Preserving Near Unanimity Terms under Direct Products

Preserving Near Unanimity Terms

under Direct Products

By

Maria Campanella B.S. Hons. Mathematics and Statistics

A Project

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

MASTER OF SCIENCE (2013)
(Mathematics)

McMaster University
Hamilton, Ontario

TITLE: Preserving Near Unanimity Terms under Direct Products

AUTHOR: Maria Campanella, B.S. Hons. (McMaster University)

SUPERVISOR: Professor Matthew A. Valeriote

NUMBER OF PAGES: iv, 24

Abstract

When studying universal algebra, one may focus on families of varieties defined by Mal'cev conditions. Varieties are classes of similar algebras closed under homomorphic images, subalgebras, and direct products. This paper deals with idempotent varieties which satisfy the Mal'cev condition of having a near unanimity term. More specifically, we consider near unanimity terms in the direct product of algebras, and the arity of these terms. Of all possible near unanimity terms in the direct product, we focus on those which have the smallest arity and find bounds for this number.

Contents

1 Introduction

Over the past half century, much focus has been placed on Mal'cev families of varieties when studying universal algebra [5]. Varieties can be characterized by certain conditions that their terms satisfy, called Mal'cev conditions. The earliest research involved families of varieties being congruence permutable [3]. Significantly, this family contains all varieties of groups and rings. According to Mal'cev's result, a variety will be congruence permutable when it has a term $t(x, y, z)$ where $t(x, y, y) \approx x$, and $t(x, x, y) \approx y$ hold in the variety. Another important Mal'cev family is the collection of varieties that satisfy the condition of having a near unanimity term of some arity n. Having an n-ary near unanimity term can be characterized by a particular system of n equations. Since many of the common Mal'cev conditions are defined by idempotent terms, we will focus on idempotent algebras in this paper.

It is known that if $\mathbf{A_1}$ and $\mathbf{A_2}$ are two idempotent, similar algebras, with n-ary and m-ary near unanimity terms p_1 and p_2 respectively, then the following will be an mn-ary near unanimity term for their direct product:

$$t = p_1(p_2(x_1, \ldots, x_m), \ldots, p_2(x_{mn-(m-1)}, \ldots, x_{mn}))$$

where the x_i are all distinct variables. Now, if the basic operation p_1 is an n-ary near unanimity term in both algebras $\mathbf{A_1}$ and $\mathbf{A_2}$, then of course p_1 will be an n-ary near unanimity term in the direct product $\mathbf{A_1} \times \mathbf{A_2}$, so we exclude this trivial case. We want to investigate whether lower arity near unanimity terms can be found which work in the direct product of idempotent algebras that have near unanimity terms.

Why would we want a term of lower arity? To solve certain types of computational problems for finite algebras, it might be advantageous to work with a smaller arity near unanimity term, or at least to know that they exist. For example, if the algebra $\mathbf{A}$ has a k-ary near unanimity term, then in order to distinguish two subalgebras $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}^n$, one need only check whether the projections of $\mathbf{B}$ and $\mathbf{C}$ onto sets of coordinates of size k are the same or not [1].

2 Preliminaries

Definition 1. An *algebra* $\mathbf{A}$ is a pair $\langle A, F \rangle$, where A is a nonempty set called the *universe* of $\mathbf{A}$, and $F = \langle f_i : i \in I \rangle$ is a family of operations indexed by some set I, called the basic operations of $\mathbf{A}$. An operation f from A^n to A has *arity* n.

Definition 2. Let $\mathbf{A} = \langle A, F \rangle$ be an algebra with $F = \langle f_i : i \in I \rangle$. The *similarity type* (or type) of $\mathbf{A}$ is a function $\rho : I \to \omega$ which assigns to each $i \in I$ the arity of f_i. Two algebras are called *similar* if they have the same similarity type.

Definition 3. A *variety* is a nonempty class of similar algebras closed under homomorphic images, subalgebras and direct products.

Definition 4. A *term operation* of an algebra $\mathbf{A}$ is any operation that can be built up from the basic operations of $\mathbf{A}$ and the projection operations via composition.

Definition 5. An algebra is *idempotent* if all of its terms are idempotent. A term f is idempotent if it satisfies the Idempotency Law, $f(x, x, \ldots, x) \approx x$. A variety is *idempotent* if every algebra in it is idempotent.

Definition 6. Let $\mathbf{A} = \langle A, F \rangle$ and $\mathbf{B} = \langle B, G \rangle$ be similar algebras of type $\rho : I \to \omega$. We call $\mathbf{B}$ a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$, called a subuniverse, and for every $i \in I$, $g_i = f_i|_B$.

Definition 7. Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be two algebras of similar type. The *direct product* $\mathbf{A_1} \times \mathbf{A_2}$ is an algebra with universe being the set of ordered pairs (a_1, a_2) with $a_i \in A_i$, for $i = 1{,}2$, and with basic operations computed coordinate wise. In other words, for an n-ary operation f,

$$f^{\mathbf{A1} \times \mathbf{A2}}((a_{11}, a_{21}), (a_{12}, a_{22}), \ldots, (a_{1n}, a_{2n})) =$$

$$(f^{\mathbf{A1}}(a_{11}, \ldots, a_{1n}), f^{\mathbf{A2}}(a_{21}, \ldots, a_{2n})).$$

For example, a group $\langle G, \circ, ^{-1}, e \rangle$ is an algebra and the class of all groups is a variety. The term $t(x, y, z) = x \circ y^{-1} \circ z$ is an idempotent term operation, however, groups are not idempotent algebras.

Definition 8. A *near unanimity* term for an algebra $\mathbf{A}$ is a term t such that for any $a, b \in A$,

$$t^{\mathbf{A}}(b, a, a, \ldots, a) = t^{\mathbf{A}}(a, b, a, \ldots, a) = \cdots = t^{\mathbf{A}}(a, a, a, \ldots, a, b) = a.$$

We call a 3-ary near unanimity term a *majority term.* We will also refer to a *near unanimity* term as a NU term in this paper.

For example, consider the variety of lattices. For $\mathbf{L} = \langle L, \vee, \wedge \rangle$ a lattice, the term $t(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ is a majority term for $\mathbf{L}$.

In this thesis, we will consider products of pairs of idempotent algebras that have possibly different near unanimity terms. The following theorems will be used throughout this paper to prove our results:

Theorem 1. *Let $\mathbf{C}$ be an algebra and $X = \{a_1, \ldots, a_n\} \subseteq \mathbf{C}$. Let $\mathbf{D}$ be the subalgebra generated by X. Then for an element $d \in C$, $d \in D$ if and only if $d = t(a_1, \ldots, a_n)$ for some term operation t of $\mathbf{C}$.*

See [3] for a proof.

Theorem 2. *Let $\mathbf{K}$ be a variety, p and q n-ary terms, $\mathbf{Y}$ a set and $y_1, y_2, \ldots, y_n$ distinct elements of $\mathbf{Y}$. Let $\mathbf{F_K(Y)}$ be the free algebra generated by Y. Then $\mathbf{K}$ satisfies $p \approx q$ if and only if $p^{\mathbf{F_K(Y)}}(y_1, \ldots, y_n) = q^{\mathbf{F_K(Y)}}(y_1, \ldots, y_n)$.*

See [2] for a proof.

Theorem 3. *An algebra $\mathbf{A}$ has an n-ary near unanimity term iff the n-tuple $(a_1, a_2, ..., a_n) \in Sg^{\mathbf{B}^n}\{(b_1, a_2, \ldots, a_n), (a_1, b_2, \ldots, a_n), \ldots, (a_1, a_2, \ldots, b_n)\}$ for all $\mathbf{B} \in HSP(\mathbf{A})$ and for all $a_i, b_i \in \mathbf{B}$.*

Proof. Suppose $\mathbf{A}$ has an n-ary NU term t. Since taking powers, subalgebras, and quotients of $\mathbf{A}$ preserves the identities that define being a NU term, then every algebra in $HSP(\mathbf{A})$ will have the same NU term t. Thus, for every $\mathbf{B} \in HSP(\mathbf{A})$ and for all $a_i, b_i \in \mathbf{B}$, we have $(a_1, a_2, ..., a_n) \in Sg^{\mathbf{B}^n}\{(b_1, a_2, \ldots, a_n), (a_1, b_2, \ldots, a_n), \ldots, (a_1, a_2, \ldots, b_n)\}$ since we can apply t to the generators coordinate-wise. Now suppose that

$$(a_1, a_2, \ldots, a_n) \in Sg^{\mathbf{B}^n}\{(b_1, a_2, \ldots, a_n), (a_1, b_2, \ldots, a_n), \ldots, (a_1, a_2, \ldots, b_n)\}$$

holds for all $\mathbf{B} \in HSP(\mathbf{A})$ and for all $a_i, b_i \in \mathbf{B}$. Consider the free algebra in HSP($\mathbf{A}$) generated by {X,Y}, and let $Sg^{\mathbf{F}^n}(X,Y)$ be the subalgebra of $\mathbf{F}^n$ generated by

$$\{(Y,X,\ldots,X),(X,Y,\ldots,X),\ldots,(X,\ldots,X,Y)\}.$$

Then, by hypothesis, $(X,X,\ldots,X) \in Sg^{\mathbf{F}^n}(X,Y)$, and by Theorem 1, there exists a term $m(x_1,\ldots,x_n)$ such that

$$(X,X,\ldots,X) = m^{\mathbf{F}^n}((Y,X,\ldots,X),(X,Y,\ldots,X),\ldots,(X,X,\ldots,Y)).$$

Since this equality operates coordinate wise, we get n equalities in the free algebra $\mathbf{F}$, namely,

$$\begin{aligned} m^{\mathbf{F}}(Y,X,\ldots,X) &= X \\ m^{\mathbf{F}}(X,Y,\ldots,X) &= X \\ &\vdots \\ m^{\mathbf{F}}(X,X,\ldots,Y) &= X \end{aligned}$$

By Theorem 2, this implies that the variety $HSP(\mathbf{A})$ satisfies the following equations,

$$\begin{aligned} m(y,x,\ldots,x) &\approx x \\ m(x,y,\ldots,x) &\approx x \\ &\vdots \\ m(x,x,\ldots,y) &\approx x \end{aligned}$$

for any variables x, y, and these precisely define an n-ary NU term. $\square$

3 Results

Definition 4. For natural numbers n and m greater than 2, define $f(n,m)$ to be the smallest k such that if $\mathbf{A_1}$ and $\mathbf{A_2}$ are idempotent algebras that have n-ary and m-ary near unanimity terms respectively, then $\mathbf{A_1} \times \mathbf{A_2}$ has a k-ary near unanimity term.

