Idempotent Maltsev Conditions

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Abstract

Varieties can be classified based on the congruence properties of the algebras contained in them. Also, they can be characterized by the existence of conditions satisfied by the terms of the variety. Such conditions are known as Maltsev conditions. Many known congruence properties of varieties are shown to be equivalent to Maltsev conditions. For example congruence permutability, congruence distributivity and congruence modularity are equivalent to Maltsev conditions [Ber11]. This paper gives a summary of the known Maltsev conditions for six families of varieties in the locally finite and the non-locally finite cases. For some varieties there exists much nicer characterizations while for others the known Maltsev conditions cannot be simplified any further.

1 Introduction

The following definitions can be found in [Ber11]. For more on the subject see [Ber11].

Definition 1.1. *An **algebra** is a pair $\langle A, F\rangle$ where A is a nonempty set and $F = \{f_i : i \in I\}$ is a family of operations on the set A indexed by some set I. The set A is called the universe of the algebra and the f_i are called the basic operations of the algebra.*

Examples 1.2.

- *Group:* $\langle G, \circ, ^{-1}, e\rangle$.
- *Vector space over some field* F : $\langle V, +, -, 0, c_f : f \in F\rangle$.

- *Lattice:* $\langle L, \vee, \wedge\rangle$. *A lattice is an algebra with two binary operations* $\vee, \wedge$ *such that the following set of identities hold:* $x \vee x \approx x, x \vee (y \vee z) \approx (x \vee y) \vee z, x \vee y \approx y \vee x$ *and* $x \vee (x \wedge y) \approx x$ *and the dual identities obtained by interchanging* $\vee$ *and* $\wedge$.

Remark 1.3.
- *Normal subgroups of a group* G *form a lattice:* $\langle NmlG, N_1 \vee N_2 = N_1N_2, N_1 \wedge N_2 = N_1 \bigcap N_2\rangle$.
- *Subspaces of a vector space* V *form a lattice:* $\langle SubV, A \vee B = A \bigoplus B, A \wedge B = A \bigcap B\rangle$.

Definition 1.4. *Let* $\boldsymbol{A}$ *be an algebra and let* θ *be a binary relation on* $\boldsymbol{A}$.

1. *We say* θ *has the* ***substitution property*** *if for every basic operation* f *on* $\boldsymbol{A}$, *with n=arity of* f, *we have* $x_1\theta y_1$ *&* $x_2\theta y_2$ *&* $\cdots$ *&* $x_n\theta y_n \Rightarrow f(x_1, x_2, \ldots, x_n)\theta f(y_1, y_2, \ldots, y_n)$.
2. *A* ***congruence relation*** *on* $\boldsymbol{A}$ *is an equivalence relation on* $\boldsymbol{A}$ *with the substitution property.*

Remark 1.5. *[Ber11]*

1. *The collection of congruence relations on an algebra* $\boldsymbol{A}$ *naturally forms a lattice, called the congruence lattice of* $\boldsymbol{A}$ *and is denoted* Con *(*$\boldsymbol{A}$*).*
2. *For an algebra* $\boldsymbol{A}$ *let* $f : A \rightarrow B$ *be any homomorphism with domain* A. *The kernel of the homomorphism, denoted ker(f), is an equivalence relation on* $\boldsymbol{A}$ *and is defined as the set* $\{$*(a,b) : f(a) = f(b)*$\}$. *It can be easily verified that it is a congruence relation on* $\boldsymbol{A}$. *Conversely, given any congruence relation* θ *on* $\boldsymbol{A}$ *we can define a homomorphism* $f : A \rightarrow A/\theta$ *where* θ *= Ker(f) and* $f(a) = a/\theta$, *the* θ*-class that contains a.*
3. *Given any congruence relation* θ *on a group* G *the quotient structure* G/θ *is the same as the quotient group* G/N, *where* N *is the normal subgroup consisting of all elements of* G *that are* θ*-related to e. There is a bijective map between the congruences on a group* G *and the normal subgroups of* G. *There is a similar correspondence between ring congruences and ideals and between vector space congruences and subspaces.*

