

Fast CDO computations in the affine Markov chain model

T. R. Hurd* and A. Kuznetsov*

hurdt@mcmaster.ca, kuznets@math.mcmaster.ca

905-525-9140

Department of Mathematics and Statistics

McMaster University

Hamilton ON L8S 4K1

Canada

First version: March 1, 2005

Current version: October 23, 2006

*Research supported by the Natural Sciences and Engineering Research Council of Canada and MITACS, Mathematics of Information Technology and Complex Systems Canada

Abstract

[NB REWRITE THIS]

Abstract: It is shown that credit basket derivatives such as CDOs which depend on a large number M of firms ($M \geq 100$ is typical in some contexts) can be modeled in a parsimonious and computationally efficient manner within the affine Markov chain (AMC) framework for multifirm credit migration introduced in a companion paper [Hurd and Kuznetsov (2006)]. The proposed method has a number of merits. First, since our AMC models extend the intensity based doubly stochastic framework for multifirm default to a credit migration setting, they can be flexibly fit to observed market bond data for the individual constituent firms, and they can in principle explain the dynamics of this data. Second, the method handles some of the variations of CDOs such as nonhomogeneous hazard rates and unequal notational amounts that industry practitioners need to use. Thirdly, the approximation schemes we use can be verified in special cases, and prove to perform to basis point accuracy with typical parameter choices. Finally, in our model, prices and sensitivities for such derivatives are reduced to low dimensional integrals which can often be computed on a desktop computer in fractions of seconds. In this paper we develop an illustrative version of the modeling framework and present a number of sample CDO computations which illustrate the power of the method.

Key words: CDO, multifirm credit migration, stochastic intensity, default correlation, credit spread

1 Introduction

Standard industry approaches to pricing and hedging CDOs (collateralized debt obligations), of which the best known versions are those of [Laurent and Gregory (2003), Hull and White (2004), Andersen et al. (2003)], are based on the copula or factor multifirm credit model initiated by [Li (2000)]. In these models advantages of flexibility, computational speed and ease of calibration are offset by the deficiency that hazard rates and default correlations are introduced without regard to the dynamics of the underlying companies (essentially by fitting functional forms such as a one factor normal copula to current market data). This means that these methods provide no means to include dynamics of default correlations and credit spread changes, making them intrinsically unreliable when credit markets become stressed, for example as they were in May 2005. In contrast, models for credit dynamics, such as multifirm structural credit models, should be more reliable, but pricing CDOs seems to be a difficult unsolved problem. The purpose of this paper is to demonstrate that CDOs can in fact be computed very efficiently (in principle faster than in factor models), in a modelling framework for dynamic multifirm credit we have called the affine Markov chain model.

The affine Markov chain (AMC) framework introduced in [Hurd and Kuznetsov (2006)] is an approach to the joint modeling of multifirm credit migration and default which generalizes the single firm credit migration methods initiated in [Jarrow et al. (1997), Lando (1998), Arvanitis et al. (1999)]. The structure of our model was also inspired by the affine credit default framework of [Chen and Filipović (2004)]. The basic modelling premise which underlies the AMC framework is that firms undergo credit rating migrations which are independent conditioned on one or more intensities which can be viewed simply as the speed of a stochastic time change. In the simplest versions of the framework, we assume that the rating classes are “ideal” in that they represent the best estimate of a firm’s credit worthiness at any instant in time and therefore that firms with the same rating have identical default risk. In the specification we adopt here, default dependence is introduced by a single stochastic time change which represents the overall “speed” of the credit market. This stochastic time change and other market factors such as the interest rate, is jointly modelled in terms of a multivariate affine process. To illustrate the range of effects the framework can handle, we include various simple components of the time change which build in distinct default correlation mechanisms: a diffusive component represents the “normal market”, a component in which the speed jumps leads to jumps in hazard rates, while jumps in the time itself lead to the possibility of simultaneous defaults (i.e. a “contagion” effect). The concept of a stochastic market time has been used by numerous authors to explain various models of equity prices, for example, the well known VG model [Madan et al. (1999)] can be thought of as geometric Brownian motion with a gamma–distributed stochastic time change. Stochastic volatility models also have a similar interpretation. The time–change concept in the AMC framework is in the same spirit.

