TESTING FOR A SEMILATTICE TERM

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Abstract. This paper investigates the computational complexity of deciding if a given finite algebra is an expansion of a semilattice. In general this problem is known to be EXP-TIME complete, and we show that even for idempotent algebras, this problem remains hard. This result is in contrast to a series of results that show that similar decision problems turn out to be tractable.

1. Introduction

In this paper we investigate the computational complexity of deciding if a given finite algebra is an expansion of a semilattice. That is, the problem of deciding if an algebra has a binary term operation $x \wedge y$ such that the following equations hold:

$$\begin{aligned} x \wedge x &\approx x, \\ x \wedge y &\approx y \wedge x, \\ x \wedge (y \wedge z) &\approx (x \wedge y) \wedge z. \end{aligned}$$

This type of existential condition on the set of term operations of an algebra is known as a strong Maltsev condition. Maltsev conditions play a central role in the classification of algebraic structures and the equational classes that they determine, and have been studied intensively over the past several decades [8, 12]. More recently, attention has been given to computational issues related to Maltsev conditions and deep connections with well-studied combinatorial problems have been developed [2]. In some instances, efficient polynomial-time algorithms for deciding if a given Maltsev condition is satisfied by a finite algebra have been found and implemented [6, 7, 14].

To simplify the exposition, we will use the following as a working definition of a strong Maltsev condition.

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Definition 1.1. A *strong Maltsev condition* Σ consists of a finite sequence of function symbols $d_1, \dots, d_k$ and a finite set of equations involving them. An algebra $\mathbf{A}$ *satisfies* Σ if it has a sequence of term operations $f_1, \dots, f_k$, with the arity of f_i equal to the arity of d_i, for $1 \leq i \leq k$, such that the given equations are satisfied by $\mathbf{A}$ after substituting f_i for d_i, for $1 \leq i \leq k$.

So the property of having a binary term that satisfies the semilattice equations is an example of a strong Maltsev condition.

Example 1.2. The strong Maltsev condition $\mathsf{CD}(4)$ consists of three ternary function symbols d_1, d_2, d_3 along with the equations:

$$\begin{aligned} d_1(x,x,y) &\approx x, \\ d_1(x,y,y) &\approx d_2(x,y,y), \\ d_2(x,x,y) &\approx d_3(x,x,y), \\ d_3(x,y,y) &\approx y, \\ d_i(x,y,x) &\approx x, \quad \text{for } i = 1,2,3 \end{aligned}$$

In [1] it is shown that if $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a lattice, then the algebra $\langle L, x \wedge (y \vee z) \rangle$ satisfies $\mathsf{CD}(4)$. In [10] Jónsson shows that if an algebra $\mathbf{A}$ satisfies this condition then it generates a variety that is congruence distributive. In [7] it is shown that deciding if a given finite algebra satisfies $\mathsf{CD}(4)$ is $\mathsf{EXP\text{-}TIME}$ complete, while in [11] it is shown that for finite idempotent algebras, there is a polynomial-time algorithm to decide it.

This example is typical in that it turns out that for many familiar strong Maltsev conditions, deciding if a finite algebra satisfies it is an $\mathsf{EXP\text{-}TIME}$ complete problem [7, 9]. On the other hand when restricted to finite idempotent algebras the decision problems become tractable [7, 9, 11].

Note that if Σ is a strong Maltsev condition, then there will always be an $\mathsf{EXP\text{-}TIME}$ algorithm to decide if a given finite algebra $\mathbf{A}$ satisfies it. One need only construct the set of at most n-ary term operations of $\mathbf{A}$, where n is the largest arity of the function symbols that appear in Σ, and check to see if there are term operations in this set that can be used to witness Σ. For a fixed n, this set of operations can be constructed by an algorithm whose run time can be bounded by an exponential function in the size of $\mathbf{A}$.

The types of strong Maltsev conditions for which complexity results have been obtained satisfy two additional conditions: idempotency and linearity. An operation $f(\bar{x})$ on a set A is *idempotent* if $f(a, a, \dots, a) =$

a for all $a \in A$. An algebra is *idempotent* if all of its term operations are, and a strong Maltsev condition Σ is *idempotent* if its equations imply that all of the associated operations are idempotent. A strong Maltsev condition is *linear* if its equations do not involve compositions of operations. So, the strong Maltsev condition of having a semilattice term operation is idempotent but not linear, while the condition $\mathsf{CD}(4)$ is both idempotent and linear.

It is natural to wonder whether the condition of having a semilattice term operation is equivalent to a linear strong Maltsev condition. Using the criterion from Theorem 2.1 of [13] it can be shown, without too much effort, that it is not. We note that this result has been independently observed by several researchers.

In [11] it is conjectured that if Σ is a strong, idempotent, linear Maltsev condition then there is a polynomial time algorithm to decide if a finite idempotent algebra satisfies it. In that paper, and also in [14], the polynomial-time algorithms produced involve checking if there are enough "local versions" of the terms from Σ. It is shown that for certain types of Maltsev conditions, this will imply the satisfaction of Σ.