Definition 1.6. 1. *A **variety** is a class of similar algebras that is closed under the formation of homomorphic images, subalgebras and direct products.*

2. *An algebra is called **locally finite** if every finitely generated subalgebra is finite. A variety is locally finite if every member of the variety is locally finite.*

3. *A variety is **finitely generated** if it is of the form $V(K)$ for some finite set of finite algebras K.*

Remark 1.7. *If K is a class of similar algebras then the variety generated by K is the smallest variety containing K and is denoted $V(K)$. Birkhoff's theorem shows that a class of algebras is a variety iff it can be defined by a set of equations. For example groups and rings are defined by sets of equations. For more on the subject see Theorem 4.41 in [Ber11].*

Definition 1.8. *[Val, KK09]*

1. *An operation $f(\vec{x})$ on a set A is **idempotent** if the equation $f(x, x, \ldots, x) \approx x$ holds. A term $t(\vec{x})$ of an algebra or variety is idempotent if the associated operation is, and we call an algebra or variety idempotent if all of its terms are.*

2. *A term $t(\vec{x})$ of a variety V is a **weak near unanimity term** if it is idempotent and V satisfies the following equations: $t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, x, \ldots, x, y)$.*

Definition 1.9. *[KK09, Val]*

1. *A **Maltsev condition** is a countably infinite disjunction $\bigvee_{i\in\omega} \sigma_i$ where σ_i are strong Maltsev conditions. A **strong Maltsev condition** is a first order sentence of the form:*
   $\exists \bigwedge$(atomic sentence) in the language of clones of operations.

   Remark 1.10. *Informally a Maltsev condition refers to the existence of special term operations that satisfy a set of equations. For example a variety V is congruence permutable if the following Maltsev condition holds: there exists a term p(x,y,z) in the variety V such that $p(x, y, y) \approx x \approx p(y, y, x)$ holds.*

2. *Let U and V be varieties and let $\{f_i : i \in I\}$ be the set of basic operations of U. We say U is* ***interpretable*** *in V if for every $i \in I$ there is a V-term t_i of the same arity as f_i such that for all $A \in V$, the algebra $\langle A, t_i^A (i \in I)\rangle$ is a member of U.*

   Examples 1.11. *The variety of groups is interpretable in the variety of rings.*

   Remark 1.12. *The concept of interpretability can be used to describe Maltsev conditions as well. For example if we let V be the variety that consists of one ternary operation $q(x, y, z)$ s.t. $q(x, y, y) \approx x \approx q(y, y, x)$ then any variety W is congruence permutable iff V is interpretable in W.*

3. *If the variety U is finitely presented (i.e. has finitely many basic operation and is finitely axiomatized) then the class of all varieties V such that U is interpretable in V is called the* ***strong Maltsev class*** *defined by U.*

4. *If $U_i : i > 0$ is a decreasing sequence of finitely presented varieties, relative to interpretability, then the class $\{V : U_i \leq V \text{ for some } i\}$ is called the* ***Maltsev class*** *defined by this sequence, and the associated condition on varieties is called the Maltsev condition defined by this sequence.*

Lemma 1.13. *[KK09] Any idempotent strong Maltsev condition is equivalent to one of the form $\exists F \bigwedge \Sigma$ where $F = \{h, k\}$, h is n-ary and k is n^2-ary, and Σ consists of identities:*