It is known that n, m are lower bounds, and nm is an upper bound for $f(n,m)$.

3.1 Lower Bound

Theorem 5. *For any natural numbers n and m greater than 2, there exists two similar, idempotent algebras $\mathbf{A_1}$ and $\mathbf{A_2}$ with n-ary and m-ary near unanimity terms respectively, such that the direct product $\mathbf{A} = \mathbf{A_1} \times \mathbf{A_2}$ does not have a near unanimity term with arity less than or equal to $n+m-2$. In other words, $f(n,m) > n+m-2$.*

We illustrate this result with an example before proving the general case. To show that $4 \neq f(3,3)$, we consider the following algebras.

Let $\mathbf{A_1} = \langle \{0,1\}, p_1^{\mathbf{A_1}}, p_2^{\mathbf{A_1}} \rangle$ and $\mathbf{A_2} = \langle \{0,1\}, p_1^{\mathbf{A_2}}, p_2^{\mathbf{A_2}} \rangle$ where

$$p_1^{\mathbf{A_1}}(x_1,x_2,x_3) = \begin{cases} 1 & \text{if } (x_1,x_2,x_3) \text{ contains 2 or 3 1's} \\ 0 & \text{else} \end{cases}$$
$$p_2^{\mathbf{A_1}}(x_1,x_2,x_3) = \begin{cases} 1 & \text{if } (x_1,x_2,x_3) = (1,1,1) \\ 0 & \text{else} \end{cases}$$
$$p_1^{\mathbf{A_2}}(x_1,x_2,x_3) = \begin{cases} 0 & \text{if } (x_1,x_2,x_3) = (0,0,0) \\ 1 & \text{else} \end{cases}$$
$$p_2^{\mathbf{A_2}}(x_1,x_2,x_3) = \begin{cases} 0 & \text{if } (x_1,x_2,x_3) \text{ contains 2 or 3 0's} \\ 1 & \text{else} \end{cases}$$

and let $\mathbf{A} = \mathbf{A_1} \times \mathbf{A_2}$.

Consider the set $\mathbf{C} = \{\mathbf{0},\mathbf{2}\} \times \{\mathbf{0},\mathbf{2}\} \times \{\mathbf{0},\mathbf{1}\} \times \{\mathbf{0},\mathbf{1}\}$ of $\mathbf{A}^4 \subseteq \mathrm{HSP}(\mathbf{A})$, where 0=(0,0), 1=(1,0), 2=(0,1), 3=(1,1). We want to show that $\mathbf{C}$ is a subalgebra of $\mathbf{A}^4$. To start, we first observe that $\{\mathbf{0},\mathbf{2}\}$ and $\{\mathbf{0},\mathbf{1}\}$ are subalgebras of $\mathbf{A}$. This follows since $\{\mathbf{0},\mathbf{2}\} = \{\mathbf{0}\} \times \mathbf{A_2}$ and $\{\mathbf{0},\mathbf{1}\} = \mathbf{A_1} \times \{\mathbf{0}\}$, and $\{\mathbf{0}\}$ is a subuniverse of $\mathbf{A_1}$ and $\mathbf{A_2}$ (since both algebras are idempotent). Since $\mathbf{C}$ is the direct product of subalgebras of $\mathbf{A}$, then it is a subalgebra of $\mathbf{A^4}$, as claimed.

Let $\mathbf{S} = Sg^{\mathbf{A}^4}\{(2,0,1,1),(0,2,1,1),(0,0,0,1),(0,0,1,0)\}$. By construction, $\mathbf{S}$ is a subalgebra of $\mathbf{C}$, since the generators are elements of $\mathbf{C}$. We want to show that $\mathbf{S}$ does not contain the element $(0,0,1,1)$. We do this first by demonstrating that $\mathbf{C} \setminus \{(0,0,1,1)\}$ is a subalgebra of $\mathbf{C}$. This is done by showing that the only way to attain the element $(0,0,1,1)$ from a basic operation of $\mathbf{C}$ is by applying it to (0,0,1,1) and two other elements. And second, we observe that the generators of $\mathbf{S}$ are elements of $\mathbf{C} \setminus \{(0,0,1,1)\}$ which immediately gives us containment. Thus $\mathbf{S} \subseteq \mathbf{C} \setminus \{(0,0,1,1)\}$ and $(0,0,1,1) \notin \mathbf{S}$. Recall that all elements in $\{\mathbf{0},\mathbf{2}\}$ have the element 0 in their first coordinate, and all elements in $\{\mathbf{0},\mathbf{1}\}$ have the element 0 in their second coordinate. Then, applying $p_1^{\mathbf{A}}$ to three elements of $\mathbf{C}$, we get

$$p_1^{\mathbf{A}}\left(\begin{pmatrix}(0,a_1)\\(0,b_1\\(c_1,0)\\(d_1,0)\end{pmatrix}\begin{pmatrix}(0,a_2)\\(0,b_2)\\(c_2,0)\\(d_2,0)\end{pmatrix}\begin{pmatrix}(0,a_3)\\(0,b_3)\\(c_3,0)\\(d_3,0)\end{pmatrix}\right)$$
$$=\begin{pmatrix}(p_1^{\mathbf{A_1}}(0,0,0),p_1^{\mathbf{A_2}}(a_1,a_2,a_3))\\(p_1^{\mathbf{A_1}}(0,0,0),p_1^{\mathbf{A_2}}(b_1,b_2,b_3))\\(p_1^{\mathbf{A_1}}(c_1,c_2,c_3),p_1^{\mathbf{A_2}}(0,0,0))\\(p_1^{\mathbf{A_1}}(d_1,d_2,d_3),p_1^{\mathbf{A_2}}(0,0,0))\end{pmatrix}=\begin{pmatrix}(0,p_1^{\mathbf{A_2}}(a_1,a_2,a_3))\\(0,p_1^{\mathbf{A_2}}(b_1,b_2,b_3))\\(p_1^{\mathbf{A_1}}(c_1,c_2,c_3),0)\\(p_1^{\mathbf{A_1}}(d_1,d_2,d_3),0)\end{pmatrix}$$

since $p_1^{\mathbf{A_1}}$ and $p_1^{\mathbf{A_2}}$ are idempotent. In order for this to equal

$$\begin{pmatrix}(0,0)\\(0,0)\\(1,0)\\(1,0)\end{pmatrix}$$

we must have $p_1^{\mathbf{A_2}}(a_1,a_2,a_3)=p_1^{\mathbf{A_2}}(b_1,b_2,b_3)=0$ which only happens when $(a_1,a_2,a_3)=(b_1,b_2,b_3)=(0,0,0)$. We need $p_1^{\mathbf{A_1}}(c_1,c_2,c_3)=1$, which happens when (c_1,c_2,c_3) has at most one 0. We must also have $p_1^{\mathbf{A_1}}(d_1,d_2,d_3)=1$. In other words, this must happen in the last 2 of the 4 coordinates, but in each of these, we have $p_1^{\mathbf{A_1}}$ which is a 3-ary operation. We can choose from (1,1,0), (1,0,1), or (0,1,1), i.e we have $\binom{3}{2}$ possible tuples to choose from, and we only need 2. So which ever tuples we choose, we have $c_i=d_i=1$ for some i.

Suppose $i=1$, this leaves us with the following,

$$\begin{pmatrix}(0,0)\\(0,0)\\(p_1^{\mathbf{A_1}}(1,1,0),0)\\(p_1^{\mathbf{A_1}}(1,0,1),0)\end{pmatrix}$$

which translates back to having

$$p_1^{\mathbf{A}}\left(\begin{pmatrix}(0,0)\\(0,0)\\(1,0)\\(1,0)\end{pmatrix}\begin{pmatrix}(0,0)\\(0,0)\\(1,0)\\(0,0)\end{pmatrix}\begin{pmatrix}(0,0)\\(0,0)\\(0,0)\\(1,0)\end{pmatrix}\right)=\begin{pmatrix}(0,0)\\(0,0)\\(1,0)\\(1,0)\end{pmatrix}$$

A similar process shows closure under $p_2^{\mathbf{A}}$. We now observe that $\mathbf{S} \subseteq \mathbf{C} \setminus \{(0,0,1,1)\}$, so clearly

$$(0,0,1,1) \notin Sg^{\mathbf{A^4}}\{(2,0,1,1),(0,2,1,1),(0,0,0,1),(0,0,1,0)\},$$

and by Theorem 3, we see that $\mathbf{A_1} \times \mathbf{A_2}$ has no 4-ary NU term.