Example 1.3. A Maltsev term operation for an algebra $\mathbf{A}$ is a term operation $p(x, y, z)$ that satisfies the equations

$$p(y, x, x) \approx y \quad \text{and} \quad p(x, x, y) \approx y.$$

In [9] it is shown that a finite idempotent algebra $\mathbf{A}$ will have a Maltsev term operation if and only if for every set of elements $S = \{a, b, c, d\} \subseteq A$ there is a term operation $p_S(x, y, z)$ such that $a = p_S(a, b, b)$ and $p_S(c, c, d) = d$. So, to decide if $\mathbf{A}$ has a Maltsev term operation, one need only check, for each 4 element subset S of A, if a term operation p_S as above exists. This is equivalent to checking if the pair (a, d) is in the subuniverse of $\mathbf{A}^2$ generated by $\{(a, c), (b, c), (b, d)\}$. By [7], this test can be carried out by an algorithm whose run-time can be bounded by a polynomial in the size of $\mathbf{A}$.

In the next section we will show that no such local term testing algorithm can exist for deciding if a finite idempotent algebra has a semilattice term operation, while in Section 3 we show that testing for a flat semilattice term operation can be accomplished in this manner. In Section 4, we prove that there is no polynomial-time algorithm of any type to decide if a finite idempotent algebra has a semilattice term operation, and in fact show that this problem is $\mathsf{EXP\text{-}TIME}$ complete.

For background on the basic algebraic notions mentioned in this work, the reader may consult [5] or [3].

2. Local semilattice term operations

In this section we show that it is not sufficient for a finite idempotent algebra to have enough term operations that act locally as semilattice operations in order to conclude that it has an actual semilattice term operation. For each $n > 2$ we present a finite idempotent algebra $\mathbf{A}_n$ that does not have a semilattice term operation but such that for each subset of size $n-1$ there is a term operation that when restricted to the subset satisfies the semilattice equations.

The existence of such a family of finite idempotent algebras rules out the possibility of there being a polynomial time algorithm for deciding if a finite idempotent algebra has a semilattice term operation that is based on checking for enough local semilattice term operations. This is in contrast to the situation for many strong, idempotent, linear Maltsev conditions [7, 11].

For integers n and i with $0 \le i < n$, define $b_i(x, y)$ to be the following operation on the set $\{0, 1, 2, \dots, n-1\}$:

$$b_i(x,y) = \begin{cases} i & \text{if } \{x,y\} = \{i-1, i\} \\ \max_i(x,y) & \text{otherwise} \end{cases}$$

where the operation $\max_i(x, y)$ selects the largest of the two elements x and y with respect to the linear order

$$i < i+1 < \cdots < n-1 < 0 < 1 \cdots < i-1.$$

In the case $i = 0$, $i-1$ is set to $n-1$ in the above. Note that b_i is a commutative idempotent operation. In fact it is conservative: $b_i(x, y) \in \{x, y\}$ for all x, y. Further, the restriction of b_i to $\{0, 1, \dots, n-1\} \setminus \{i\}$ (or to $\{0, 1, \dots, n-1\} \setminus \{i-1\}$) is a semilattice operation with respect to the $\max_i$ linear order.

For $n > 2$, define $\mathbf{A}_n$ to be the algebra with universe $\{0, 1, 2, \dots, n-1\}$ with basic operations $b_0, b_1, \dots, b_{n-1}$. Then by construction $\mathbf{A}_n$ is an idempotent algebra such that for every subset S of size $n-1$ there is a binary term operation of $\mathbf{A}_n$ whose restriction to S is a semilattice operation.

In fact $\mathbf{A}_n$ is a conservative algebra: if $f(x_1, \dots, x_k)$ is a term operation of $\mathbf{A}_n$ then $f(a_1, \dots, a_k) \in \{a_1, \dots, a_k\}$ for all $a_i \in A$.

We show that in spite of having plenty of "local" semilattice term operations, the algebra $\mathbf{A}_n$ does not have a semilattice term operation. This result is a consequence of the following lemma.

Lemma 2.1. *Let $n > 2$ and $b(x, y)$ a binary term of $\mathbf{A}_n$. If $b^{\mathbf{A}_n}(x, y)$ is not equal to one of the binary projection maps $\pi_1(x, y)$ or $\pi_2(x, y)$ then $b^{\mathbf{A}_n}(i, i+1) = i + 1 \mod n$ for all $0 \le i < n$.*

Proof. We prove this by induction on the length of the term $b(x, y)$. If b is just a projection term, there is nothing to prove. So, we may assume that $b(x, y) = b_i(s_1(x, y), s_2(x, y))$ for some i and some shorter binary terms s_1 and s_2. By symmetry, we may assume that $i = 0$. If $s_1^{\mathbf{A}_n}$ and $s_2^{\mathbf{A}_n}$ are both projection operations, then $b^{\mathbf{A}_n}(x, y)$ is equal to one of the operations $b_0(x, x) = x$, $b_0(y, y) = y$, which are both binary projection operations, or to $b_0(x, y)$. In the latter case, it can be readily seen that $b_0(i, i + 1) = i + 1 \mod n$ for all i.