In [Hurd and Kuznetsov (2006)], it was also shown how essential default com-

putations and credit derivative prices for one and two firms could be reduced to explicit formulas or one dimensional integrals. The present paper addresses how in the AMC setting, the prices of CDOs on M firms are expressible as an integral of the so-called tranche function H^U of the conditional default loss distribution, integrated over the condition, in this case the stochastic time τ . The integrand, the conditional tranche function, is in general computationally intensive, but we show it can often be computed efficiently using one of several approximation schemes based on the law of large numbers. To explain the speedup of our algorithm relative to a similar approach used in factor models, one can observe that our use of time itself as a conditioning variable reduces the dimensionality of the conditioning integral by one, with a consequent reduction in the number of evaluations of the conditional tranche function.

In view of their different structure, a direct comparison of the AMC modeling approach to the copula approach is difficult. However, we show how the benchmark CDO computations found in [Hull and White (2004)] are at least qualitatively similar to those achieved in a very simple version of the AMC framework. We also compute CDO prices in a far more realistic model, leading to results which are qualitatively consistent with, but much more extensive than those coming from the copula/factor methods. Even in this more complicated model, it is a simple matter to plot the dependency of the CDO prices on the underlying dynamic variables and on parameters.

The organization of the paper is as follows. Section 2 introduces the main modeling ingredients for multifirm credit migration in the AMC framework, and illustrates the method by computing one-firm transition and default probabilities in terms of an explicit multivariate Laplace transform function. Section 3 describes the basic structure of a CDO tranche as a swap between a premium leg and an insurance leg. The main results of this section are Theorems 3.1 and 3.2 which give formulas for the two legs. Section 4 gives some approximation schemes that enable fast computation of CDO prices. Section 5 compares the benchmark computations for the normal copula model from [Hull and White (2004)] to those of a very simple version of our model. Section 6 presents numerical results for CDO spreads in the more typical AMC setting introduced in [Hurd and Kuznetsov (2006)]. Section 7 discusses sensitivity computations for CDOs.

In summary, the present paper introduces an efficient and flexible approach for evaluating prices and hedge ratios for the large scale credit derivative securities known as CDOs, in a model with dynamic credit migration and default correlation. The results pass visual inspection to be a plausible description of real credit markets, but much more research will be needed to validate the quantitative aspects of the model.

2 The affine Markov chain model

For our present purpose of understanding CDOs in a model of dynamic default correlation and credit spreads, we adopt a particular variation of the AMC multifirm credit model of [Hurd and Kuznetsov (2006)]. Following their notation, we assume Assumptions 1 to 5, restrict attention to the risk-neutral measure Q , and suppress stochastic recovery by fixing the recovery rate to be $R_0 \in [0, 1]$. We shall have two affine market risk factors $\mathbf{Z}_t = (Z_t^{(1)}, Z_t^{(2)})$, plus an additional pure jump process $Z_t^{(3)}$ (i.e. η_t from section 6.4 of the above paper). Thus our model for M firms has the following basic ingredients:

1. A vector of independent $K + 1$ -state Markov chains $\tilde{\mathbf{Y}}_t = (\tilde{Y}_t^1, \tilde{Y}_t^2, \dots, \tilde{Y}_t^M)$ where $\tilde{Y}_t^k \in \{0, 1, \dots, K\}$ and 0 is an absorbing state.
2. The “market time” defined to be a stochastic time change process $\tau_t = \int_0^t \mathbf{m}^{(\tau)} \cdot \mathbf{Z}_s ds + m^{(3)} Z_t^{(3)}$, where $\mathbf{m}^{(\tau)} = (m^{(\tau,1)}, m^{(\tau,2)})$, $m^{(3)}$ is a two-vector of non-negative coefficients.
3. The spot interest rate process $r_t = \mathbf{m}^{(r)} \cdot \mathbf{Z}_t$ where $\mathbf{m}^{(r)} = (m^{(r,1)}, m^{(r,2)})$ are non-negative. The processes $\tilde{\mathbf{Y}}_t, \mathbf{Z}_t, Z_t^{(3)}$ are assumed to be mutually independent under Q .
4. The credit migration process is $\mathbf{Y}_t = \tilde{\mathbf{Y}}_{\tau_t}$.
5. The time of default of the i th firm is the stopping time $t_i^* = \inf\{t | Y_t^i = 0\}$.