Proof. To prove $f(n,m) > n+m-2$, we find a suitable counter example in the general case. Let $\mathbf{A_1} = \langle \{0,1\}, p_1^{\mathbf{A_1}}, p_2^{\mathbf{A_1}} \rangle$ and $\mathbf{A_2} = \langle \{0,1\}, p_1^{\mathbf{A_2}}, p_2^{\mathbf{A_2}} \rangle$, where $p_1^{\mathbf{A_i}}$ have arity n and $p_2^{\mathbf{A_i}}$ have arity m for $i = 1,2$, and are defined as follows:

$$p_1^{\mathbf{A_1}}(x_1,x_2,\ldots,x_n) = \begin{cases} 1 & \text{if } (x_1,x_2,\ldots,x_n) \text{ contains at least (n-1) 1's} \\ 0 & \text{else} \end{cases}$$

$$p_2^{\mathbf{A_1}}(x_1,x_2,\ldots,x_n) = \begin{cases} 1 & \text{if } (x_1,x_2,\ldots,x_n) = (1,1,\ldots,1) \\ 0 & \text{else} \end{cases}$$

$$p_1^{\mathbf{A_2}}(x_1,x_2,\ldots,x_m) = \begin{cases} 0 & \text{if } (x_1,x_2,\ldots,x_m) = (0,0,\ldots,0) \\ 1 & \text{else} \end{cases}$$

$$p_2^{\mathbf{A_2}}(x_1,x_2,\ldots,x_m) = \begin{cases} 0 & \text{if } (x_1,x_2,\ldots,x_m) \text{ contains at least (m-1) 0's} \\ 1 & \text{else} \end{cases}$$

and let $\mathbf{A} = \mathbf{A_1} \times \mathbf{A_2}$.

Then the universe $A = \{0 = (0,0), 1 = (1,0), 2 = (0,1), 3 = (1,1)\}$. Using the same logic as in the example above, it is easy to see that $\{\mathbf{0},\mathbf{2}\}$ and $\{\mathbf{0},\mathbf{1}\}$ are subalgebras of $\mathbf{A}$. Let $\mathbf{C} = \{\mathbf{0},\mathbf{2}\} \times \cdots \times \{\mathbf{0},\mathbf{2}\} \times \{\mathbf{0},\mathbf{1}\} \times \cdots \times \{\mathbf{0},\mathbf{1}\}$ be the subalgebra of $\mathbf{A}^{n+m-2}$, where the first $m-1$ elements come from the subalgebra $\{\mathbf{0},\mathbf{2}\}$ and the next $n-1$ elements come from the subalgebra $\{\mathbf{0},\mathbf{1}\}$. As seen in the previous example, $\mathbf{C}$ is indeed a subalgebra of $\mathbf{A}^{n+m-2}$.

Let $l = (0,\ldots,0,1,\ldots,1)$, the tuple with the first $m-1$ coordinates being 0, and the next $n-1$ being 1, an element of $\mathbf{C}$. First, we must show that $\mathbf{C} \setminus \{l\}$ is again a subalgebra of $\mathbf{A}^{n+m-2}$. As demonstrated earlier, we do this by showing that the only way to attain l from a basic operation of $\mathbf{A}^{n+m-2}$, is if at least one of the variables in the term is set to l. When considering $p_1^{\mathbf{A}}$ applied to elements of $\mathbf{C} \setminus \{l\}$, since all basic operations are idempotent, we end up with the following,

$$p_1^{\mathbf{A}} \left(\begin{pmatrix} (0, y_1^1) \\ \vdots \\ (0, y_{m-1}^1) \\ (x_1^1, 0) \\ \vdots \\ (x_{n-1}^1, 0) \end{pmatrix} \begin{pmatrix} (0, y_1^2) \\ \vdots \\ (0, y_{m-1}^2) \\ (x_1^2, 0) \\ \vdots \\ (x_{n-1}^2, 0) \end{pmatrix} \cdots \begin{pmatrix} (0, y_1^n) \\ \vdots \\ (0, y_{m-1}^n) \\ (x_1^n, 0) \\ \vdots \\ (x_{n-1}^n, 0) \end{pmatrix} \right)$$

$$= \begin{pmatrix} (0, p_1^{\mathbf{A_2}}(y_1^1, y_1^2, \ldots, y_1^n)) \\ \vdots \\ (0, p_1^{\mathbf{A_2}}(y_{m-1}^1, y_{m-1}^2, \ldots, y_{m-1}^n)) \\ (p_1^{\mathbf{A_1}}(x_1^1, x_1^2, \ldots, x_1^n), 0) \\ \vdots \\ (p_1^{\mathbf{A_1}}(x_{n-1}^1, x_{n-1}^2, \ldots, x_{n-1}^n), 0) \end{pmatrix}$$

where y_j^i and x_s^r are all either 0 or 1.

Since we want the right hand side to equal l, we must have that $p_1^{\mathbf{A_2}}(y_j^1, y_j^2, \ldots, y_j^n) = 0$ for $1 \leq j \leq m-1$. Recall that this will only happen when $(y_j^1, y_j^2, \ldots, y_j^n) = (0, 0, \ldots, 0)$. We also need $p_1^{\mathbf{A_1}}(x_j^1, x_j^2, \ldots, x_j^n) = 1$, for $1 \leq j \leq n-1$. This will happen when there is at most one $x_j^i = 0$ in the tuple, for each j. By the Pigeon Hole Principle, there will be some i such that $x_j^i = 1$, for all j. Thus, we get that for some i,

$$\begin{pmatrix} (0, y_1^i) \\ \vdots \\ (0, y_{m-1}^i) \\ (x_1^i, 0) \\ \vdots \\ (x_{n-1}^i, 0) \end{pmatrix} = \begin{pmatrix} (0, 0) \\ \vdots \\ (0, 0) \\ (1, 0) \\ \vdots \\ (1, 0) \end{pmatrix} = l$$

We can use a similar argument when $p_2^{\mathbf{A}}$ is applied to elements of $\mathbf{C} \setminus \{l\}$. We have thus shown that $\mathbf{C} \setminus \{l\}$ is indeed a subalgebra of $\mathbf{A}^{n+m-2}$.

Now consider the subalgebra $\mathbf{S}$ of $\mathbf{A}^{n+m-2}$ generated by the tuples x_i and y_j where $1 \leq i \leq m-1$ and $m \leq j \leq n+m-2$, with $x_i = (0, \dots, 0, 2, 0, \dots 0, 1, \dots, 1)$ the tuple that varies from l only in the i-th coordinate, and $x_j = (0, \dots, 0, 1, \dots, 1, 0, 1, \dots, 1)$ varies only in the $(m-1+j)$-th coordinate. Since all of the generators of $\mathbf{S}$ are elements of $\mathbf{C} \setminus \{l\}$, immediately we see that $\mathbf{S} \subseteq \mathbf{C} \setminus \{l\}$, and thus, $l \notin \mathbf{S}$. Using $\mathbf{S}$ and the x_i's and y_j's, it follows from one direction of Theorem 3 that $\mathbf{A}$ does not have an $(n+m-2)$-ary near unanimity term. $\square$

3.2 Upper Bound

We will now consider the upper bound for the function $f(n, m)$. We are aware of a loose upper bound, which is $f(n, m) \leq nm$. However, when considering the direct product of two similar algebras, we want to find a near unanimity term with a smaller arity.

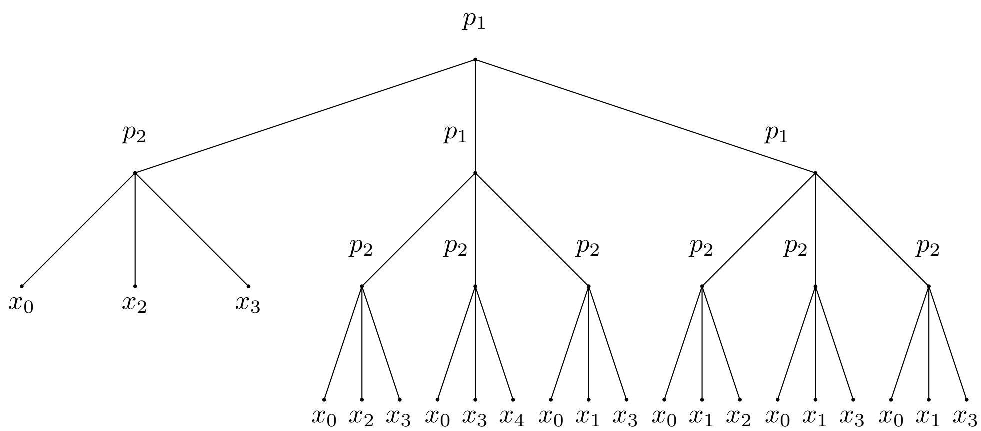
Conjecture 1. *Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be two similar, idempotent algebras with n-ary and m-ary near unanimity terms respectively, where n and m are natural numbers greater than 2. Then the direct product $\mathbf{A_1} \times \mathbf{A_2}$ has a near unanimity term with arity $n+m-1$. In other words, $f(n, m) = n+m-1$.*

