MATH 4L03 Assignment #4 Solutions

Due: Friday, November 8, 11:59pm.

1. Do exercise 4.11 (b) on page 150 of the textbook.

   Solution: The following occurrences of variables are free: the second occurrence of x_1 and the first and third occurrences of x_2. All other occurrences of variables are bound.

2. Do exercise 4.12 from page 150 of the textbook.

   Solution:

   - The scope of $\forall y$ is the subformula $(\exists x(R(y,z) \rightarrow \exists y R(x,y)) \wedge \neg\forall z R(x,y))$. The first and third occurrences of y in this subformula are bound by this quantifier.
   - The scope of $\exists x$ is the subformula $(R(y,z) \rightarrow \exists y R(x,y))$. The occurrence of x in this subformula is bound by this quantifier.
   - The scope of $\forall y$ is the first occurrence of the subformula $R(x,y)$. The occurrence of y in this subformula is bound by this quantifier.
   - The scope of $\forall z$ is the second occurrence of the subformula $R(x,y)$. No variables are bound by this quantifier.

3. Do exercise 4.13 from page 151 of the textbook.

   Solution: The values of the three terms, using the given structure and interpretation of variables is:

   (a) 4

   (b) 8

   (c) 37

4. Let L be the first order language that has a single 2 place relation symbol E. Show that if $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ is an L-structure that satisfies the following two sentences:

$$\forall x\,(E(x,x))$$

$$\forall x \forall y \forall z\, ((E(x,y) \wedge E(y,z)) \rightarrow E(z,x))$$

then it also satisfies the sentence

$$\forall x \forall y\, (E(x,y) \rightarrow E(y,x))\,.$$

Note that this provides a slightly shorter axiomatization of the class of equivalence relation structures.

Solution:

Suppose that $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ is a structure that satisfies the two sentences given in the problem. Let $a,\ b \in A$ and suppose that $(a,b) \in E^{\mathcal{A}}$. We must show that $(b,a) \in E^{\mathcal{A}}$. From the first sentence we know that $(a,a) \in E^{\mathcal{A}}$ and from the second we know that $(a,a) \in E^{\mathcal{A}}$ and $(a,b) \in E^{\mathcal{A}}$ implies $(b,a) \in E^{\mathcal{A}}$ as required.

5. (a) Using the same language L as in the previous question, show that there is no L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ which satisfies the sentences:

$$\exists x \forall y\, (E(x,y))$$
$$\exists x \forall y\, (\neg E(x,y))$$
$$\forall x \forall y ((E(x,y) \rightarrow E(y,x))\,.$$

(b) Is there an L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ that satisfies the sentences:

$$\forall x \exists y\, (E(x,y))$$
$$\forall x \forall y\, (E(x,y) \rightarrow \neg E(y,x))$$
$$\forall x \forall y \forall z\, ((E(x,y) \wedge E(y,z)) \rightarrow E(x,z))?$$

Solution: Note that when the context is clear, the interpretation of a relation symbol R by a structure $\mathcal{A}$ will be written as just R rather than $R^{\mathcal{A}}$. The same will apply to interpretations of function symbols.

(a) If $\langle A, E \rangle$ is a structure which satisfies the three given sentences, then there is some element of A, call it ∞ with $(\infty, a) \in E$ for all a in A. From the second statement, we see that there is another element of A, call it $\clubsuit$, with $(\clubsuit, b) \notin E$ for all $b \in A$. So we have, in particular, that $(\infty, \clubsuit) \in E$ and $(\clubsuit, \infty) \notin E$. Finally, the third statement shows that since $(\infty, \clubsuit) \in E$ then $(\clubsuit, \infty) \in E$, a contradiction.

(b) The structure $\langle \mathbb{N}, < \rangle$ satisfies all three of the sentences.

6. Let $\underline{\mathbb{N}}$ be the structure $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$, where the function symbols $+$ and $\cdot$, the constant symbols 0 and 1 and the predicate symbol $\leq$ have their usual interpretations on the set of natural numbers. Determine which of the following sentences are satisfied by the structure $\underline{\mathbb{N}}$:

   i) $\forall x \exists y (x = y + y \vee x = (y + y) + 1)$,
   ii) $\forall x \forall y \exists z (x + z = y)$
   iii) $\forall x \forall y (x \leq y \leftrightarrow \exists z (x + z = y))$.

   Solution:

   The first sentence is true in $\mathbb{N}$ since it expresses the fact that every natural number is either even or odd.

   The second statement is not true in $\mathbb{N}$ since by setting x to 1 and y to 0 we see that there is no natural number z with $x + z = y$.

   The third statement is true in $\mathbb{N}$ since it expresses the fact that a natural number is less than or equal to another iff their difference is a natural number.

7. Let L be a first order language with equality. For each natural number n, find sentences α_n, β_n and γ_n such that for all normal L-structures $\mathcal{A}$:

   (a) $\mathcal{A} \models \alpha_n$ iff A has exactly n elements,
   (b) $\mathcal{A} \models \beta_n$ iff A has at least n elements,
   (c) $\mathcal{A} \models \gamma_n$ iff A has at most n elements.

   Find a set Σ of sentences such that a normal L-structure $\mathcal{A} \models \Sigma$ iff A is infinite. Note that the set Σ must consist of infinitely many sentences (we will prove this later).