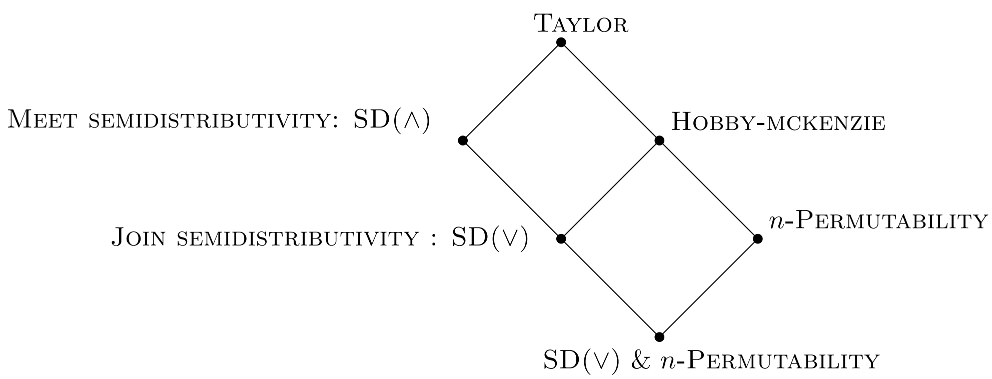
1. $h(x, x, \ldots, x) \approx x$,
2. $k(x_{11}, \ldots, x_{nn}) \approx h(h(x_{11}, \ldots, x_{1n}), \ldots, h(x_{n1}, \ldots, x_{nn}))$,
3. *finitely many identities of the form k(variables)≈ k(variables).*

Remark 1.14. *In[HM88] Hobby and Mckenzie state that the local behaviour of finite algebras can be divided into five types:*

- *Type 1 is Unary,*
- *Type 2 is Affine,*

- *Type 3 is the 2-element Boolean algebra,*
- *Type 4 is the 2-element lattice and*
- *Type 5 is the 2-element semi-lattice.*

An algebra omits a particular type if locally, that algebra does not allow that type. Similarly we can define a variety to omit a particular type if every algebra in that variety omits that type. Hobby and Mckenzie show that some omitting type conditions for locally finite varieties can be characterized by idempotent Maltsev conditions. Hobby and Mckenzie give six omitting type conditions for locally finite varieties which are represented in the following figure.



2 Taylor Term

Definition 2.1. *A term $f(\vec{x})$ of a variety is* ***Taylor*** *if it is idempotent and for each $1 \leq i \leq n$, V satisfies an equation of the form: $f(a_1, \ldots, a_n) \approx f(b_1, \ldots, b_n)$, where the equations are in variables $\{x, y\}$ and $a_i = x$ and $b_i = y$. The given set of equations can be represented by the following matrix form.:*

$$f\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} \quad = \quad f\begin{bmatrix} y & & & \\ & y & & \\ & & \ddots & \\ & & & y \end{bmatrix}$$

All the entries in the matrix belong to the set $\{x, y\}$.

The following theorem follows from the results found in [KK09], lemma 9.4 of [HM88] and [Val].

Theorem 2.2. *Let V be any variety. The following are equivalent:*

1. *V satisfies a nontrivial idempotent Maltsev condition (i.e. one that fails in some variety).*

2. *V satisfies an idempotent Maltsev condition that fails in the variety of sets.*

3. *V is not interpretable in the variety of sets.*

4. *For some n > 1, V has an idempotent n-ary Taylor term $f(x_1, \ldots, x_n)$.*

Furthermore, if V is assumed to be locally finite, then the following are equivalent to the above:

5. *V omits the unary type.*

6. *For some $n > 1$ V has an n-ary weak near unanimity term.*

7. *V has a 4-ary idempotent term t such that V satisfies:*

   t(y,y,x,x)≈ t(x,y,y,x)≈ t(x,x,x,y)

8. *V has two 3-ary idempotent terms $p(x, y, z)$,$q(x, y, z)$ such that V satisfies:*

   p(x,x,y)≈ p(y,x,x) ≈ q(x,y,y) and
   p(x,y,x)≈ q(x,y,x).

The following lemma establishes part of the above theorem namely $4 \Rightarrow 2$.