Proposition 1. *Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be two similar, idempotent algebras with majority terms p_1 and p_2 respectively. Then the direct product $\mathbf{A} = \mathbf{A_1} \times \mathbf{A_2}$ has a near unanimity term with arity 5. In other words, $f(3, 3) = 5$.*

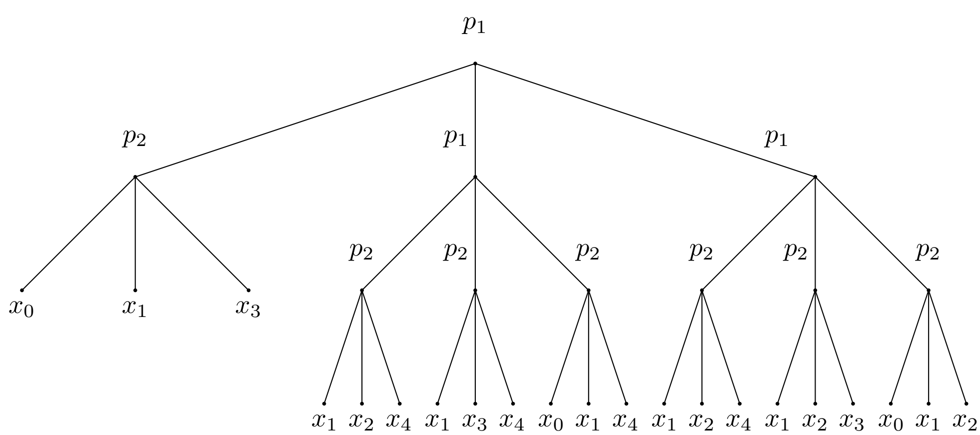
Proof. To support this, we show that $f(3, 3) = 5$ by constructing a suitable 5-ary term.

Let this 5-ary term be $t(x_0, x_1, x_2, x_3, x_4) = p_1(t_0, t_1, t_2)$, where the x_i are variables, and the t_i are the three subterms described by the following parse trees:

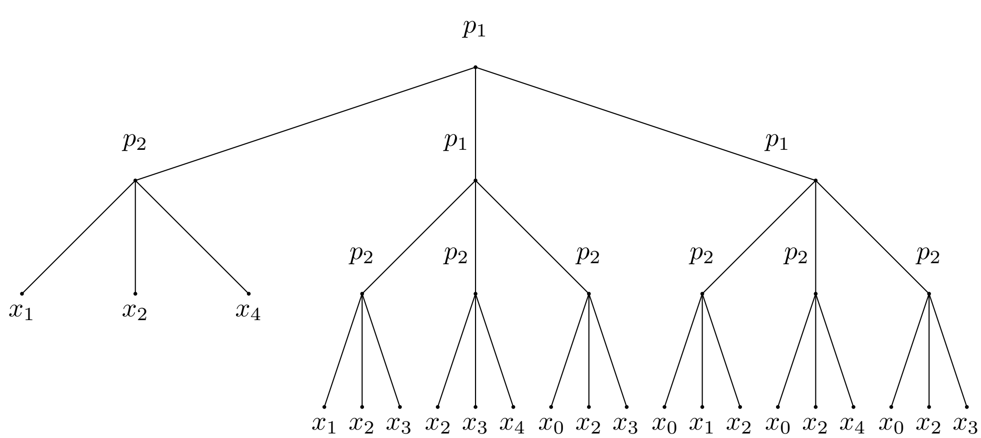
$t_0 =$


p_1
p_2 p_1 p_1
p_2 p_2 p_2 p_2 p_2 p_2
x_0 x_2 x_3
x_0 x_2 x_3 x_0 x_3 x_4 x_0 x_1 x_3 x_0 x_1 x_2 x_0 x_1 x_3 x_0 x_1 x_3

$t_1 =$


p_1
p_2 p_1 p_1
p_2 p_2 p_2 p_2 p_2 p_2
x_0 x_1 x_3
x_1 x_2 x_4 x_1 x_3 x_4 x_0 x_1 x_4 x_1 x_2 x_4 x_1 x_2 x_3 x_0 x_1 x_2

$t_2 =$


p_1
p_2 p_1 p_1
p_2 p_2 p_2 p_2 p_2 p_2
x_1 x_2 x_4
x_1 x_2 x_3 x_2 x_3 x_4 x_0 x_2 x_3 x_0 x_1 x_2 x_0 x_2 x_4 x_0 x_2 x_3

These subterms can also be defined as follows:

$$\begin{aligned}
t_0 = p_1(&p_2(x_0, x_2, x_3),\\
&p_1(p_2(x_0, x_2, x_3), p_2(x_0, x_3, x_4), p_2(x_0, x_1, x_3)),\\
&p_1(p_2(x_0, x_1, x_2), p_2(x_0, x_1, x_3), p_2(x_0, x_1, x_3)))\\
t_1 = p_1(&p_2(x_0, x_1, x_3),\\
&p_1(p_2(x_1, x_2, x_4), p_2(x_1, x_3, x_4), p_2(x_0, x_1, x_4)),\\
&p_1(p_2(x_1, x_2, x_4), p_2(x_1, x_2, x_3), p_2(x_0, x_1, x_2)))\\
t_2 = p_1(&p_2(x_1, x_2, x_4),\\
&p_1(p_2(x_1, x_2, x_3), p_2(x_2, x_3, x_4), p_2(x_0, x_2, x_3)),\\
&p_1(p_2(x_0, x_1, x_2), p_2(x_0, x_2, x_4), p_2(x_0, x_2, x_3)))
\end{aligned}$$

Now to verify that t is indeed a 5-ary NU term for $\mathbf{A}$, we need only show this for $\mathbf{A_1}$ and $\mathbf{A_2}$ separately. Recall that p_1 and p_2 are majority terms for the algebras $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively, and that both algebras are idempotent.

First consider $\mathbf{A_1}$. Now, $t(x_0, x_1, x_2, x_3, x_4) = p_1(t_0, t_1, t_2)$, and we must show that $\mathbf{A_1}$ satisfies the following five equations:

$$\begin{aligned}
&t(y, x, x, x, x) \approx t(x, y, x, x, x) \approx t(x, x, y, x, x)\\
&\approx t(x, x, x, y, x) \approx t(x, x, x, x, y) \approx x
\end{aligned}$$

Let $a, b \in A_1$ and consider $t^{\mathbf{A_1}}(b, a, a, a, a)$. We need at least two of the three t_i's to evaluate to a since p_1 is a 3-ary NU term for $\mathbf{A_1}$. Again, each subterm $t_i = p_1(s_0^i, s_1^i, s_2^i)$ so we must have that two of the three s_j^i's evaluate to a. Notice that in the subterm t_0, the term s_0^0 evaluates to $p_2(b, a, a)$ which may not equal a, since p_2 is just assumed to be idempotent. Also, $s_1^0 = p_1(p_2(x_0, x_2, x_3), p_2(x_0, x_3, x_4), p_2(x_0, x_1, x_3))$ evaluates to

$$p_1(p_2(b, a, a), p_2(b, a, a), p_2(b, a, a))$$

and for similar reasons, its value is undetermined. Similarly for s_2^0. Thus, $t_0 = p_1(s_0^0, s_1^0, s_2^0)$ may or may not evaluate to a. So, in order for $t^{\mathbf{A_1}}(b, a, a, a, a) = a$, we need both t_1 and t_2 to evaluate to a.

Looking at $t_1 = p_1(s_0^1, s_1^1, s_2^1)$, we see that $s_0^1 = p_2(x_0, x_1, x_3)$ may or may not evaluate to a. However, $s_1^1 = p_1(p_2(x_1, x_2, x_4), p_2(x_1, x_3, x_4), p_2(x_0, x_1, x_4))$

and when evaluated at (b,a,a,a,a) is equal to

$$p_1(p_2(a,a,a), p_2(a,a,a), p_2(b,a,a)) = p_1(a,a,p_2(b,a,a)) = a.$$

A similar process shows this for s_2^1.

Considering $t_2 = p_1(s_0^2, s_1^2, s_2^2)$, $s_0^2 = p_1(x_1, x_2, x_4)$ evaluates to

$$p_1(a,a,a) = a,$$

so we only need one of s_1^2 and s_2^2 to evaluate to a. We use $s_1^2 = p_1(p_2(x_1, x_2, x_3), p_2(x_2, x_3, x_4), p_2(x_0, x_2, x_3))$ since once evaluated, we get

$$p_1(p_2(a,a,a), p_2(a,a,a), p_2(b,a,a)) = p_1(a,a,p_2(b,a,a)) = a.$$

Thus, $t^{\mathbf{A_1}}(b,a,a,a,a) = p_1(t_0(b,a,a,a,a), a, a) = a$, and the first of the five equations is satisfied. The remaining equations are similar. We now have that $t(x_0, x_1, x_2, x_3, x_4)$ is a 5-ary NU term for $\mathbf{A_1}$.