If one of $s_1^{\mathbf{A}_n}(x, y)$ or $s_2^{\mathbf{A}_n}(x, y)$ is not a projection operation, then since b_0 is commutative, we may assume that $s_2^{\mathbf{A}_n}(x, y)$ is not a projection. Then by induction, for any i, $s_2^{\mathbf{A}_n}(i, i + 1) = i + 1 \mod n$ and so $b^{\mathbf{A}_n}(i, i + 1)$ is equal to $b_0(i, i + 1) = i + 1 \mod n$, or to $b_0(i + 1, i + 1) = i + 1 \mod n$ (since $s_i^{\mathbf{A}_n}$ is a conservative operation). Thus, in all cases we have shown that $b^{\mathbf{A}_n}(i, i + 1) = i + 1$. □

Theorem 2.2. *For any $n > 2$, the algebra $\mathbf{A}_n$ does not have a semilattice term operation.*

Proof. If $x \wedge y$ is a semilattice term operation of $\mathbf{A}_n$, then it is not a projection operation and so from the previous lemma we know that $i \wedge (i + 1) = i + 1 \mod n$ for all i. In terms of the semilattice ordering determined by the operation $\wedge$, this translates to $i > i + 1 \mod n$. Applying transitivity and using that we are operating modulo n, we reach the contradiction that $0 > 0$. □

3. Flat semilattices

In this section we show that there is a polynomial-time algorithm to determine if a given finite idempotent algebra has a *flat semilattice term operation*, i.e., a binary term operation $b(x_0, x_1)$ and an element $0 \in A$ such that $b(a, a) = a$ for all $a \in A$ and $b(a, a') = 0$ for all a, $a' \in A$ with $a \neq a'$. The element 0 is called an *absorbing element* for the operation b. This is in contrast to the non-idempotent case. In [7] it is shown that in general, testing for such a binary term in a finite algebra is an EXP-TIME complete problem.

We first show that there is a polynomial-time algorithm to determine, given a finite idempotent algebra $\mathbf{A}$ and an element $0 \in A$, if there is a binary term operation $b(x_0, x_1)$ of $\mathbf{A}$ such that $b(0, a) = b(a, 0) = 0$ for all $a \in A$.

Definition 3.1. Let $\mathbf{A}$ be a finite idempotent algebra and let 0, u, $v \in A$. For any $S \subseteq A \times \{\mathbf{0}, \mathbf{1}\}$, we say that a term operation $b(x_0, x_1)$ of $\mathbf{A}$ is *local for S and (u, v)* if:

- $b(a, 0) = 0$ whenever $(a, \mathbf{0}) \in S$,

- $b(0, a) = 0$ whenever $(a, \mathbf{1}) \in S$, and
- $b(u, v) = 0$.

Lemma 3.2. *Let* $\mathbf{A}$ *be a finite idempotent algebra and let* $0 \in A$. *Then for all* $(u, v) \in A^2$ *with* $u \neq v$, $\mathbf{A}$ *has a binary term operation* $b_{(u,v)}(x, y)$ *such that for all* $a \in A$,

$$b_{(u,v)}(0, a) = b_{(u,v)}(a, 0) = b_{(u,v)}(u, v) = 0$$

if and only if for each $(a_0, a_1), (c, d) \in A^2$ *with* $c \neq d$, $\mathbf{A}$ *has a term operation* $b(x_0, x_1)$ *that is local for* $\{(a_0, \boldsymbol{0}), (a_1, \boldsymbol{1})\}$ *and* (c, d).

Proof. Suppose that for all $(u, v) \in A^2$ with $u \neq v$, $\mathbf{A}$ has a binary term operation $b_{(u,v)}(x, y)$ such that for all $a \in A$,

$$b_{(u,v)}(0, a) = b_{(u,v)}(a, 0) = b_{(u,v)}(u, v) = 0.$$

Then for any $(a_0, a_1), (c, d) \in A^2$ with $c \neq d$, the term $b_{(c,d)}(x_0, x_1)$ is local for $\{(a_0, \mathbf{0}), (a_1, \mathbf{1})\}$ and (c, d) since it is local for $A \times \{\mathbf{0}, \mathbf{1}\}$ and (c, d).

Conversely, suppose that for each $(a_0, a_1), (c, d) \in A^2$ with $c \neq d$, $\mathbf{A}$ has a term operation $b(x_0, x_1)$ that is local for $\{(a_0, \mathbf{0}), (a_1, \mathbf{1})\}$ and (c, d). We will prove by induction on the size of $S \subseteq A \times \{\mathbf{0}, \mathbf{1}\}$, that

> for any $(c, d) \in A^2$, with $c \neq d$, $\mathbf{A}$ has a term operation that is local for S and (c, d).

By assumption, this condition holds when $|S| = 1$.