Lemma 2.3. *Let V be the variety of sets. Then V does not have a Taylor term.*

Proof. Assume for contradiction that the variety of sets V has a Taylor term $f(x_1,\ldots,x_n)$ for some $n > 1$. The only n-ary operations on sets are the n-ary projection operations denoted π_k^n where $k \leq n$. Let $f(x_1,\ldots,x_n) = \pi_k^n = x_k$ for some $k \leq n$. The Taylor term $f(x_1,\ldots,x_n)$ satisfies a set of equations of the form $f(x_{i1},\ldots,x_{in}) = f(y_{i1},\ldots,y_{in})$ where 1≤ i ≤ n and $x_{ij}, y_{ij} \in \{x, y\}$ and $x_{ii} = x, y_{ii} = y$. The k-th equation in the given matrix form is $f(x_{k1},\ldots,x_{kn}) = f(y_{k1},\ldots,y_{kn})$. The right hand side is equal to y_{kk} and the left hand side is equal to x_{kk}. This forces $x = y$ which contradicts the definition of a Taylor term. □

Examples 2.4. *Groups have a Maltsev term $p(x,y,z) = x \circ y^{-1} \circ z$ which is also a Taylor term where the following set of equations satisfies the matrix condition of the Taylor term : p(x,x,y)=p(y,y,y), p(y,x,x)=p(y,y,y) and p(y,y,x)=p(x,y,y).*

Remark 2.5. *Any variety that satisfies a nontrivial idempotent Maltsev condition has a Taylor term. For the locally finite case this class is the largest non-trivial idempotent class and it can be characterized by a strong Maltsev condition. The result due to Siggers shows that the Taylor term condition can be represented by a 4-ary Taylor term i.e. it can be represented by a 4×4 matrix. Also, this strong Maltsev condition cannot be described by any term of smaller arity [Val].*

3 Hobby-Mckenzie Term

Definition 3.1. *A term $f(\vec{x})$ of a variety is **Hobby-Mckenzie** if it is idempotent and for each $I \subseteq \{1,\ldots,n\}$, V satisfies an equation of the form: $f(a_1,\ldots,a_n) \approx f(b_1,\ldots,b_n)$, where the equations are in variables $\{x,y\}$ and $\{a_i : i \in I\} \neq \{b_i : i \in I\}$. These set of equations are equivalent (after possibly rearranging the variables in f) to the following matrix representation:*

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \ldots & x \end{bmatrix} \quad = \quad f\begin{bmatrix} y & & & \\ & y & & \\ & & \ddots & \\ & & & y \end{bmatrix}$$

All the entries in the matrix belong to the set $\{x,y\}$.

The following theorem follows from results found in theorem 2.16, 5.28, 8.13 of [KK09] and lemma 9.5 and theorem 9.8 of [HM88].

Theorem 3.2. *Let V be any variety. The following are equivalent:*

1. *V satisfies a nontrivial congruence identity.*
2. *V satisfies an idempotent Maltsev condition that fails in the variety of semi-lattices.*
3. *V is not interpretable in the variety of semi-lattices.*
4. *There is a positive integer k and ternary term $d_0, \ldots, d_{2k+1}$ and $e_0, \ldots, e_{2k+1}$ and p such that V satisfies the following equations:*
   (a) $d_0(x, y, z) \approx p(x, y, z) \approx e_0(x, y, z)$
   (b) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$ *and* $e_i(x, x, y) \approx e_{i+1}(x, x, y)$ *for even i.*
   (c) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$,$d_i(x, y, x) \approx d_{i+1}(x, y, x)$ *and* $e_i(x, y, y) \approx e_{i+1}(x, y, y)$,$e_i(x, y, x) \approx e_{i+1}(x, y, x)$ *for odd i.*
   (d) $d_{2k+1}(x, y, z) \approx x$ *and* $e_{2k+1}(x, y, z) \approx z$.
5. *V has a sequence of terms $f_i(x, y, u, v)$ where $0 \leq i \leq 2m + 1$ such that V satisfies the following equations:*
   (a) $f_0(x, y, u, v) \approx x$.
   (b) $f_i(x, y, y, y) \approx f_{i+1}(x, y, y, y)$ *for even i.*
   (c) $f_i(x, x, y, y) \approx f_{i+1}(x, x, y, y)$ *and* $f_i(x, y, x, y) \approx f_{i+1}(x, y, x, y)$ *for odd i.*
   (d) $f_{2m+1}(x, y, u, v) \approx v$
6. *For some $n > 1$, V has an n-ary Hobby-Mckenzie term $f(x_1, \ldots, x_n)$ of V.*

Furthermore, if V is assumed to be locally finite, then the following are equivalent to the above:

7. *V omits the unary and the semi-lattice type.*

The following lemma establishes part of the theorem above namely $6 \Rightarrow 2$.