As for $\mathbf{A_2}$, as described above, p_2 is a 3-ary NU term, and p_1 is an idempotent operation. Within the term $t(x_0, x_1, x_2, x_3, x_4)$, p_2 is always evaluated at 3 distinct variables. For each of the five equations we must satisfy, there is only one b occurring in the tuple. This means that at most one b will occur in any given p_2 instance. For example, $p_2(x_0, x_2, x_3)$ evaluates to $p_2(b,a,a) = a$ when t is evaluated at (b,a,a,a,a) in $\mathbf{A_2}$, since p_2 is a 3-ary NU term.

Now, $t(x_0, x_1, x_2, x_3, x_4) = p_1(t_0, t_1, t_2)$ so we must have that all t_i's evaluate to a, in order for t to evaluate to a. Let's look at $t_0 = p_1(s_1^0, s_2^0, s_3^0)$. Again, each s_j^0 must evaluate to a since p_1 is idempotent. Recall that

$$\begin{aligned}
s_0^0 =& p_2(x_0, x_2, x_3)\\
s_1^0 =& p_1(p_2(x_0, x_2, x_3), p_2(x_0, x_3, x_4), p_2(x_0, x_1, x_3))\\
s_2^0 =& p_1(p_2(x_0, x_1, x_2), p_2(x_0, x_1, x_3), p_2(x_0, x_1, x_3))
\end{aligned}$$

For any tuple $(x_0, x_1, x_2, x_3, x_4)$ consisting of four a'a and one b, we will get that s_j^0 evaluates to a. Similarly for t_1 and t_2. Thus,

$$t^{\mathbf{A_2}}(b,a,a,a,a) = \cdots = t^{\mathbf{A_2}}(a,a,a,a,b) = a,$$

and we have shown that t is a 5-ary NU term for $\mathbf{A_2}$. Therefore, it is a 5-ary NU term for $\mathbf{A}$. □

We began the search for an upper bound by working with small algebras and testing if their direct products had NU terms. This was tested using the Universal Algebra Calculator [4] and some software produced by Dr. Valeriote. We found a 5-ary NU term when both algebras consisted of two, 3-ary basic operations. After manipulating the tree structure of this term to suit algebras with higher arity NU terms, we were able to find an upper bound for f, namely that $f(n,m) \leq \lceil \frac{nm}{2} \rceil$.

Theorem 6. *Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be two similar, idempotent algebras with n-ary and m-ary near unanimity terms respectively, where n and m are natural numbers greater than 2. Then the direct product $\mathbf{A_1} \times \mathbf{A_2}$ has a near unanimity term with arity $\lceil \frac{nm}{2} \rceil$. In other words, $f(n,m) \leq \lceil \frac{nm}{2} \rceil$.*

Proof. Consider $f(n,m)$ where $m = 2p$ for simplicity, then $d = \lceil \frac{nm}{2} \rceil = \lceil \frac{n2p}{2} \rceil = np$. The structure of the np-ary NU term t that we construct is as follows:

$$\begin{aligned} t(x_0, x_1, \ldots, x_{np-1}) =& p_1(t_0, t_1, \ldots, t_{n-1}) \\ t_i =& p_1(s_0^i, s_1^i, \ldots, s_{n-1}^i) \\ s_0^i =& p_2(y_1, y_2, \ldots, y_m) \\ s_j^i =& p_1(p_2(z_1^1, \ldots, z_m^1), \ldots, p_2(z_1^n, \ldots, z_m^n)) \end{aligned}$$

for $0 \leq i \leq n-1$ and $1 \leq j \leq n-1$.

Because the order of the placement of variables in each instance of p_2 in t will not matter, we will describe the set of m variables that appear in each instance. We now divide the variables into n cosets, based on the modulus of their index. Let $\{\bar{i}\} = \{x_i, x_{n+i}, x_{2n+i}, \ldots\}$ for $0 \leq i \leq n-1$. There will be p elements in each coset. Then $s_0^i = p_2(\{\bar{i}\}, \{\overline{i+1}\})$, $s_0^{n-1} = p_2(\{\overline{n-1}\}, \{\bar{0}\})$, and we can start filling in the variables in each s_j^i as follows:

$$s_j^i = p_1(p_2(\{\bar{i}\}, \ldots), \ldots, p_2(\{\bar{i}\}, \ldots))$$

We have now filled in np of the $2np$ variables in each s_j^i. Now, for each coset $\{\bar{k}\}$, divide it into $n-1$ subsets as evenly as possible, call them $\{\bar{k}_j\}$ with $1 \leq j \leq n-1$. Then we continue to fill in each s_j^i.

$$s_j^i = p_1(p_2(\{\bar{k}_j\}_{k \neq i, i+1}, \{\bar{i}\}, \ldots), \ldots, p_2(\{\bar{k}_j\}_{k \neq i, i+1}, \{\bar{i}\}, \ldots))$$

Of the remaining np variables, we have filled $\frac{np(n-2)}{n-1}$ variables in for each

s^i_j. There is now only $(np - \frac{np(n-2)}{n-1})$ left to fill. Let $D = \{x_0, x_1, \ldots, x_{np-1}\}$ and let $R^i_j = D \setminus \{\{\bar{i}\} \cup \{\bar{k}_j\}_{k \neq i, i+1}\}$. We will choose from the sets R^i_j to fill in the remaining variables. The cardinality of R^i_j is $(np - p - \frac{(n-2)p}{n-1})$ and we may only use each variable once. So, we must have that $np - p - \frac{(n-2)p}{n-1} \geq np - \frac{n(n-2)p}{n-1}$, or that

$$\begin{aligned} np - p - \frac{(n-2)p}{n-1} - np + \frac{np(n-2)}{n-1} &\geq 0 \\ -p + p[\frac{n(n-2)}{n-1} - \frac{(n-2)}{n-1}] &\geq 0 \\ -p + p(n-2) &\geq 0 \\ p(n-3) &\geq 0 \end{aligned}$$

and this is always true since $n \geq 3$ and $p \geq 0$. Therefore, we can fill the remaining positions in s^i_j for all $0 \leq i, j \leq n-1$ by cycling through R^i_j. This ensures that p_2 is always evaluated at distinct variables. We leave it to the reader to verify that t is indeed a np-ary NU term for both algebras.

When m is odd, we may overfill or under fill s^i_0. So we then rearrange the cosets by moving one variable to another coset, and ensuring that no variable appears in more than 2 of the n s^i_0's. It may also happen that when dividing each coset into $n-1$ subsets, $\{\bar{k}_j\}$ may be empty for larger j. This will leave R^i_j with a higher cardinality, which will compensate for the missing variables.

□

We illustrate this construction for $f(3, 8) = 12$. Let

$$\begin{aligned} \{\bar{0}\} &= \{x_0, x_3, x_6, x_9\} \\ \{\bar{1}\} &= \{x_1, x_4, x_7, x_{10}\} \\ \{\bar{2}\} &= \{x_2, x_5, x_8, x_{11}\} \end{aligned}$$

$$\{\bar{0}_1\} = \{x_0, x_6\},\ \{\bar{0}_2\} = \{x_3, x_9\}$$
$$\{\bar{1}_1\} = \{x_1, x_7\},\ \{\bar{1}_2\} = \{x_4, x_{10}\}$$
$$\{\bar{2}_1\} = \{x_2, x_8\},\ \{\bar{2}_2\} = \{x_5, x_{11}\}$$

We begin by partially filling in each s^i_j as follows:

$$
\begin{aligned}
s^0_0 &= p_2(x_0, x_3, x_6, x_9, x_1, x_4, x_7, x_{10}) \\
s^0_1 &= p_1(p_2(x_0, x_3, x_6, x_9, x_2, x_8, \dots), p_2(x_0, x_3, x_6, x_9, x_2, x_8, \dots), \\
&\qquad p_2(x_0, x_3, x_6, x_9, x_2, x_8, \dots)) \\
s^0_2 &= p_1(p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, \dots), p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, \dots), \\
&\qquad p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, \dots))
\end{aligned}
$$

$$
\begin{aligned}
s^1_0 &= p_2(x_1, x_4, x_7, x_{10}, x_2, x_5, x_8, x_{11}) \\
s^1_1 &= p_1(p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, \dots), p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, \dots), \\
&\qquad p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, \dots)) \\
s^1_2 &= p_1(p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, \dots), p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, \dots), \\
&\qquad p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, \dots))
\end{aligned}
$$

$$
\begin{aligned}
s^2_0 &= p_2(x_2, x_5, x_8, x_{11}, x_0, x_3, x_6, x_9) \\
s^2_1 &= p_1(p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, \dots), p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, \dots), \\
&\qquad p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, \dots)) \\
s^2_2 &= p_1(p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, \dots), p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, \dots), \\
&\qquad p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, \dots))
\end{aligned}
$$

We now describe the set of remaining variables that we will fill each s^i_j with, and cycle through these sets, using each variable only once.

$$
\begin{aligned}
R_1^0 &= \{x_1, x_4, x_7, x_{10}, x_5, x_{11}\}, \;\; R_2^0 = \{x_1, x_4, x_7, x_{11}, x_2, x_8\} \\
R_1^1 &= \{x_2, x_5, x_8, x_{11}, x_3, x_9\}, \;\; R_2^1 = \{x_2, x_5, x_8, x_{11}, x_0, x_6\} \\
R_1^2 &= \{x_0, x_3, x_6, x_9, x_4, x_{10}\}, \;\; R_2^2 = \{x_0, x_3, x_6, x_9, x_1, x_7\}
\end{aligned}
$$