Suppose that $|S| > 1$ and assume that the condition holds for all sets that are smaller in size than S. If S is of the form $\{(a_0, \mathbf{0}), (a_1, \mathbf{1})\}$ for some elements a_0, $a_1 \in A$ then by assumption the condition holds for S. Otherwise, there must be elements a, $a' \in A$ such that $a \neq a'$ and either $\{(a, \mathbf{0}), (a', \mathbf{0})\} \subseteq S$ or $\{(a, \mathbf{1}), (a', \mathbf{1})\} \subseteq S$. Without loss of generality, we may assume that the former holds. Let $S' = S \setminus \{(a, \mathbf{0})\}$ and let $b'(x_0, x_1)$ be a term operation of $\mathbf{A}$ that is local for S' and (c, d). Let $b(x_0, x_1)$ be a term operation of $\mathbf{A}$ that is local for the set

$$\{(w, \mathbf{1}) \mid (w, \mathbf{1}) \in S\} \cup \{(b'(a, 0), \mathbf{0})\}$$

and the pair $(0, d)$. By our induction hypothesis, the term operations b' and b are guaranteed to exist.

Let $t(x_0, x_1) = b(b'(x_0, x_1), x_1)$. We claim that t is local for S and (c, d). To see this, let $(w, \mathbf{1}) \in S$. Then

$$t(0, w) = b(b'(0, w), w) = b(0, w) = 0,$$

since both b and b' are local for $\{(w, \mathbf{1})\}$. If $(w, \mathbf{0}) \in S$ with $w \neq a$, then

$$t(w, 0) = b(b'(w, 0), 0) = b(0, 0) = 0,$$

since b' is local for $\{(w, \mathbf{0})\}$ and b is idempotent. Finally,

$$t(a, 0) = b(b'(a, 0), 0) = 0$$

and

$$t(c, d) = b(b'(c, d), d) = b(0, d) = 0,$$

since b' is local for (c, d) and b is local for $\{(b'(a, 0), \mathbf{0})\}$ and $(0, d)$.

By induction, we can conclude that for any $(u, v) \in A^2$ with $u \neq v$, $\mathbf{A}$ has a binary term operation $b_{(u,v)}(x_0, x_1)$ that is local for $A \times \{\mathbf{0}, \mathbf{1}\}$, i.e., that for all $a \in A$,

$$b_{(u,v)}(0, a) = b_{(u,v)}(a, 0) = b_{(u,v)}(u, v) = 0.$$

□

Corollary 3.3. *The problem of deciding, given a finite idempotent algebra* $\mathbf{A}$ *and an element* $0 \in A$*, if for all* $(u, v) \in A^2$ *with* $u \neq v$*,* $\mathbf{A}$ *has a binary term operation* $b_{(u,v)}(x, y)$ *such that for all* $a \in A$*,*

$$b_{(u,v)}(0, a) = b_{(u,v)}(a, 0) = b_{(u,v)}(u, v) = 0$$

is in the complexity class P.

Proof. Given idempotent $\mathbf{A}$ and $0 \in A$, to determine if such term operations exist we need only verify the local condition of the previous Lemma. That is, we need to check, given two elements $(a_0, \mathbf{0})$ and $(a_1, \mathbf{1})$ from $A \times \{\mathbf{0}, \mathbf{1}\}$ and some pair (c, d) from A^2 with $c \neq d$, that there is a binary term operation $b(x_0, x_1)$ of $\mathbf{A}$ such that $b(a_0, 0) = 0$, $b(0, a_1) = 0$, and $b(c, d) = 0$. This amounts to checking, for each a_0, a_1, c, $d \in A$ that the tuple $(0, 0, 0)$ lies in the subuniverse of $\mathbf{A}^3$ generated by $\{(a_0, 0, c), (0, a_1, d)\}$. By [7] we know that this can be carried out via an algorithm that runs in time bounded by a polynomial in the size of $\mathbf{A}$. □

Theorem 3.4. *The problem of deciding, given a finite idempotent algebra* $\mathbf{A}$*, whether or not it has a flat semilattice term operation, is in* P.

Proof. We will show that $\mathbf{A}$ will have such a term operation, with $0 \in A$ as its absorbing element, if and only if for each $(u, v) \in A^2$ with $u \neq v$ there is a term operation $b_{(u,v)}(x_0, x_1)$ of $\mathbf{A}$ such that for all $a \in A$,

$$b_{(u,v)}(0, a) = b_{(u,v)}(a, 0) = b_{(u,v)}(u, v) = 0.$$

Of course, if $\mathbf{A}$ has a flat semilattice term operation $b(x_0, x_1)$ with absorbing element 0 then this condition can be seen to hold by setting $b_{(u,v)}(x_0, x_1) = b(x_0, x_1)$ for all $(u, v) \in A^2$ with $u \neq v$.

Conversely, assuming that the above condition holds, we will show by induction on the size of $S \subseteq A^2 \setminus \{(a, a) \mid a \in A\}$ that $\mathbf{A}$ has a term operation $b_S(x_0, x_1)$ such that $b_S(0, a) = b_S(a, 0) = 0$ for all $a \in A$ and $b_S(u, v) = 0$ for all $(u, v) \in S$.

By assumption this condition holds for any S with $|S| = 1$. Suppose that $|S| > 1$ and assume that the condition holds for all sets that are smaller in size than S. We can write S as $S' \cup \{(u, v)\}$ for some $S' \subset S$ and $u \neq v \in A$ with $(u, v) \notin S'$.