Lemma 3.3. *Let V be the variety of semi-lattices. Then V does not have a Hobby-Mckenzie term.*

Proof. We need to show that there does not exist any n-ary term $f(x_1, \dots, x_n)$ of V such that it satisfies the following set of equations in matrix form:

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \dots & x \end{bmatrix} \quad = \quad f\begin{bmatrix} y & & & \\ & y & & \\ & & \ddots & \\ & & & y \end{bmatrix}$$

All the entries in the matrix belong to the set $\{x, y\}$.

Assume that such a term f exists in the variety of semi-lattices, $\langle S, \wedge \rangle$. Then $f(x_1, \dots, x_n)$ = meet of some of the x_i's. Let k be the largest index that f depends on. The equation from the k-th row of the matrix shows $f(x_1, \dots, x_k, \dots, x_n) = f(\dots, y, \dots)$ where $x_1 = \dots = x_k = x$ on the left hand side and y is the k-th indexed variable on the right hand side of the equation. Since k is the largest index that f depends on the left hand side gives us $f(x_1, \dots, x_k, \dots, x_n) = x$. The right hand side gives us f(...,y,...)= $x \wedge y$ or just y, since all the entries in the matrix belong to the set $\{x, y\}$. This gives us $x = x \wedge y$ or $x = y$ holds for all x,y in the variety of semi-lattices which is a contradiction. □

Remark 3.4. *Congruence modularity is a stronger condition than having a Hobby-Mckenzie term. According to a theorem in [Ber11] the congruence lattice of a group is modular. The Maltsev term $p(x, y, z) = x \circ y^{-1} \circ z$ of a group defined earlier is also a Hobby-Mckenzie term. It satisfies the following set of equations in matrix form:*

$$f\begin{bmatrix} x & y & y \\ x & x & y \\ x & x & x \end{bmatrix} \quad = \quad f\begin{bmatrix} y & y & x \\ y & y & y \\ x & y & y \end{bmatrix}$$

4 Congruence Meet Semidistributivity

A variety is congruence meet semidistributive i.e. $V \models SD(\wedge)$ if:

> $\forall$ algebra $\mathbf{A} \in V$ and all α,β,$\gamma \in Con(\mathbf{A})$,
> $(\alpha \wedge \beta)$= $(\alpha \wedge \gamma) \rightarrow (\alpha \wedge (\beta \vee \gamma))$= $(\alpha \wedge \beta)$.

This is a weaker condition than distributivity.

Definition 4.1. *A term $f(\vec{x})$ of a variety is a **Barto-Kozik term** if it is idempotent and satisfies a system of equations represented by the following matrices:*

$$f\begin{bmatrix} x & & & & \\ p_{2,1} & x & & & \\ p_{3,1} & p_{3,2} & x & & \\ \vdots & \vdots & \ddots & \ddots & \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n-1} & x \end{bmatrix} = f\begin{bmatrix} y & & & & \\ p_{2,1} & y & & & \\ p_{3,1} & p_{3,2} & y & & \\ \vdots & \vdots & \ddots & \ddots & \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n-1} & y \end{bmatrix}$$

All the entries of the matrix belong to the set {x,y}.

The following theorem follows from results found in theorem 8.1 of [KK09], theorem 9.1 of [HM88], [Val] and unpublished work of Barto and Kozik.