The Markov chains $\tilde{\mathbf{Y}}_t$ have identical Markov generators \mathcal{L}_Y (a $K + 1 \times K + 1$ matrix) and thus identical transition probabilities $Q(\tilde{Y}_t = j | \tilde{Y}_0 = k)$ given by the semigroup $e^{t\mathcal{L}_Y}$. The state of $Y_t^i \in \{0, 1, \dots, K\}$ represents the credit rating of firm i at time t , where the absorbing state 0 is the default state. For our present illustrative purposes, we typically take $K = 7$ and map these states to Standard and Poor’s rating classes:

$$\{0, 1, \dots, 7\} \leftrightarrow \{\text{‘default’}, \text{CCC}, \text{B}, \text{BB}, \text{BBB}, \text{A}, \text{AA}, \text{AAA}\}.$$

The Markov chain generator \mathcal{L}_Y has the meaning of the historical transition generator (i.e. the matrix logarithm of the one-year historical transition matrix). If we neglect the effect of $Z^{(3)}$, the risk-neutral transition generator is the time dependent scalar multiple $\mathbf{m}^{(\tau)} \cdot \mathbf{Z}_t \mathcal{L}_Y$, and we can say we have a credit risk premium which is a scalar stochastic process. In more general versions of the AMC framework, the credit risk premium will be a time dependent matrix valued process.

Remark 1. The picture which describes the basic AMC model is that firms of the same rating have identical migration and default probabilities: We say that the M firms are *exchangeable*. Conditioned on τ_t , firms undergo independent credit migration with identical transition probabilities: Eventually every firm defaults. The

stochastic time change τ_t leads to correlations between firm defaults. When the stochastic clock is running fast, migration and hence defaults happen relatively frequently; if the stochastic clock jumps, then simultaneous defaults may occur.

To create an interesting range of possibilities we make the following choices for the market factors:

- $Z^{(1)}$ is a CIR process satisfying

$$dZ_t = a_1(1 - Z_t)dt + \sqrt{2c_1 Z_t}dW_t, \quad (1)$$

and thus with Markov generator

$$\mathcal{L}^{(1)}f(x) = a_1(1 - x)f'(x) + c_1xf''(x), \quad (2)$$

- $Z^{(2)}$ is a mean-reverting affine process with exponentially distributed jumps. It is defined by its stochastic differential equation

$$dZ_t = -b_2Z_tdt + dJ_t \quad (3)$$

where J_t is the pure jump increasing Lévy process with exponential jump distribution

$$\nu^{(2)}(dx) = c_2^2b_2e^{-c_2x}dx \quad (4)$$

with parameters $b_2, c_2 > 0$. The Markov generator of $Z^{(2)}$ is

$$\mathcal{L}^{(2)}f(x) = \int_0^\infty (f(x+y) - f(x))\nu^{(2)}(dy) - b_2xf'(x). \quad (5)$$

- The jump part of the time change $Z_t^{(3)}$ is taken to be a pure jump increasing Lévy process with exponential jump distribution

$$\nu^{(3)}(dx) = c_3^2e^{-c_3x}dx, \quad d > 0 \quad (6)$$

We have chosen parameters that normalize $Z_t^{(1)}, Z_t^{(2)}, Z_t^{(3)}$ to have long term means of 1, 1, t respectively, and thus $m^{(\tau,1)} + m^{(\tau,2)} + m^{(3)}$ is equal to the average speed of the time change.

The main computational building blocks are the functions G^{CIR}, G^{MR} computed in the Appendix of [Hurd and Kuznetsov (2006)] and

$$G^{(3)}(t, w) := E^Q[e^{-wZ_t^{(3)}}] = e^{-t\Psi(w)}$$

where $\Psi(w) = c_3w/(c_3 = w)$. In terms of these functions, we have for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $w \in \mathbb{R}$ the explicit Laplace transform:

$$\begin{aligned} G^Q(t, \mathbf{Z}; \mathbf{u}, \mathbf{v}, w) &:= E_{0, \mathbf{Z}}^Q \left[e^{-\int_0^t \langle \mathbf{u}, \mathbf{Z}_s \rangle ds} e^{-\langle \mathbf{v}, \mathbf{Z}_t \rangle} e^{-wZ_t^{(3)}} \right] \\ &= G^{CIR}(t, z_0^{(1)}; u^{(1)}, v^{(1)})G^{MR}(t, z_0^{(2)}; u^{(2)}, v^{(2)})G^{(3)}(t, w) \end{aligned} \quad (7)$$

2.1 Transition probabilities for the process Y_t

As we showed in [Hurd and Kuznetsov (2006)], this setup is computationally efficient: the affine structure of the stochastic time change works beautifully with the Markov chains. To illustrate, we show how one can compute credit migration probabilities for each firm.