Let $b_{S'}(x_0, x_1)$ be a term operation of $\mathbf{A}$ that satisfies the condition for the set S' and let $b_{(u,v)}(x_0, x_1)$ be a term operation that satisfies the condition for the set $\{(u, v)\}$. By the induction hypothesis $b_{S'}$ exists and by assumption $b_{(u,v)}$ exists. Let

$$b(x_0, x_1) = b_{S'}(b_{S'}(x_0, x_1), b_{(u,v)}(x_0, x_1)).$$

Then for $(c, d) \in S'$,

$$b(c, d) = b_{S'}(b_{S'}(c, d), b_{(u,v)}(c, d)) = b_{S'}(0, b_{(u,v)}(c, d)) = 0,$$

and

$$b(u, v) = b_{S'}(b_{S'}(u, v), b_{(u,v)}(u, v)) = b_{S'}(b_{S'}(u, v), 0) = 0.$$

Finally, for $a \in A$,

$$b(0, a) = b_{S'}(b_{S'}(0, a), b_{(u,v)}(0, a)) = b_{S'}(0, 0) = 0$$

and

$$b(a, 0) = b_{S'}(b_{S'}(a, 0), b_{(u,v)}(a, 0)) = b_{S'}(0, 0) = 0.$$

By induction we can conclude that $\mathbf{A}$ will have a binary term operation that satisfies the condition for $S = A^2 \setminus \{(a, a) \mid a \in A\}$. Such an operation is a flat semilattice operation on A with absorbing element 0.

We can now use Corollary 3.3 to show that there is a polynomial time algorithm to determine, for a given $0 \in A$, if $\mathbf{A}$ has a flat semilattice term operation with absorbing element 0. To determine if $\mathbf{A}$ has such an operation for some $0 \in A$, we need only run this algorithm at most $|A|$ times. $\square$

4. A hardness result

In this section we show that the problem of deciding if a finite idempotent algebra has a semilattice term operation is EXP-TIME complete. We first show that the following decision problem is EXP-TIME complete:

BOUNDED-SEMILATTICE: Given a finite **idempotent** algebra $\mathbf{A}$ and element $1 \in A$, is there a binary term operation $x \wedge y$ of $\mathbf{A}$ such that $\langle A, \wedge, 1\rangle$ is a bounded semilattice, i.e., $\langle A, \wedge\rangle$ is a semilattice and $a \wedge 1 = 1 \wedge a = a$ for all $a \in A$?

Our proof of the hardness of BOUNDED-SEMILATTICE will actually show that the problem of deciding whether a given finite idempotent algebra has an element 1 and a binary term operation $b(x, y)$ such that $b(x, 1) = b(1, x) = x$ holds for all elements x is an EXP-TIME complete problem.

To obtain our result, we reduce the clone membership problem to BOUNDED-SEMILATTICE. From [4] and [7] we know that the following version of clone membership is EXP-TIME complete.

GEN-CLO$'$: Given a finite set A, a finite set of operations $\mathcal{F}$ on A and a **unary** operation h on A, is h in the clone on A generated by $\mathcal{F}$?

Theorem 4.1. *The problem* BOUNDED-SEMILATTICE *is* EXP-TIME *complete.*

Proof. We prove this theorem by reducing GEN-CLO$'$ to BOUNDED-SEMILATTICE. Let $\langle A, \mathcal{F}, h(x)\rangle$ be an instance of GEN-CLO$'$ and let $A' = A \cup \{0, 1\}$ for two distinct elements 0, 1 that are not in A. Without loss of generality, we may assume that $\mathcal{F}$ contains the unary identity function on A. Let $\leq$ be the partial order on A' with $0 < a < 1$ for all $a \in A$ and such that any two elements of A are incomparable. Let $x \wedge y$ be the bounded meet semilattice operation with respect to this ordering.

For $g(x_1, \dots, x_k)$ a k-ary operation on A, define $g'(x_1, \dots, x_k, y)$ to be the following $(k + 1)$-ary operation on A':

$$g'(x_1, \dots, x_k, y) = \begin{cases} g(x_1, \dots, x_k) & \text{if } \{x_1, \dots, x_k\} \subseteq A \text{ and } y = 1; \\ y & \text{if } x_i = y \text{ for all } 1 \leq i \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

Define $t_h(x, y, z)$ on A' as follows:

$$t_h(x, y, z) = \begin{cases} x \wedge y, & \text{if } z = h'(x, y); \\ 0, & \text{otherwise.} \end{cases}$$

Note that t_h can be described as the ternary **idempotent** operation on A' such that for $a \in A$, $t_h(a, 1, h(a)) = t_h(1, a, 0) = a$ and is equal to 0 in all other cases not covered by idempotency.

Let $\mathbf{A}'$ be the algebra with universe A' and with basic operations $t_h(x, y, z)$ and f', for all $f \in \mathcal{F}$. Note that $\mathbf{A}'$ is an idempotent algebra, and it can be constructed from the given instance of GEN-CLO$'$ by an algorithm whose run time can be bounded by a polynomial in the size of the instance. Observe that the basic operations of $\mathbf{A}'$ depend on all of their variables. We will show that $\langle A, \mathcal{F}, h(x)\rangle$ is a "yes" instance of GEN-CLO$'$ if and only if $\mathbf{A}'$ and 1 is a "yes" instance of BOUNDED-SEMILATTICE.