The result is a 12-ary term, with the following structure:

$$t(x_0, x_1, \ldots, x_{11}) = p_1(t_0, t_1, t_2)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i)$, and

$$
\begin{aligned}
s_0^0 &= p_2(x_0, x_3, x_6, x_9, x_1, x_4, x_7, x_{10}) \\
s_1^0 &= p_1(p_2(x_0, x_3, x_6, x_9, x_2, x_8, x_1, x_{10}), p_2(x_0, x_3, x_6, x_9, x_2, x_8, x_4, x_5), \\
&\qquad p_2(x_0, x_3, x_6, x_9, x_2, x_8, x_7, x_{11})) \\
s_2^0 &= p_1(p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, x_1, x_{10}), p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, x_4, x_2), \\
&\qquad p_2(x_0, x_3, x_6, x_9, x_5, x_{11}, x_7, x_8))
\end{aligned}
$$

$$
\begin{aligned}
s_0^1 &= p_2(x_1, x_4, x_7, x_{10}, x_2, x_5, x_8, x_{11}) \\
s_1^1 &= p_1(p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, x_2, x_{11}), p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, x_5, x_3), \\
&\qquad p_2(x_1, x_4, x_7, x_{10}, x_0, x_6, x_8, x_9)) \\
s_2^1 &= p_1(p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, x_2, x_{11}), p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, x_5, x_0), \\
&\qquad p_2(x_1, x_4, x_7, x_{10}, x_3, x_9, x_8, x_6))
\end{aligned}
$$

$$
\begin{aligned}
s_0^2 &= p_2(x_2, x_5, x_8, x_{11}, x_0, x_3, x_6, x_9) \\
s_1^2 &= p_1(p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, x_0, x_9), p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, x_3, x_4), \\
&\qquad p_2(x_2, x_5, x_8, x_{11}, x_1, x_7, x_6, x_{10})) \\
s_2^2 &= p_1(p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, x_0, x_9), p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, x_3, x_1), \\
&\qquad p_2(x_2, x_5, x_8, x_{11}, x_4, x_{10}, x_6, x_7))
\end{aligned}
$$

It is left as an exercise to the reader to verify that this is a 12-ary NU term.

However, these upper bounds are not always in accordance with our conjecture. With these NU terms, we can reduce the arity by deepening the parse tree structure, or in other words, increasing the length of the term by composing more basic operations within this term. By adding one more layer to the parse tree structure, which is done by replacing a variable by a basic operation and evaluating this at existing variables, we were able to achieve the following:

- $f(3,3) = 5$
- $f(3,4) = 6$
- $f(3,5) = 7$
- $f(4,4) = 7$
- $f(4,5) = 8$

For this one additional layer, we replace the unwanted variable by the basic operation p_1 and evaluate this at the set of variables complementary to the variables in the original tuple.

For $f(3,4)$ we have

$$t(x_0, x_1, x_2, x_3, x_4, x_5) = p_1(t_0, t_1, t_2)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i)$ and

$$\begin{aligned}
s_0^0 &= p_2(x_0, x_2, x_3, x_5) \\
s_1^0 &= p_1(p_2(x_0, x_3, x_4, x_5), p_2(x_0, x_2, x_3, x_4), p_2(x_0, x_1, x_3, x_4)) \\
s_2^0 &= p_1(p_2(x_0, x_1, x_3, x_5), p_2(x_0, x_1, x_3, x_4), p_2(x_0, x_1, x_2, x_3))
\end{aligned}$$

$$\begin{aligned}
s_0^1 &= p_2(x_0, x_1, x_3, x_4) \\
s_1^1 &= p_1(p_2(x_1, x_3, x_4, x_5), p_2(x_1, x_2, x_4, x_5), p_2(x_0, x_1, x_4, x_5)) \\
s_2^1 &= p_1(p_2(x_1, x_2, x_3, x_4), p_2(x_1, x_2, x_4, x_5), p_2(x_0, x_1, x_2, x_4))
\end{aligned}$$

$$\begin{aligned}
s_0^2 &= p_2(x_1, x_2, x_4, x_5) \\
s_1^2 &= p_1(p_2(x_2, x_3, x_4, x_5), p_2(x_1, x_2, x_3, x_5), p_2(x_0, x_2, x_3, x_5)) \\
s_2^2 &= p_1(p_2(x_0, x_2, x_4, x_5), p_2(x_0, x_2, x_3, x_5), p_2(x_0, x_1, x_2, x_5))
\end{aligned}$$

For $f(3,5)$ we have

$$t(x_0,x_1,x_2,x_3,x_4,x_5,x_6) = p_1(t_0,t_1,t_2)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i)$ and

$$\begin{aligned}
s_0^0 &= p_2(x_0,x_2,x_3,x_5,x_6)\\
s_1^0 &= p_1(p_2(x_0,x_3,x_4,x_5,x_6), p_2(x_0,x_2,x_3,x_4,x_6), p_2(x_0,x_1,x_3,x_4,x_6))\\
s_2^0 &= p_1(p_2(x_0,x_1,x_3,x_4,x_6), p_2(x_0,x_1,x_3,x_5,x_6), p_2(x_0,x_1,x_2,x_3,x_6))
\end{aligned}$$

$$\begin{aligned}
s_0^1 &= p_2(x_0,x_1,x_3,x_4,x_6)\\
s_1^1 &= p_1(p_2(x_1,x_3,x_4,x_5,p_1(x_0,x_2,x_6)), p_2(x_1,x_2,x_4,x_5,p_1(x_0,x_3,x_6)),\\
&\qquad p_2(x_0,x_1,x_4,x_5,x_6))\\
s_2^1 &= p_1(p_2(x_1,x_2,x_3,x_4,p_1(x_0,x_5,x_6)), p_2(x_1,x_2,x_4,x_5,p_1(x_0,x_3,x_6)),\\
&\qquad p_2(x_0,x_1,x_2,x_4,x_6))
\end{aligned}$$

$$\begin{aligned}
s_0^2 &= p_2(x_1,x_2,x_4,x_5,p_1(x_0,x_3,x_6))\\
s_1^2 &= p_1(p_2(x_2,x_3,x_4,x_5,p_1(x_0,x_1,x_6)), p_2(x_1,x_2,x_3,x_5,p_1(x_0,x_4,x_6)),\\
&\qquad p_2(x_0,x_2,x_3,x_5,x_6))\\
s_2^2 &= p_1(p_2(x_0,x_2,x_4,x_5,x_6), p_2(x_0,x_2,x_3,x_5,x_6), p_2(x_0,x_1,x_2,x_5,x_6))
\end{aligned}$$

For $f(4,4)$ we have

$$t(x_0,x_1,x_2,x_3,x_4,x_5,x_6) = p_1(t_0,t_1,t_2,t_3)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i, s_3^i)$ and

$$\begin{aligned}
s_0^0 &= p_2(x_0,x_3,x_4,p_1(x_1,x_2,x_5,x_6))\\
s_1^0 &= p_1(p_2(x_0,x_2,x_3,x_6), p_2(x_0,x_2,x_5,x_6), p_2(x_0,x_2,x_4,x_6), p_2(x_0,x_1,x_2,x_6))\\
s_2^0 &= p_1(p_2(x_0,x_2,x_5,p_1(x_1,x_3,x_4,x_6)), p_2(x_0,x_5,x_6,p_1(x_1,x_2,x_3,x_4)),\\
&\qquad p_2(x_0,x_3,x_5,p_1(x_1,x_2,x_4,x_6)), p_2(x_0,x_1,x_4,x_5)\\
s_3^0 &= p_1(p_2(x_0,x_1,x_3,p_1(x_2,x_4,x_5,x_6)), p_2(x_0,x_1,x_5,p_1(x_2,x_3,x_4,x_6)),\\
&\qquad p_2(x_0,x_1,x_4,x_6), p_2(x_0,x_1,x_2,p_1(x_3,x_4,x_5,x_6))
\end{aligned}$$

$$\begin{aligned}
s_0^1 &= p_2(x_0,x_1,x_4,x_5)\\
s_1^1 &= p_1(p_2(x_1,x_2,x_3,x_4),p_2(x_1,x_2,x_4,x_6),p_2(x_1,x_2,x_4,x_5),p_2(x_0,x_1,x_2,x_4))\\
s_2^1 &= p_1(p_2(x_1,x_2,x_4,x_6),p_2(x_1,x_4,x_5,x_6),p_2(x_1,x_3,x_4,x_6),p_2(x_0,x_1,x_4,x_6))\\
s_3^1 &= p_1(p_2(x_1,x_2,x_3,x_4),p_2(x_1,x_3,x_4,x_5),p_2(x_1,x_3,x_4,x_6),p_2(x_0,x_1,x_3,x_4))
\end{aligned}$$

$$\begin{aligned}
s_0^2 &= p_2(x_1,x_2,x_5,x_6)\\
s_1^2 &= p_1(p_2(x_1,x_2,x_3,x_5),p_2(x_2,x_3,x_4,x_5),p_2(x_2,x_3,x_5,x_6),p_2(x_0,x_2,x_3,x_5))\\
s_2^2 &= p_1(p_2(x_1,x_2,x_4,x_5),p_2(x_2,x_4,x_5,x_6),p_2(x_2,x_3,x_4,x_5),p_2(x_0,x_2,x_4,x_5))\\
s_3^2 &= p_1(p_2(x_0,x_1,x_2,x_5),p_2(x_0,x_2,x_4,x_5),p_2(x_0,x_2,x_5,x_6),p_2(x_0,x_2,x_3,x_5))
\end{aligned}$$

$$\begin{aligned}
s_0^3 &= p_2(x_2,x_3,x_6,p_1(x_0,x_1,x_4,x_5))\\
s_1^3 &= p_1(p_2(x_1,x_2,x_3,x_6),p_2(x_1,x_3,x_4,x_6),p_2(x_1,x_3,x_5,x_6),p_2(x_0,x_1,x_3,x_6))\\
s_2^3 &= p_1(p_2(x_1,x_3,x_4,x_5),p_2(x_2,x_3,x_4,x_5),p_2(x_3,x_4,x_5,x_6),p_2(x_0,x_3,x_4,x_5))\\
s_3^3 &= p_1(p_2(x_0,x_1,x_3,x_6),p_2(x_0,x_3,x_4,x_6),p_2(x_0,x_3,x_5,x_6),p_2(x_0,x_2,x_3,x_6))
\end{aligned}$$