Theorem 4.2. *Let V be any variety. The following are equivalent:*

1. *V is congruence meet semidistributive.*
2. *V satisfies a single idempotent Maltsev condition that fails in every nontrivial variety of modules.*
3. *V is not interpretable in the variety of modules.*

Furthermore, if V is assumed to be locally finite, then the following are equivalent to the above:

4. *V omits the unary and the affine type.*
5. *For all $n > 2$, V has an n-ary weak near unanimity term.*
6. *V has a 3-ary and a 4-ary weak near unanimity terms v(x,y,z) and w(x,y,z,w) that satisfy the equation $v(y,x,x) \approx w(y,x,x,x)$.*
7. *For some $n > 0$ there exists an n-ary Barto-Kozik term $f(x_1, x_2, \ldots, x_n)$ of V.*

Remark 4.3. *It is not known whether the last condition of the above theorem holds in the non-locally finite case. For the locally-finite case there is an even nicer characterization for congruence meet semidistributivity. The*

This has not been shown. All that we know is that SD(meet) is a strong maltsev condition in the locally finite case.

above condition can be represented by a 12-ary term, i.e. we can represent this condition by a 12×12 matrix. The 12-ary term comes from the 3-ary and 4-ary weak near unanimity terms described in the above theorem. Hence for the locally finite case congruence meet semidistributivity can be described by a strong Maltsev condition.

The following lemma establishes part of the theorem above namely $7 \Rightarrow 2$.

Lemma 4.4. *Let M be a nontrivial variety of modules. Then there does not exist an n-ary Barto- Kozik term $f(x_1, \ldots, x_n)$ in M.*

Proof. Let M be a variety of modules over the ring R and let $f(x_1, \ldots, x_n)$ be a term of M. Then $\forall i \leq n\ \exists r_i \in R$ such that $f(x_1, \ldots, x_n) = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. Assume for contradiction that f is a Barto-Kozik term. Since f is idempotent we have $f(x, \ldots, x) = x$. This means $r_1x + r_2x + \cdots + r_nx = x$. This forces $\sum_{i=1}^{i=n} r_i = 1$ in R. The last equation in the matrix representation of the Barto-Kozik term gives us:

$r_1p_{n,1} + r_2p_{n,2} + \cdots + r_nx = r_1p_{n,1} + r_2p_{n,2} + \cdots + r_ny.$
$\Rightarrow r_nx = r_ny$ holds for $\forall$ x and y.
$\Rightarrow r_n = 0$

Using a similar argument the second last equation gives us $r_{n-1} = 0$. Inductively we get $r_i = 0, \forall i$. Hence there does not exist such a term $f(x_1, \ldots, x_n)$ in the variety of modules that satisfy the above set of equations. ☐

5 Congruence join semidistributivity

A variety is congruence join semidistributive i.e. $V \models SD(\vee)$ if:

$\forall$ algebra $\mathbf{A} \in V$ and all $\alpha, \beta, \gamma \in \mathrm{Con}\,(\mathbf{A})$,
$(\alpha \vee \beta) = (\alpha \vee \gamma) \rightarrow (\alpha \vee (\beta \wedge \gamma)) = (\alpha \vee \beta)$.

Any congruence join-semidistributive variety is also congruence meet-semidistributive [KK09].

Definition 5.1. *A term $f(\vec{x})$ of a variety is* ***SD($\vee$)*** *if it is idempotent and satisfies a system of equations represented by the following matrices:*

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \dots & x \end{bmatrix} \qquad = \qquad f\begin{bmatrix} y & & & \\ x & y & & \\ \vdots & \ddots & \ddots & \\ x & \dots & x & y \end{bmatrix}$$

All the entries in the matrix are in the variables $\{x,y\}$.

The following theorem follows from results found in theorem 8.14 of [KK09], [Val] and theorem 9.11 of [HM88].