One can prove that the matrix \mathcal{L}_Y is diagonalizable $\mathcal{L}_Y = VDV^{-1}$ and that the diagonal eigenvalue matrix $D = -\text{diag}\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_K\}$ has $\alpha_0 = 0$ and $\alpha_1, \alpha_2, \dots, \alpha_K$ all with positive real parts. The eigenvector matrix $V = (v_{ij})_{i,j=0\dots K}$ is a matrix whose columns are the corresponding eigenvectors of \mathcal{L}_Y . The elements of V^{-1} will be denoted as $V^{-1} = (\tilde{v}_{ij})_{i,j=0\dots K}$.

Solving the Kolmogorov equation shows that the probability semigroup for the process Y_t is given (in matrix form) by $e^{t\mathcal{L}_Y} = Ve^{tD}V^{-1}$ and the transition probabilities for the process \tilde{Y}^i starting in state y at $t = 0$ are given by

$$q_{yj}(t) := Q_{0,y}(\tilde{Y}_t^i = j) = \sum_{k=0}^K v_{yk}\tilde{v}_{kj}e^{-\alpha_k t} \quad (8)$$

Here is the result for the credit migration process Y_t :

Lemma 1. *The credit migration probabilities for the i th firm are given by*

$$Q_{0,\mathbf{z}_0,y}(Y_t^i = j) = \sum_{k=0}^K v_{yk}\tilde{v}_{kj}G^Q(t, \mathbf{Z}_0; \alpha_k \mathbf{m}^{(\tau)}, \mathbf{0}, \alpha_k m^{(3)}). \quad (9)$$

Proof. By the law of iterated expectations and (8)

$$\begin{aligned} Q_{0,\mathbf{z}_0,y}(Y_t^i = j) &= E_{0,\mathbf{z}_0,y}^Q[E^Q[I\{\tilde{Y}_{\tau_t} = j\}|\{\mathbf{Z}_s\}_{s \leq t}]] \\ &= E_{0,\mathbf{z}_0,y}^Q[\exp(\tau_t \mathcal{L}_Y)_{yj}] = E_{0,\mathbf{z}_0,y}^Q\left[\sum_{k=0}^K v_{yk}\tilde{v}_{kj} \exp(-\tau_t \alpha_k)\right] \end{aligned}$$

The result follows since

$$E_{0,\mathbf{z}_0}^Q[e^{-\alpha_k \tau_t}] = E_{0,\mathbf{z}_0}^Q\left[e^{-\alpha_k \int_0^t \mathbf{m}^{(\tau)} \cdot \mathbf{Z}_s ds - \alpha_k m^{(3)} Z_t^{(3)}}\right] = G^Q(t, \mathbf{Z}_0; \alpha_k \mathbf{m}^{(\tau)}, \mathbf{0}, \alpha_k m^{(3)}).$$

□

3 Pricing CDOs

Collateralized debt obligations (CDOs) are basket credit derivatives involving a large number of companies. The underlying security is a portfolio of coupon paying corporate bonds on M firms: the i th bond is taken to have face value N_i (called

the “notional”). A “synthetic CDO tranche” can be regarded as a credit swap between two parties, the insured and the insurer. The two components of the swap, the “premium leg” and the “insurance leg”, are the basic credit contingent claims. We now show how the price at time $t = 0$ of each leg can be priced separately by risk neutral expectation. These prices will depend on the initial credit ratings $y_0 = (y_0^1, y_0^2, \dots, y_0^M)$, and the initial values Z_0 .

The components of a CDO tranche are derivatives on the total loss of the bond portfolio due to default of the constituent names. Under the assumption of a constant recovery rate R_0 , the loss at time t as a fraction of the total notional is given by

$$L_t = \sum_{i=1}^M (1 - R_0) \frac{N_i}{N} I\{t_i^* \leq t\} \quad (10)$$

where

- M is the number of firms;
- N_i is the notional of firm i ;
- $N = \sum_{i=1}^M N_i$ is the total notional of the portfolio;
- t_i^* is the default time of the firm i .