For $f(4,5)$ we have

$$t(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7) = p_1(t_0,t_1,t_2,t_3)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i, s_3^i)$ and

$$\begin{aligned}
s_0^0 &= p_2(x_0,x_3,x_4,x_7,p_1(x_1,x_2,x_5,x_6))\\
s_1^0 &= p_1(p_2(x_0,x_4,x_5,x_6,x_7),p_2(x_0,x_3,x_4,x_5,x_6),p_2(x_0,x_2,x_4,x_5,x_6),\\
&\qquad p_2(x_0,x_1,x_4,x_5,x_6))\\
s_2^0 &= p_1(p_2(x_0,x_2,x_4,x_5,p_1(x_1,x_3,x_6,x_7)),p_2(x_0,x_2,x_4,x_6,p_1(x_1,x_3,x_5,x_7)),\\
&\qquad p_2(x_0,x_2,x_3,x_4,p_1(x_1,x_5,x_6,x_7)),p_2(x_0,x_1,x_2,x_4,x_7))\\
s_3^0 &= p_1(p_2(x_0,x_1,x_4,x_5,p_1(x_2,x_3,x_6,x_7)),p_2(x_0,x_1,x_4,x_6,p_1(x_2,x_3,x_5,x_7)),\\
&\qquad p_2(x_0,x_1,x_3,x_4,p_1(x_2,x_5,x_6,x_7)),p_2(x_0,x_1,x_2,x_4,x_7))
\end{aligned}$$

$$
\begin{aligned}
s_0^1 &= p_2(x_0, x_1, x_4, x_5, p_1(x_2, x_3, x_6, x_7))\\
s_1^1 &= p_1(p_2(x_1, x_4, x_5, x_6, p_1(x_0, x_2, x_3, x_7)), p_2(x_1, x_3, x_5, x_6, p_1(x_0, x_2, x_4, x_7)),\\
&\qquad p_2(x_1, x_2, x_5, x_6, p_1(x_0, x_3, x_4, x_7)), p_2(x_0, x_1, x_5, x_6, x_7))\\
s_2^1 &= p_1(p_2(x_1, x_3, x_4, x_5, p_1(x_0, x_2, x_6, x_7)), p_2(x_1, x_3, x_5, x_6, p_1(x_0, x_2, x_4, x_7)),\\
&\qquad p_2(x_1, x_2, x_3, x_5, p_1(x_0, x_4, x_6, x_7)), p_2(x_0, x_1, x_3, x_5, x_7))\\
s_3^1 &= p_1(p_2(x_1, x_2, x_4, x_5, x_7), p_2(x_1, x_2, x_5, x_6, x_7), p_2(x_1, x_2, x_3, x_5, x_7),\\
&\qquad p_2(x_0, x_1, x_2, x_5, x_7))
\end{aligned}
$$

$$
\begin{aligned}
s_0^2 &= p_2(x_1, x_2, x_5, x_6, p_1(x_0, x_3, x_4, x_7))\\
s_1^2 &= p_1(p_2(x_2, x_4, x_5, x_6, p_1(x_0, x_1, x_3, x_7)), p_2(x_2, x_3, x_4, x_6, p_1(x_0, x_1, x_5, x_7)),\\
&\qquad p_2(x_1, x_2, x_4, x_6, p_1(x_0, x_3, x_5, x_7)), p_2(x_0, x_2, x_4, x_6, x_7))\\
s_2^2 &= p_1(p_2(x_2, x_3, x_5, x_6, p_1(x_0, x_1, x_4, x_7)), p_2(x_2, x_3, x_4, x_6, p_1(x_0, x_1, x_5, x_7)),\\
&\qquad p_2(x_1, x_2, x_3, x_6, p_1(x_0, x_4, x_5, x_7)), p_2(x_0, x_2, x_3, x_6, x_7))\\
s_3^2 &= p_1(p_2(x_0, x_2, x_5, x_6, x_7), p_2(x_0, x_2, x_4, x_6, x_7), p_2(x_0, x_2, x_3, x_6, x_7),\\
&\qquad p_2(x_0, x_1, x_2, x_6, x_7))
\end{aligned}
$$

$$
\begin{aligned}
s_0^3 &= p_2(x_2, x_3, x_6, x_7, p_1(x_0, x_1, x_4, x_5))\\
s_1^3 &= p_1(p_2(x_3, x_4, x_5, x_6, x_7), p_2(x_2, x_3, x_4, x_7, p_1(x_0, x_1, x_5, x_6)),\\
&\qquad p_2(x_1, x_3, x_4, x_7, p_1(x_0, x_2, x_5, x_6)), p_2(x_0, x_3, x_4, x_7, p_1(x_1, x_2, x_5, x_6)))\\
s_2^3 &= p_1(p_2(x_1, x_3, x_5, x_6, x_7), p_2(x_1, x_3, x_4, x_7, p_1(x_0, x_2, x_5, x_6)),\\
&\qquad p_2(x_1, x_2, x_3, x_7, p_1(x_0, x_4, x_5, x_6)), p_2(x_0, x_1, x_3, x_7, p_1(x_2, x_4, x_5, x_6)))\\
s_3^3 &= p_1(p_2(x_0, x_3, x_5, x_6, x_7), p_2(x_0, x_3, x_4, x_5, x_7), p_2(x_0, x_2, x_3, x_5, x_7),\\
&\qquad p_2(x_0, x_1, x_3, x_5, x_7))
\end{aligned}
$$

For $f(5,5)$, with one additional layer, we were able to find a NU term with arity 10. This term is as follows:

$$t(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = p_1(t_0, t_1, t_2, t_3, t_4)$$

where $t_i = p_1(s_0^i, s_1^i, s_2^i, s_3^i, s_4^i)$ and

$$
\begin{aligned}
s_0^0 &= p_2(x_0, x_4, x_5, x_9, p_1(x_1, x_2, x_3, x_6, x_7)) \\
s_1^0 &= p_1(p_2(x_0, x_1, x_2, x_5, x_8), p_2(x_0, x_1, x_4, x_5, x_8), p_2(x_0, x_1, x_5, x_7, x_8), \\
&\qquad p_2(x_0, x_1, x_5, x_6, x_8), p_2(x_0, x_1, x_3, x_5, x_8)) \\
s_2^0 &= p_1(p_2(x_0, x_2, x_5, x_6, p_1(x_1, x_3, x_4, x_7, x_8)), p_2(x_0, x_5, x_6, x_7, p_1(x_1, x_2, x_3, x_4, x_8)), \\
&\qquad p_2(x_0, x_4, x_5, x_6, p_1(x_1, x_2, x_3, x_7, x_8)), p_2(x_0, x_3, x_5, x_6, x_8), \\
&\qquad p_2(x_0, x_1, x_5, x_6, x_9)) \\
s_3^0 &= p_1(p_2(x_0, x_2, x_3, x_5, p_1(x_1, x_4, x_6, x_7, x_8)), p_2(x_0, x_3, x_5, x_7, p_1(x_1, x_2, x_4, x_6, x_8)), \\
&\qquad p_2(x_0, x_3, x_5, x_6, x_8), p_2(x_0, x_3, x_4, x_5, p_1(x_1, x_2, x_6, x_7, x_8)), \\
&\qquad p_2(x_0, x_1, x_3, x_5, x_9)) \\
s_4^0 &= p_1(p_2(x_0, x_2, x_3, x_5, x_7), p_2(x_0, x_2, x_5, x_7, x_8), \\
&\qquad p_2(x_0, x_2, x_5, x_6, x_7), p_2(x_0, x_2, x_4, x_5, x_7), p_2(x_0, x_1, x_2, x_5, x_7))
\end{aligned}
$$