Claim 4.2. *Let $t(x_1, \dots, x_k)$ be a term operation of $\mathbf{A}'$ and let $u_i \in A'$ for $1 \leq i \leq k$.*

(1) *$A \cup \{0\}$ and $\{0, 1\}$ are subuniverses of $\mathbf{A}'$ and the element 0 is an absorbing element in both of these subuniverses (but not in A').*
(2) *If t depends on all of its variables and $t(u_1, \dots, u_k) = 1$ then $u_i = 1$ for all i.*
(3) *If $t(a_1, \dots, a_{k-1}, 1) \in A$ for all $a_i \in A$ then the restriction of the operation $t(x_1, \dots, x_{k-1}, 1)$ to A is a member of the clone on A generated by $\mathcal{F}$.*
(4) *If $g(x)$ is in the clone on A generated by $\mathcal{F}$ then the operation $g'(x, y)$ is in the clone of $\mathbf{A}'$.*

Statement (1) can be verified by examining the basic operations of $\mathbf{A}'$. Statement (2) can be proved by induction on the length of a term that defines the operation t, but it suffices to note that the basic operations of $\mathbf{A}'$ have this property and it is preserved under the composition of functions.

Statement (3) can be proved by induction on the length of a term that defines the operation t. If t is a projection operation then by our assumption, it follows that it projects onto x_i for some $i < k$ and in this case, the statement holds. If t is not a projection operation then it can be written as

$$f'(s_1(\vec{x}), \dots, s_m(\vec{x}), r(\vec{x})) \text{ or } t_h(s_1(\vec{x}), s_2(\vec{x}), r(\vec{x}))$$

for some term operations s_i and r of $\mathbf{A}'$ that have shorter definitions than t does and some $f \in \mathcal{F}$. If the former holds, then it follows that $s_j(a_1, \dots, a_{k-1}, 1) \in A$ and $r(a_1, \dots, a_{k-1}, 1) = 1$ for all j and all $a_i \in A$. By induction, we can apply the claim to the s_j's and then compose their restrictions to A with the operation f to establish the claim for t. Note: this part of the argument uses part (2) of the claim. We need to consider the possible values for $r(a_1, \dots, a_{k-1}, 1)$. If it ever takes on the value 1, then it can only depend on its last variable, by (2), in which case, we apply the induction hypothesis to the s_j. If $r(a_1, \dots, a_{k-1}, 1)$ never takes on the value 1, then it must always take on values in A (or else $t(a_1, \dots, a_{k-1}, 1)$ would be equal to 0 for some a_i). But then all of the values $s_j(a_1, \dots, a_{k-1}, 1)$ must equal $r(a_1, \dots, a_{k-1}, 1)$ for all $a_i \in A$, implying that $t(a_1, \dots, a_{k-1}, 1) = r(a_1, \dots, a_{k-1}, 1)$. Since r has a shorter definition than t, the claim follows in this case.

For the latter case, from $t(a_1, \dots, a_{k-1}, 1) \in A$ we can deduce that both of $s_1(a_1, \dots, a_{k-1}, 1)$ and $s_2(a_1, \dots, a_{k-1}, 1)$ must belong to $A \cup \{1\}$. There are two subcases to consider. If these two values always lie in A (and so never take on the value 1) then we can apply our induction hypothesis to both s_1 and s_2. The fact that the value of t is always in A implies that $s_1(a_1, \dots, a_{k-1}, 1) = s_2(a_1, \dots, a_{k-1}, 1)$ and that $t(a_1, \dots, a_{k-1}, 1)$ is equal to this value, for all $a_i \in A$. But then the restriction of $t(x_1, \dots, x_{k-1}, 1)$ to A is equal to the restriction of $s_1(x_1, \dots, x_{k-1}, 1)$ to A, which, by induction, we know lies in the clone generated by $\mathcal{F}$.

On the other hand, if for some $a_i \in A$ and some $j = 1$ or 2, $s_j(a_1, \dots, a_{k-1}, 1)$ is equal to 1 then using (2) we can conclude that this term operation is equal to its projection onto the variable x_k. It follows that for all $a_i \in A$, $s_p(a_1, \dots, a_{k-1}, 1) \in A$ (where $\{j, p\} = \{1, 2\}$) and that $t(a_1, \dots, a_{k-1}, 1)$ is equal to this value. By induction, we conclude that the restriction of $s_p(x_1, \dots, x_{k-1}, 1)$ to A and hence the restriction of $t(x_1, \dots, x_{k-1}, 1)$ to A, lies in the clone generated by $\mathcal{F}$.

The last part of this claim can be proved by induction on the construction of g from the operations in $\mathcal{F}$. Just replace every occurrence of a function from $f \in \mathcal{F}$ by f'. For the base of the induction, when $g(x) = x$, recall that we are assuming that $\mathcal{F}$ contains g and so, by construction, $g'(x, y)$ is in the clone of $\mathbf{A}'$.

