Some propositional logic deriviations

In class on Thursday, September 26, the following derivation of $\neg\neg\theta \vdash \theta$ was presented:

(1)	$(\neg\neg\theta \to (\neg\neg\neg\neg\theta \to \neg\neg\theta))$	Ax. 1
(2)	$\neg\neg\theta$	Ass.
(3)	$(\neg\neg\neg\neg\theta \to \neg\neg\theta)$	MP, 2, 1.
(4)	$((\neg\neg\neg\neg\theta \to \neg\neg\theta) \to (\neg\theta \to \neg\neg\neg\theta))$	Ax. 3
(5)	$(\neg\theta \to \neg\neg\neg\theta)$	MP, 3, 4
(6)	$((\neg\theta \to \neg\neg\neg\theta) \to (\neg\neg\theta \to \theta))$	Ax. 3
(7)	$(\neg\neg\theta \to \theta)$	MP, 5, 6
(8)	θ	MP, 2, 7

Afterwards, we discussed a related derivation, $\theta \vdash \neg\neg\theta$ and left it as an exercise to find one. We then discussed Theorem 3.1 from the text and worked through Exercise 3.8 c): if $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$, for some formula ψ, then $\Gamma \vdash \theta$ for all formulas θ.

During the lecture on Wednesday, October 2, the following derivations were considered:

- $(\neg\phi \to \phi) \vdash \phi$,
- $\vdash (\phi \to \phi)$.

The first can be shown by using the Deduction Theorem as follows: From Exercise 3.8 c) (mentioned earlier), we have

$$\phi, \neg\phi \vdash \neg(\neg\phi \to \phi),$$

with $\Gamma = \{\phi, \neg\phi\}$, $\psi = \phi$, and $\theta = \neg(\neg\phi \to \phi)$.

By the Deduction Theorem, we conclude that

$$\neg\phi \vdash (\phi \to \neg(\neg\phi \to \phi)),$$

and, with one more application, that

$$\vdash (\neg\phi \to (\phi \to \neg(\neg\phi \to \phi))).$$

Using this, the following is a derivation of $(\neg\phi \to \phi) \vdash \phi$:

(1)	$(\neg\phi \to (\phi \to \neg(\neg\phi \to \phi)))$	Just deduced
(2)	$((\neg\phi \to (\phi \to \neg(\neg\phi \to \phi))) \to ((\neg\phi \to \phi) \to (\neg\phi \to \neg(\neg\phi \to \phi))))$	Ax. 2
(3)	$((\neg\phi \to \phi) \to (\neg\phi \to \neg(\neg\phi \to \phi)))$	MP, 1, 2.
(4)	$(\neg\phi \to \phi)$	Ass.
(5)	$(\neg\phi \to \neg(\neg\phi \to \phi))$	MP, 3, 4
(6)	$((\neg\phi \to \neg(\neg\phi \to \phi)) \to ((\neg\phi \to \phi) \to \phi))$	Ax. 3
(7)	$((\neg\phi \to \phi) \to \phi)$	MP, 5, 6
(8)	ϕ	MP, 4, 7

The following is a derivation of $\vdash (\phi \to \phi)$ (that doesn't use the Deduction Theorem):

(1)	$(\phi \to ((\phi \to \phi) \to \phi))$	Ax. 1
(2)	$((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$	Ax. 2
(3)	$((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$	MP, 1, 2.
(4)	$(\phi \to (\phi \to \phi))$	Ax. 1
(5)	$(\phi \to \phi)$	MP, 3, 4