$$
\begin{aligned}
s_0^1 &= p_2(x_0, x_1, x_5, x_6, p_1(x_2, x_3, x_4, x_7, x_8)) \\
s_1^1 &= p_1(p_2(x_1, x_2, x_3, x_6, x_8), p_2(x_1, x_3, x_6, x_7, x_8), p_2(x_1, x_3, x_5, x_6, x_8), \\
&\qquad p_2(x_1, x_3, x_4, x_6, x_8), p_2(x_0, x_1, x_3, x_6, x_8)) \\
s_2^1 &= p_1(p_2(x_1, x_2, x_6, x_7, p_1(x_0, x_3, x_4, x_5, x_8)), p_2(x_1, x_5, x_6, x_7, p_1(x_0, x_2, x_3, x_4, x_8)), \\
&\qquad p_2(x_1, x_4, x_6, x_7, p_1(x_0, x_2, x_3, x_5, x_8)), p_2(x_1, x_3, x_6, x_7, x_9), \\
&\qquad p_2(x_0, x_1, x_6, x_7, x_8)) \\
s_3^1 &= p_1(p_2(x_1, x_2, x_4, x_6, p_1(x_0, x_3, x_5, x_7, x_8)), p_2(x_1, x_4, x_6, x_7, p_1(x_0, x_2, x_3, x_5, x_8)), \\
&\qquad p_2(x_1, x_4, x_5, x_6, p_1(x_0, x_2, x_3, x_7, x_8)), p_2(x_1, x_3, x_4, x_6, x_9), \\
&\qquad p_2(x_0, x_1, x_4, x_6, x_8), \\
s_4^1 &= p_1(p_2(x_1, x_2, x_3, x_6, x_9), p_2(x_1, x_2, x_6, x_7, x_9), p_2(x_1, x_2, x_5, x_6, x_9), \\
&\qquad p_2(x_1, x_2, x_4, x_6, x_8), p_2(x_0, x_1, x_2, x_6, x_9))
\end{aligned}
$$

$$
\begin{aligned}
s_0^2 &= p_2(x_1, x_2, x_6, x_7, p_1(x_0, x_3, x_4, x_5, x_8)) \\
s_1^2 &= p_1(p_2(x_0, x_2, x_5, x_7, x_8), p_2(x_2, x_5, x_6, x_7, x_8), p_2(x_2, x_4, x_5, x_7, x_8), \\
&\qquad p_2(x_2, x_3, x_5, x_7, x_8), p_2(x_1, x_2, x_5, x_7, x_8)) \\
s_2^2 &= p_1(p_2(x_0, x_2, x_3, x_7, p_1(x_1, x_4, x_5, x_6, x_8)), p_2(x_2, x_3, x_4, x_7, p_1(x_0, x_1, x_5, x_6, x_8)), \\
&\qquad p_2(x_2, x_3, x_6, x_7, p_1(x_0, x_1, x_4, x_5, x_8)), p_2(x_2, x_3, x_5, x_7, x_9), \\
&\qquad p_2(x_1, x_2, x_3, x_7, x_8)) \\
s_3^2 &= p_1(p_2(x_0, x_2, x_4, x_7, p_1(x_1, x_3, x_5, x_6, x_8)), p_2(x_2, x_4, x_6, x_7, p_1(x_0, x_1, x_3, x_5, x_8)), \\
&\qquad p_2(x_2, x_4, x_5, x_7, p_1(x_0, x_1, x_3, x_6, x_8)), p_2(x_2, x_3, x_4, x_7, x_9), \\
&\qquad p_2(x_1, x_2, x_4, x_7, x_8)) \\
s_4^2 &= p_1(p_2(x_0, x_2, x_3, x_7, x_9), p_2(x_0, x_2, x_6, x_7, x_9), p_2(x_0, x_2, x_5, x_7, x_9), \\
&\qquad p_2(x_0, x_2, x_4, x_7, x_9), p_2(x_0, x_1, x_2, x_7, x_8))
\end{aligned}
$$

$$
\begin{aligned}
s_0^3 &= p_2(x_2, x_3, x_7, x_8, p_1(x_0, x_1, x_4, x_5, x_6)) \\
s_1^3 &= p_1(p_2(x_0, x_2, x_3, x_8, x_9), p_2(x_0, x_3, x_6, x_8, x_9), p_2(x_0, x_3, x_5, x_8, x_9), \\
&\qquad p_2(x_0, x_3, x_4, x_8, x_9), p_2(x_0, x_1, x_3, x_7, x_8)) \\
s_2^3 &= p_1(p_2(x_0, x_3, x_4, x_8, p_1(x_1, x_2, x_5, x_6, x_7)), p_2(x_2, x_3, x_4, x_8, p_1(x_0, x_1, x_5, x_6, x_7)), \\
&\qquad p_2(x_3, x_4, x_6, x_8, p_1(x_0, x_1, x_2, x_5, x_7)), p_2(x_3, x_4, x_5, x_8, x_9), \\
&\qquad p_2(x_1, x_3, x_4, x_7, x_8)) \\
s_3^3 &= p_1(p_2(x_0, x_3, x_5, x_6, x_8), p_2(x_3, x_5, x_6, x_7, x_8), p_2(x_3, x_4, x_5, x_6, x_8), \\
&\qquad p_2(x_2, x_3, x_5, x_6, x_8), p_2(x_1, x_3, x_5, x_6, x_8)) \\
s_4^3 &= p_1(p_2(x_0, x_1, x_3, x_8, p_1(x_2, x_4, x_5, x_6, x_7)), p_2(x_1, x_3, x_6, x_8, p_1(x_0, x_2, x_4, x_5, x_7)), \\
&\qquad p_2(x_1, x_3, x_5, x_8, p_1(x_0, x_2, x_4, x_6, x_7)), p_2(x_1, x_3, x_4, x_7, x_8), \\
&\qquad p_2(x_1, x_2, x_3, x_8, x_9))
\end{aligned}
$$

$$
\begin{aligned}
s_0^4 &= p_2(x_3, x_4, x_8, x_9, p_1(x_0, x_1, x_2, x_5, x_6)) \\
s_1^4 &= p_1(p_2(x_0, x_2, x_4, x_9, p_1(x_1, x_3, x_5, x_6, x_7)), p_2(x_0, x_4, x_5, x_9, p_1(x_1, x_2, x_3, x_6, x_7)), \\
&\qquad p_2(x_0, x_4, x_6, x_9, p_1(x_1, x_2, x_3, x_5, x_7)), p_2(x_0, x_3, x_4, x_8, x_9), \\
&\qquad p_2(x_0, x_1, x_4, x_7, x_9)) \\
s_2^4 &= p_1(p_2(x_0, x_4, x_5, x_6, x_9), p_2(x_4, x_5, x_6, x_7, x_9), p_2(x_3, x_4, x_5, x_6, x_9), \\
&\qquad p_2(x_2, x_4, x_5, x_6, x_9), p_2(x_1, x_4, x_5, x_6, x_8)) \\
s_3^4 &= p_1(p_2(x_0, x_2, x_4, x_9, p_1(x_1, x_3, x_5, x_6, x_7)), p_2(x_2, x_4, x_5, x_9, p_1(x_0, x_1, x_3, x_6, x_7)), \\
&\qquad p_2(x_2, x_4, x_6, x_8, x_9), p_2(x_2, x_3, x_4, x_9, p_1(x_0, x_1, x_5, x_6, x_7)), \\
&\qquad p_2(x_1, x_2, x_4, x_7, x_9)) \\
s_4^4 &= p_1(p_2(x_0, x_1, x_4, x_7, x_9), p_2(x_1, x_4, x_6, x_7, x_9), p_2(x_1, x_4, x_5, x_7, x_9), \\
&\qquad p_2(x_1, x_3, x_4, x_7, x_9), p_2(x_1, x_2, x_4, x_7, x_8))
\end{aligned}
$$

We want to reduce the arity of this term by one, which means we need to eliminate a variable. If we try to eliminate say x_9, we see that in some tuples, we can replace this variable by p_1 evaluated at the complement of the other variables in the tuple. However, this replacement can only occur once in each instance of p_2. To further eliminate this variable, we will need a term to satisfy certain equations which maintain the original conditions.

4 Conclusion

Throughout this report, we considered idempotent algebras with near unanimity terms and analyzed the arity of near unanimity terms in the direct product of pairs of such algebras. To do this, we introduced the function $f(n, m)$. Firstly, we have proven that the function f is bounded below by $n + m - 2$. We also proved that the function f is bounded above by $\lceil \frac{nm}{2} \rceil$. However, this upper bound has potential for improvement and we conjecture that $n + m - 1$ is the actual value of $f(n, m)$. Supporting evidence for our conjecture was obtained by computing $f(n, m)$ for small values of n and m. Our technique involved starting with a near unanimity term of arity $\lceil \frac{nm}{2} \rceil$ and then eliminating variables in an inductive manner to attain the conjectured result. This process has yet to be proven to work in all cases, however, once a thorough analysis of more examples has been completed, we hope that the inductive nature of our method will lead to a proof of the conjecture.

Bibliography

[1] Baker, K. and Pixley, A. *Polynomial interpolation and the Chinese remainder theorem for algebraic systems*. Math. Z. 143:165-174, 1975.

[2] Bergman, C.H. "Universal Algebra: Fundamentals and Selected Topics," CRC Press, Boca Raton, 2012.

[3] Burris, S. and Sankappanavar, H.P. "A course in universal algebra," Springer-Verlag, New York, 1981.

[4] Freese, R. and Kiss, E. and Valeriote, M. "Universal Algebra Calculator," Available at: `www.uacalc.org`, 2011.

[5] Freese, R. and McKenzie, R. *Maltsev families of varieties closed under join or Maltsev product*. Unpublished.

[6] Valeriote, M. Private Communication with Dr. M. Valeriote: Dept. of Mathematics and Statistics, McMaster University, 2013.