Theorem 5.2. *Let V be any variety. The following are equivalent.*

1. *V is congruence join semidistributive.*

2. *V satisfies an idempotent Maltsev condition that fails in every non trivial variety of modules and in the variety of semi-lattices.*

3. *V is not interpretable in any non trivial variety of modules and it is not interpretable in the variety of semi-lattices.*

4. *V is congruence meet semidistributive and satisfies a nontrivial congruence identity.*

5. *There is a positive integer k and ternary terms $d_0, \dots, d_k$ such that V satisfies the following equations:*

   (a) $d_0(x,y,x) \approx x$;
   (b) $d_i(x,y,y) \approx d_{i+1}(x,y,y)$ *and* $d_i(x,y,x) \approx d_{i+1}(x,y,x)$*;for even* $i < k$.
   (c) $d_i(x,x,y) \approx d_{i+1}(x,x,y)$ *for odd* $i < k$;
   (d) $d_k(x,y,z) \approx z$.

6. *V has an $SD(\vee)$ term.*

Furthermore, if V is assumed to be locally finite, then the following are equivalent to the above:

7. *V omits the unary, affine and semi-lattices type.*

Remark 5.3. *The matrix condition of congruence join semidistributivity is the combination of the matrix condition of congruence meet semidistributivity and Hobby-Mckenzie term condition. Congruence distributivity is a stronger condition than congruence join semidistributivity. Lattices have a majority term $M(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ which satisfies the following equation: $M(x,x,y) = M(x,y,x) = M(y,x,x) = x$. This implies that the congruence lattice of any lattice is distributive [Ber11]. The term M(x,y,z) is also an $SD(\vee)$ term and the set of equations it satisfies can be represented by the following matrix form:*

$$f\begin{bmatrix} x & y & x \\ x & x & y \\ x & x & x \end{bmatrix} \qquad = \qquad f\begin{bmatrix} y & x & x \\ x & y & x \\ x & x & y \end{bmatrix}$$

6 n-Permutability

Definition 6.1. *[HM88] A variety is called n-permutable iff for every $\boldsymbol{A} \in V$ and $\alpha, \beta \in Con(\boldsymbol{A})$, we have $\alpha \circ_n \beta = \beta \circ_n \alpha$, where $\alpha \circ_n \beta = \alpha \circ \beta \circ \alpha \circ \cdots$ with $n-1$ occurrences of $\circ$.*

Definition 6.2. *[Ber11] A lattice is distributive iff it satisfies $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$.*

The following theorem follows from the results found in theorem 9.14 [HM88] and [Val].

Theorem 6.3. *Let V be any variety. The following are equivalent:*

1. *V is n-permutable.*
2. *There are terms $p_1(x,y,z), \ldots, p_{n-1}(x,y,z)$ such that V satisfies:*

   $x \approx p_1(x,y,y)$, $p_i(x,x,y) \approx p_{i+1}(x,y,y)$ *for each i*, $p_{n-1}(x,x,y) \approx y$

3. *For some $n \geq 0, \exists$ an n-ary idempotent term $f(x_1, \ldots, x_n)$ of V such that f satisfies a set of equations in two variables {x,y} of the form:*

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \ldots & x \end{bmatrix} \qquad = \qquad f\begin{bmatrix} y & y & \ldots & y \\ & y & \ldots & \vdots \\ & & \ddots & \vdots \\ & & & y \end{bmatrix}$$

All the entries in the matrix belong to the set $\{x,y\}$.

Furthermore, if V is assumed to be locally finite, then the following are equivalent to the above:

4. *V omits the unary, lattice and the semi-lattice type.*

5. *The variety V is not interpretable in the variety of distributive lattices.*

Remark 6.4.
- *It is not known if condition 5 of the above theorem is equivalent to the other conditions of the theorem for the non-locally finite case.*

- *Any congruence n-permutable variety satisfies a nontrivial idempotent Maltsev condition and hence has a Taylor term.*

- *Any congruence n-permutable variety satisfies a nontrivial congruence identity and hence has a Hobby-Mckenzie term as well [KK09].*

- *A variety is congruence permutable if $\forall \boldsymbol{A} \in V$ and $\alpha, \beta \in Con(\boldsymbol{A})$, $\alpha \circ \beta = \beta \circ \alpha$. Therefore, congruence permutability implies n-permutability for all n [Ber11].*