Claim 4.3. *There is a binary term operation $b(x, y)$ of $\mathbf{A}'$ such that $b(a, 1)$, $b(1, a) \in A$ for all $a \in A$ if and only if $h(x)$ is in the clone on A generated by $\mathcal{F}$.*

One direction of this claim follows by construction. Namely, if $h(x)$ is in the clone generated by $\mathcal{F}$ then by part (4) of Claim 4.2, the operation $h'(x, y)$ is in the clone of $\mathbf{A}'$. Then the term operation $b(x, y) = t_h(x, y, h'(x, y))$ is the semilattice operation $x \wedge y$ on A' and so satisfies the conditions of this claim.

Conversely, if $\mathbf{A}'$ has such a term operation $b(x, y)$ then select one that has the shortest possible definition. Clearly it cannot be a projection operation and so must be of the form

$$f'(b_1(x, y), \dots, b_m(x, y), r(x, y)) \text{ or } t_h(b_1(x, y), b_2(x, y), r(x, y))$$

for some term operations b_i and r of $\mathbf{A}'$ that have shorter definitions than b does and some $f \in \mathcal{F}$. In the former case, because we are assuming that $b(1, a)$ and $b(a, 1)$ to belong to A for all $a \in A$, it follows from the definition of f' that $b_1(1, a)$ and $b_1(a, 1) \in A$ for all $a \in A$. This contradicts our assumption on the minimality of the length of a term defining b.

Thus we have that $b(x,y) = t_h(b_1(x,y), b_2(x,y), r(x,y))$. In this case, the only way for $b(a,1)$ and $b(1,a)$ to belong to A for all $a \in A$ is for each $i = 1$, 2 to have $b_i(a,1)$ and $b_i(1,a)$ belonging to $A \cup \{1\}$. If for some i, $b_i(a,1)$ and $b_i(1,a)$ both belong to A for all $a \in A$, then we could have used b_i in place of b, contradicting the selection of b as having the shortest definition amongst such term operations.

So, we may assume that for some $a \in A$, $b_1(a,1) = 1$ or $b_1(1,a) = 1$. Using part (2) of Claim 4.2, it follows that one of the equations $b_1(x,y) \approx x$ or $b_1(x,y) \approx y$ holds and we can draw a similar conclusion for $b_2(x,y)$. Thus, we can conclude that $b(x,y)$ is equal to $t_h(x,y,r(x,y))$ (or to $t_h(y,x,r(x,y))$) for all $x, y \in A'$. From $b(a,1) \in A$ we get that $t_h(a,1,r(a,1)) \in A$ and hence that $r(a,1) = h'(a,1) = h(a)$ for all $a \in A$. Using part (3) of Claim 4.2, it follows that the operation $h(x)$ is in the clone generated by $\mathcal{F}$. This concludes the proof of this claim.

As noted in the proof of the previous claim, if $h(x)$ is in the clone generated by $\mathcal{F}$ then the semilattice operation $x \wedge y$ on A' is a term operation of $\mathbf{A}'$ and has 1 as the largest element. Conversely, if $\mathbf{A}'$ has a semilattice term operation $x \curlywedge y$ with largest element 1, then for all $a \in A$ we have that $1 \curlywedge a = a \curlywedge 1 = a \in A$ and so by the previous claim, $h(x)$ is in the clone generated by $\mathcal{F}$. Thus our instance of $\mathsf{GEN\text{-}CLO}'$ is a "yes" instance if and only if the algebra $\mathbf{A}'$ with the element 1 is a "yes" instance of BOUNDED-SEMILATTICE. $\square$

Corollary 4.4. *The problem of deciding if a given finite idempotent algebra* $\mathbf{A}$ *has a binary term* $b(x,y)$ *and an element 1 such that* $b(x,1) = b(1,x) = x$ *holds for all* $x \in A$ *is* EXP-TIME *complete.*

In contrast to this result, we note that from the previous section we can conclude that testing for the presence in a finite idempotent algebra of a binary term $b(x,y)$ and an element 0 such that $b(x,0) = b(0,x) = 0$ is in the class P.

The main result of this section follows from Theorem 4.1.

Theorem 4.5. *The problem of deciding if a given finite idempotent algebra has a semilattice operation is* EXP-TIME *complete.*

Proof. We reduce BOUNDED-SEMILATTICE to this problem. Let $\mathbf{A} = \langle A, f_1, \ldots, f_m \rangle$ be a finite idempotent algebra with $1 \in A$, and let $\diamond$ be some element that isn't in A. For $f(x_1, \ldots, x_k)$ an operation on A, let

$f^\diamond$ be the extension of it to $A^\diamond = A \cup \{\diamond\}$, defined by:

$$f^\diamond(x_1,\dots,x_k) = \begin{cases} f(x_1,\dots,x_k) & \text{if } \{x_1,\dots,x_k\} \subseteq A; \\ a & \text{if } \{x_1,\dots,x_k\} = \{a,\diamond\} \text{ and } a \neq 1; \\ \diamond & \text{if } \{x_1,\dots,x_k\} = \{1,\diamond\} \text{ or } \{\diamond\}; \\ 1 & \text{otherwise.} \end{cases}$$

We observe that if f is idempotent, then so is $f^\diamond$ and that for each $a \in A$, the subset $\{a,\diamond\}$ is closed under $f^\diamond$. The set A is also closed under $f^\diamond$. We also note that $f^\diamond$ depends on each of its variables.

