MATH 4L03 Assignment #7 Solutions

Do the following exercises from the course textbook:

1. Exercise 6.5, page 268

   Solution: For (a) (i), this is handled earlier in the text. Let Gr be the (finite) set of these axioms. For (ii) the formula

   $$O_n(x) = \left(x^n = e \wedge \left(\bigwedge_{0<j<n} \neg x^j = e \right) \right)$$

   will hold for an element x of a group if and only if it has order n. So the sentence $\forall x \neg O_n(x)$ will hold in a group if and only if the group does not have an element of order n. For part (iii), the following set of axioms, along with the set Gr, will work:

   $$\Sigma = \{\forall x \neg O_n(x) \mid n > 1\}.$$

   For part (b), suppose that the theory T of torsion free groups is finitely axiomatizable. Then there is a single sentence σ such that a group G is torsion free if and only if it satisfies σ. But then the following set is not satisfiable:

   $$Gr \cup \Sigma \cup \{\neg\sigma\}$$

   and so by the compactness theorem, there is some finite subset Δ of Σ such that $Gr \cup \Delta \cup \{\neg\sigma\}$ is not satisfiable. Let

   $$N = \max\{n \mid \forall x \neg O_n(x) \in \Delta\}.$$

   Let p be a prime number with $p > N$ and consider the cyclic group $\mathbb{Z}_p$. It is not torsion free and so satisfies $Gr \cup \{\neg\sigma\}$ and it does not have elements of order N or less and so it also satisfies Δ, which is a contradiction.

2. Exercise 6.12, page 271

   Solution: Let Σ axiomatize T and let Δ be some finite set of axioms for T. We may assume that Δ consists of a single sentence δ. Then

$\Sigma \models \delta$ so there is some finite subset Σ' of Σ with $\Sigma' \models \delta$ (this follows from the compactness or the completeness theorem). But then Σ' is a finite subset of Σ that axiomatizes T.

3. Exercise 6.17, page 275

   Solution: For part (a) (i), a model $\mathcal{A}$ of Γ would be a well ordered A set that has a subset $\{c_n^{\mathcal{A}} \mid n > 0\}$ that has no least element. For (ii), if Δ is a finite subset of Γ then only a finite number N of sentences of the form $c_{n+1} < c_n$ will appear in it. Let $\mathcal{A}_{2N}$ be the structure with domain $\{0, 1, \ldots, 2N\}$ that interprets $<$ in the usual way on this set. Then this structure is a well ordering and so satisfies Σ. The structure will interpret the constant symbols c_n that appear in Δ with distinct elements of $\{0, 1, \ldots, 2N\}$ in a way that if c_n and c_m both appear and $n < m$, then c_m is interpreted as a smaller number than c_n is. By the choice of N this can be accomplished since at most $2N$ different constants will appear in Δ. So, we have a model of Δ.

   For (b), if Σ axiomatizes the theory of well-order then we obtain a contradiction from part (a), namely, we can produce a set Γ that is not satisfiable, but that is finitely satisfiable.

4. Exercise 6.24, page 287

   Solution: Suppose that there is some first-order language L and set of L-sentences Σ such that Σ describes $\mathbb{R}$. By using the updward Lowenheim-Skolem Theorem we can get a structure that satisfies Σ that has cardinality greater than $|\mathbb{R}|$. So, Σ does not solely describe the structure $\mathbb{R}$.

   There is another way to answer this question. We can show that Σ must have a model for which the completeness property fails. Let c be a new constant symbol and let L_c be L with c added and let Σ_c be the set

   $$\Sigma \cup \{1 < c, 1 + 1 < c, \ldots, 1 + 1 + 1 + \cdots + 1 < c, \ldots\}.$$

   Then Σ_c has a model since every finite subset of it is satisfiable. To see this, let $\mathbb{R}_c$ be the expansion of $\mathbb{R}$ to the language L_c such that c is interpreted as an integer that is bigger than any of the sums that appear in the finite subset. This expanded structure will be a model of the finite subset.

In the model $\mathcal{A}$ of Σ_c that the compactness theorem provides, consider the following set: $N = \{1, 1+1, \ldots, 1+1+1+\cdots+1, \ldots\}$, i.e., the set of natural numbers, as interpreted in the model $\mathcal{A}$. This set is bounded above by the element $c^{\mathcal{A}}$ and so by the completeness axiom there must be some least upper bound b in A for this set. But the element $b-1$ is also an upper bound for N, since if it isn't, there will be some integer m with $b-1 < m$. From this we get that $b < m+1$ and so conclude that b (and hence c) wasn't an upper bound of N in the first place.

So, no such set Σ can exist.

5. Exercise 6.25 (a), (c), page 288

   Solution: For (a), by the downward Lowenheim-Skolem theorem we can obtain a countable set that satisfies the proposed axioms.

   For (c), the issue here is that $\mathbb{R}$ is uncountable, but we are working in a countable language. We know that any non-trivial real vector space is uncountably infinite, since it will have a 1-dimensional subspace (and this subspace is isomorphic to $\mathbb{R}$, considered as a 1-dimensional real vector space. So the subspace and hence the entire space is uncountably infinite. But, by the downward Lowenheim-Skolem Theorem, our axioms will have a countable non-trivial model. This is a contradiction.

6. Exercise 6.29, page 291

   Solution: We can use a similar technique used in the solution to 6.24 above. We can build a model that satisfies that same sentences as $\mathbb{R}$ but that has an element that is bigger than all of the integers. Using the Lowenheim-Skolem Theorem, we can obtain such a model that has the same cardinality as $\mathbb{R}$. These two models cannot be isomorphic.

7. Exercise 6.37, page 299

   Solution: $Th(\mathcal{A})$ is complete, since it is consistent (it has a model) and for every sentence ϕ, ϕ will be in $Th(\mathcal{A})$ if $\mathcal{A} \models \phi$ or $\neg\phi$ will be in $Th(\mathcal{A})$ otherwise (since then $\mathcal{A} \models \neg\phi$.