Examples 6.5. *Any congruence permutable variety has a Maltsev term $p(x, y, z)$ such that $p(x, y, y) \approx p(y, y, x) \approx x$. For example the variety of vector spaces and the variety of groups are congruence permutable and hence are also n-permutable. The Maltsev term satisfies the following matrix condition of n-permutability:*

$$f \begin{bmatrix} x & x & y \\ x & x & x \\ x & x & x \end{bmatrix} \qquad = \qquad f \begin{bmatrix} y & y & y \\ x & y & y \\ x & y & y \end{bmatrix}$$

7 Congruence join semidistributivity and n-Permutability

The following theorem follows from results found in theorem 9.15 of [HM88].

Theorem 7.1. *Let V be a locally finite variety. The following are equivalent.*

1. *V omits the unary type, affine type, lattice type, and the semi-lattice type.*

2. *V is not interpretable in the variety of distributive lattices and V is not interpretable in any nontrivial variety of modules.*

3. *V is n-permutable for some n and is congruence join semidistributive.*

4. *There are terms* $f_0(x,y,z,u),\ldots,f_n(x,y,,z,u) \in V$ *such that V satisfies:*

   (a) $x \approx f_0(x,y,y,z)$,

   (b) $f_i(x,x,y,x) \approx f_{i+1}(x,y,y,x)$ *and* $f_i(x,x,y,y) = f_{i+1}(x,y,y,y)$ $\forall i < n$,

   (c) $f_n(x,x,y,z) \approx z$

Lemma 7.2. *Let V be a locally finite variety. Assume that for some* $n \geq 1, \exists$ *an n-ary idempotent term* $f(x_1,\ldots,x_n)$ *of* V*, such that f satisfies the following set of equations in two variables {x,y} represented by this matrix form:*

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \ldots & x \end{bmatrix} \qquad = \qquad f\begin{bmatrix} y & y & \ldots & y \\ x & y & \ldots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x & \ldots & x & y \end{bmatrix}$$

Then V is congruence join semidistributive and n-permutable.

Proof. This follows immediately from the matrix representation of congruence join semidistributivity and n-permutablility. The term $f(x_1,\ldots,x_n)$ of V satisfies:

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \ldots & x \end{bmatrix} = f\begin{bmatrix} y & y & \ldots & y \\ & y & \ldots & \vdots \\ & & \ddots & \vdots \\ & & & y \end{bmatrix}$$

which is equivalent to n-permutability, and $f(x_1,\ldots,x_n)$ of V also satisfies:

$$f\begin{bmatrix} x & & & \\ x & x & & \\ \vdots & \vdots & \ddots & \\ x & x & \dots & x \end{bmatrix} = f\begin{bmatrix} y & & & \\ x & y & & \\ \vdots & \ddots & \ddots & \\ x & \dots & x & y \end{bmatrix}$$

which is equivalent to congruence join semidistributivity. □

Remark 7.3. *It is an open question whether the matrix condition given in the lemma is equivalent to the conditions in theorem 7.*

8 Conclusion

This report provided Maltsev conditions characterizing six special families of varieties. The results stated in this report illustrates that for the locally finite case these characterizations are even "nicer". It is seen that the Taylor term condition and the congruence meet semidistributive condition can be characterized by strong Maltsev conditions in the locally finite case. This result also has applications in computational complexity. In some situations, the presence of strong Maltsev conditions can lead to more efficient algorithms for determining if a given property holds. Finding the complexity of determining if a given finite algebra generates a variety that has a Maltsev term or a majority term are one of the many open problems related to this field [Val]. For the remaining four classes, it has been proven that for the locally finite cases these conditions can not be described by strong Maltsev conditions.

Also, there are a few open problems related to this report. It is to be determined if all of the matrix term conditions stated in this report hold for the non-locally finite cases. For example, it is not known if the Barto-Kozik term condition is equivalent to congruence meet semidistributivity in the general case. Also, it is not proven if the matrix condition given in lemma 7.2 is equivalent to congruence join semidistributivity and n-permutability in the locally finite case [Val].

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