Let $\mathbf{A}^\diamond = \langle A^\diamond, f_1^\diamond, \dots, f_m^\diamond\rangle$. We will show that $\mathbf{A}^\diamond$ has a semilattice term operation if and only if $\mathbf{A}$ has a semilattice term operation with largest element 1.

Claim 4.6. *Let $b(x,y)$ be a binary term operation of $\mathbf{A}^\diamond$ that depends on the variable x. Then for $a \in A \setminus \{1\}$, $b(a,\diamond) = a$ and $b(\diamond,1) = \diamond$.*

We prove this claim by induction on the length of a term that defines the operation b. Clearly if $b(x,y)$ is a projection operation, then it is equal to the first projection and the claim is true. Otherwise, $b(x,y)$ can be written as $f_j^\diamond(b_1(x,y),\dots,b_k(x,y))$ for some $j \leq m$ and for some binary term operations b_i of $\mathbf{A}^\diamond$ that have shorter definitions than b, where $f_j^\diamond$ is k-ary.

Since b depends on x, then for at least one i, $b_i(x,y)$ also depends on x, and then by induction we have that $b_i(a,\diamond) = a$ when $a \neq 1$, and $b_i(\diamond,1) = \diamond$. As noted earlier, $\{a,\diamond\}$ and $\{1,\diamond\}$ are subuniverses of $\mathbf{A}^\diamond$ and so for any $l \leq k$, $b_l(a,\diamond) \in \{a,\diamond\}$ and $b_l(\diamond,1) \in \{1,\diamond\}$. So $b(a,\diamond)$ is equal to $f_j^\diamond$ applied to some sequence of elements from $\{a,\diamond\}$ with at least one of them equal to a. From the definition of $f_j^\diamond$ we conclude that $b(a,\diamond) = a$. A similar argument shows that $b(\diamond,1) = \diamond$.

The following claim follows from the fact that $\mathbf{A}$ is a subalgebra of $\mathbf{A}^\diamond$.

Claim 4.7. *If $b(x,y)$ is a term operation of $\mathbf{A}^\diamond$, then its restriction to A is a term operation of $\mathbf{A}$. Furthermore, if $s(x,y)$ is a term operation of $\mathbf{A}$, then it is the restriction to A of some binary term operation of $\mathbf{A}^\diamond$.*

Now, suppose that $\mathbf{A}^\diamond$ has a semilattice term operation $x \wedge y$. By the previous claim, its restriction to A is a semilattice term operation $x \curlywedge y$ of $\mathbf{A}$. Since $x \wedge y$ depends on both x and y then by Claim 4.6 we have that for every $a \in A \setminus \{1\}$,

$$1 \curlywedge a = 1 \wedge a = 1 \wedge (\diamond \wedge a) = (1 \wedge \diamond) \wedge a = \diamond \wedge a = a.$$

This establishes that $\langle A, \curlywedge, 1\rangle$, is a bounded semilattice.

Conversely, suppose that $\mathbf{A}$ has a binary term operation $x \curlywedge y$ such that $\langle A, \curlywedge, 1\rangle$, is a bounded semilattice. Then by the previous claim, there is a binary term operation $x \wedge y$ of $\mathbf{A}^\diamond$ whose restriction to A is $x \curlywedge y$. From Claim 4.6 we have that $1 \wedge \diamond = \diamond \wedge 1 = \diamond$ and for any $a \in A \setminus \{1\}$, $a \wedge \diamond = \diamond \wedge a = a$. It follows that $\wedge$ is a semilattice operation on $A^\diamond$ that extends $\curlywedge$ such that the element $\diamond$ lies below the element 1 and above all of the other elements from A. □

5. Conclusion

The hardness result from the previous section demonstrates that testing for a rather simple, strong, idempotent, non-linear Maltsev condition can be difficult, even for idempotent algebras. It would be interesting to find a strong, idempotent Maltsev condition that is not equivalent to a linear one, but which can be tested in polynomial time for idempotent algebras. A possible candidate is the 2-semilattice condition: a binary function that satisfies all of the 2-variable equations satisfied in the variety of semilattices. This is the strong idempotent Maltsev condition of having a binary term $x \wedge y$ that satisfies the equations: $x \wedge x \approx x$, $x \wedge y \approx y \wedge x$, and $x \wedge (x \wedge y) \approx x \wedge y$.

Should it turn out that this condition is $\mathsf{EXP\text{-}TIME}$ hard to decide for finite idempotent algebras, then one would be led to consider the following problem that complements the conjecture that deciding strong, idempotent, linear Maltsev conditions for finite idempotent algebras can be done with polynomial-time algorithms.

Problem 5.1. Is it the case that if Σ is a strong idempotent Maltsev condition that is not equivalent to a strong idempotent **linear** Maltsev condition, then the problem of deciding if a finite idempotent algebra satisfies Σ is $\mathsf{EXP\text{-}TIME}$ complete?

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