   Solution:

   For $n > 1$ let β_n be the formula:

$$\exists x_1 \exists x_2 \cdots \exists x_n \left(\bigwedge_{i \neq j} x_i \neq x_j \right).$$

Then a normal structure $\mathcal{A}$ satisfies β_n if and only if it has at least n distinct elements. With the β_n's we can define $\alpha_n = (\beta_n \wedge \neg\beta_{n+1})$ and $\gamma_n = \neg\beta_{n+1}$.

Let $\Sigma = \{\beta_n \mid n > 1\}$.

B1 In this problem, we assume that McMaster has a **countably infinite** number of students $S = \{s_0, s_1, \ldots, s_n, \ldots\}$ and that C is the set of courses that are on offer to them. Due to resource limitations, each student in S will be assigned to exactly one class from C. Also, each course $c \in C$ has its enrolment capped at some finite number e_c. Each student $s \in S$ provides a **finite** set $C_s \subseteq C$ of the courses that they are willing to register in.

For $A \subseteq S$, a function $\alpha : A \to C$ is a **good** assignment for A if

- For each $s \in A$, $\alpha(s) \in C_s$ (so α assigns to s one of the courses they selected), and
- for each class $c \in C$, $|\alpha^{-1}(c)| \leq e_c$ (so no class is over-enrolled by α).

Suppose that for each **finite** subset A of S there is some good assignment $\alpha : A \to C$ for A. Prove that there is some good assignment $\alpha : S \to C$ for the entire set S. In your solution you should formulate this situation within propositional logic and then use the Compactness Theorem.

Solution:

For each $s \in S$ and $c \in C$, introduce a new propositional variable $P_{s,c}$. The intended meaning of this variable is that s is assigned to the course c.

Given S, C, C_s, and e_c as above, for $A \subseteq S$, let Γ_A be the following (infinite) set of propositional formulas:

- for each $s \in A$, the formula

$$\bigvee_{c \in C_s} P_{s,c}.$$

  (So if true, each student gets assigned to at least one of their preferred courses.)

- for each $s \in A$ and $c, d \in C$ with $c \neq d$, the formula $\neg(P_{s,c} \wedge P_{s,d})$. (So if true, no student gets assigned to two different courses.)
- for each $c \in C$ and each subset $B \subseteq A$ with $|B| = e_c + 1$, the formula
$$\neg\left(\bigwedge_{b\in B} P_{b,c}\right).$$
(So if true, there will never be more than e_c students enrolled in the course c.)

We claim that for $A \subseteq S$, the set of formulas Γ_A is satisfiable if and only if there is a good assignment $f : A \to C$.

For one direction, suppose that $f : A \to C$ is good. Let ν_f be the following truth assignment:

$$\nu_f(P_{s,c}) = \begin{cases} T & \text{if } f(s) = c \\ F & f(s) \neq c \text{ or } s \notin A \end{cases}.$$

It can be seen that each of the formulas in Γ_A are satisfied by this assignment, since $f(s) \in C_s$ for each $s \in A$, and $|f^{-1}(c)| \leq e_c$ for each $c \in C$.

Conversely, suppose that ν satisfies Γ_A. Define $f_\nu(x) : A \to C$ by: $f_\nu(s) = c$ if and only if $\nu(P_{s,c}) = T$. Since ν satisfies Γ_A, then for each $s \in A$, there is a unique class $c \in C$ such that $\nu(P_{s,c}) = T$, and this c belongs to C_s. Furthermore, for each $c \in C$, $|f_\nu^{-1}(c)| \leq e_c$, for if not, then one of the formulas of the third type above would be not satisfied by ν.

So, to show that there is a good assignment $f : S \to C$ it suffices to show that the set Γ_S is satisfiable. By the Compactness Theorem, it suffices to show that each finite subset Δ of Γ_S is satisfiable. Given such a subset Δ, it follows that there is some finite subset $A \subset S$ such that $\Delta \subseteq \Gamma_A$. Just let A consist of all $s \in S$ such that the variable $P_{s,c}$ occurs in some formula in Δ, for some $c \in C$.

So, it suffices to show that for each finite subset A of S, the set Γ_A is satisfiable. But this is exactly the assumption that has been made, namely, that for each finite subset A of S, there is a good assignment $f : A \to C$ (equivalently, that Γ_A is satisfiable).

B2 Do exercise 4.24 from the textbook.

Solution: We use Theorem 4.1 to solve this problem. Suppose $\mathcal{A}$, $\vec{a}$, and ϕ are given as in the problem. If

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} \phi$$

then

$$\mathcal{A} \models_{[\vec{x}/\vec{a}][x_i/a_i]} \phi$$

so by definition,

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} \exists x_i \phi.$$

Conversely, suppose that

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} \exists x_i \phi.$$

Then by definition, for some $c \in A$,

$$\mathcal{A} \models_{[\vec{x}/\vec{a}][x_i/c]} \phi.$$

By Theorem 4.1, it follows that

$$\mathcal{A} \models_{[\vec{x}/\vec{a}][x_i/a_i]} \phi$$

since x_i doesn't occur freely in ϕ. Thus

$$\mathcal{A} \models_{[\vec{x}/\vec{a}]} \phi.$$