. As noted above, a loop in this relation gives us the desired strong realization of a diamond, so it only remains to verify the assumptions of Theorem 32. By Corollary 37, the pattern $\mathbb{Q}^3$ has a proper realization. The images of copies of vertices w_1 and w_2 in such a realization yield a directed walk in $\mathrm{proj}_{w_1,w_2}(S)$ of length 3. Next, since S 4-absorbs T, then $\mathrm{proj}_{w_1,w_2}(S)$ 4-absorbs $\mathrm{proj}_{w_1,w_2}(T)$, so it is enough to verify that the latter relation contains $=_A$ and $\mathrm{proj}_{w_1,w_2}(S)^{-1}$. We only look at the latter property. Consider any $(b_1, b_2) \in \mathrm{proj}_{w_1,w_2}(S)^{-1}$. By the definition of S, the pattern $\mathbb{Q}$ has a realization α such that $\alpha(w_1) = b_2$ and $\alpha(w_2) = b_1$. We flip the values $\alpha(w_1)$ and $\alpha(w_2)$, restrict α to $\{w_1, w_2\}$ together with the middle vertex of the bow tie, and then extend this assignment to a realization of $\mathbb{Q}$, giving us $(b_1, b_2) \in \mathrm{proj}_{w_1,w_2}(T)$. ◀

There are two major adjustments needed for the general case. First, the "if" direction of Lemma 34 (and its analogue for a general k) is no longer true over infinite domains. This is resolved by working directly with the realizability of k-trees and proving a more general claim by induction: instead of "a $(k, k+1)$-instance is sensitive" we prove, roughly, that any evaluation, which extends to a sufficiently deep k-tree, extends to a solution. Second, for higher values of k than 2 we do not prove strong realizability in one step as in, e.g., Lemma 35, but rather go through a sequence of intermediate steps between realizability and strong realizability.

6 Conclusion

We have characterized varieties that have sensitive $(k, k+1)$-instances of the CSP as those that possess a near unanimity term of arity $k+2$. From the computational perspective, the following corollary is perhaps the most interesting consequence of our results.

▶ **Corollary 39.** *Let $\mathbb{A}$ be a finite* CSP *template whose relations all have arity at most k and which has a near unanimity polymorphism of arity $k+2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing $(k, k+1)$-consistency, is sensitive.*

Therefore not only is the $(k, k+1)$-consistency algorithm sufficient to detect global inconsistency, we also additionally get the sensitivity property. Let us compare this result to some previous results as follows. Consider a template $\mathbb{A}$ that, for simplicity, has only unary and binary relations and that has a near unanimity polymorphism of arity $k+2 \geq 4$. Then any instance of the CSP over $\mathbb{A}$ satisfies the following.

1. After enforcing $(2, 3)$-consistency, if no contradiction is detected, then the instance has a solution [4] (this is the bounded width property).
2. After enforcing $(k, k+1)$-consistency, every partial solution on k variables extends to a solution (this is the sensitivity property).
3. After enforcing $(k+1, k+2)$-consistency, every partial solution extends to a solution [13] (this is the bounded strict width property).

For $k+2 > 4$ there is a gap between the first and the second item. Are there natural conditions that can be placed there?

The properties of a template $\mathbb{A}$ from the first and the third item (holding for every instance) can be characterized by the existence of certain polymorphisms: a near unanimity polymorphism of arity $k+2$ for the third item [13] and weak near unanimity polymorphisms of all arities greater than 2 for the first item [5, 11, 17]. This paper does not give such a direct characterization for the second item (essentially, since Theorem 21 involves a square). Is there any? Moreover, there are characterizations for natural extensions of the first and the third to relational structures with higher arity relations [13, 3]. This remains open for the second item as well.

In parallel with the flurry of activity around the CSP over finite templates, there has been much work done on the CSP over infinite ω-categorical templates [9, 19]. These templates cover a much larger class of computational problems but, on the other hand, share some pleasant properties with the finite ones. In particular, the $(k, k+1)$-consistency of an instance can still be enforced in polynomial time. Corollary 39 can be extended to this setting as follows.

▶ **Corollary 40.** *Let $\mathbb{A}$ be an ω-categorical* CSP *template whose relations all have arity at most k and which has local idempotent near unanimity polymorphisms of arity $k+2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing the $(k, k+1)$-consistency, is sensitive.*

Bounded strict width k of an ω-categorical template was characterized in [10] by the existence of a *quasi-near unanimity* polymorphism n of arity $k+1$, i.e.,

$$n(y, x, \ldots, x) \approx n(x, y, \ldots, x) \approx \cdots \approx n(x, x, \ldots, y) \approx n(x, x, \ldots, x),$$

which is, additionally, *oligopotent*, i.e., the unary operation $x \mapsto n(x, x, \dots, x)$ is equal to an automorphism on every finite set. This result extends the characterization of Feder and Vardi since an oligopotent quasi-near unanimity polymorphism generates a near unanimity polymorphism as soon as the domain is finite. On an infinite domain, however, oligopotent quasi-near unanimity polymorphisms generate local near unanimity polymorphisms which, unfortunately, do not need to be idempotent on the whole domain. Our results thus fall short of proving the following natural generalization of Corollary 39 to the infinite.

▶ **Conjecture 41.** *Let $\mathbb{A}$ be an ω-categorical* CSP *template whose relations all have arity at most k and which has an oligopotent quasi-near unanimity polymorphism of arity $k+2$. Then every instance of the* CSP *over $\mathbb{A}$, after enforcing $(k, k+1)$-consistency, is sensitive.*

To confirm the conjecture, a new approach, that does not use a loop lemma, will be needed since there are examples of ω-categorical structures having oligopotent quasi-near unanimity polymorphisms for which the counterpart to Theorem 30 does not hold. Indeed, one such an example is the infinite clique.

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