The total loss process can be rewritten as

$$L_t = \tilde{L}_{\tau_t}, \quad (11)$$

where

$$\tilde{L}_t = \sum_{i=1}^M (1 - R_0) \frac{N_i}{N} I\{\tilde{Y}_t^i = 0\}.$$

The main results on CDO prices will be expressed in terms of the integrated cumulative distribution function of \tilde{L}_t

$$\tilde{C}(x, t) = E[(x - \tilde{L}_t)^+] = \int_0^x \int_0^z p_t(dy) dz.$$

3.1 Credit premium

A generalized premium leg for a CDO can be regarded as a contingent claim which is paid by the insured to the insurer as a fee for default insurance. It is assumed that the insured party pays continuously in time over the period $[0, T]$ at a stochastic rate U_t , and that U_t depends only on the loss process L_t , i.e. $U_t = U(L_t)$ for some nonincreasing function $U(x)$. The usual choice defines the premium leg for a CDO tranche for fractional losses in a range $[\underline{x}, \bar{x}] \subset [0, 1]$:

$$U(x) = \frac{1}{\bar{x} - \underline{x}} [(\bar{x} - x)^+ - (\underline{x} - x)^+] \quad (12)$$

Special cases are the senior tranches with $\bar{x} = 1, \underline{x} > 0$ and equity (or ‘base’) tranches with $\underline{x} = 0, \bar{x} < 1$. In sections 5 and 6 we take the following six standard tranches:

$$[0, 0.3], [0.03, 0.07], [0.07, 0.10], [0.10, 0.15], [0.15, 0.30], [0.30, 1.0].$$

The assumption of continuous payments in time is for simplicity of exposition only. A more pragmatic choice such as quarterly payments with an accrual term can be easily accommodated.

The price of the premium leg is given by

$$V^U = E_{0, \mathbf{Z}_0, \mathbf{y}_0}^Q \left[\int_0^T e^{-\int_0^t r_s ds} U(L_t) dt \right], \quad (13)$$

and our first main result is the following formula:

Theorem 2. *If U is a bounded measurable function then*

$$V^U = \int_0^\infty H^U(\tau; \mathbf{y}_0) F^P(\tau; \mathbf{Z}_0) d\tau, \quad (14)$$

where the function $H^U(\tau; \mathbf{y}_0)$ is given by

$$H^U(\tau; \mathbf{y}_0) = E_{\mathbf{y}_0}^Q[U(\tilde{L}_\tau)] = \frac{1}{\bar{x} - \underline{x}} [\tilde{C}(\bar{x}, \tau) - \tilde{C}(\underline{x}, \tau)] \quad (15)$$

and thus depends only on the parameters of the loss process \tilde{L}_t (such as \mathbf{y}_0) and the payoff function U . The function $F^P(\tau; \mathbf{Z}_0)$ is given by

$$\begin{aligned} F^P(\tau; \mathbf{Z}_0) &= \int_0^T E_{0, \mathbf{Z}_0}^Q \left[e^{-\int_0^t r_s ds} \delta(\tau_t - \tau) \right] dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i w \tau} \left[\int_0^T G^Q(t, \mathbf{Z}_0; \mathbf{m}^{(\tau)} + i w \mathbf{m}^{(\tau)}, \mathbf{0}, i w \mathbf{m}^{(3)}) dt \right] dw, \end{aligned} \quad (16)$$

and thus depends only on the parameters of the interest rate and time change processes.

Proof. Again by iterated expectations,

$$\begin{aligned} V^U &= \int_0^T E_{0, \mathbf{Z}_0, \mathbf{y}_0}^Q \left[E_{0, \mathbf{y}_0}^Q [U(\tilde{L}_{\tau_t} | \{\mathbf{Z}_s\}_{s \leq t}) e^{-\int_0^t r_s ds}] \right] dt \\ &= \int_0^T E_{0, \mathbf{Z}_0} \left[H^U(\tau_t; \mathbf{y}_0) e^{-\int_0^t r_s ds} \right] dt \end{aligned} \quad (17)$$

where $H^U(\tau; \mathbf{y}_0)$ given by (15) is bounded and measurable. It is now sufficient to prove (14) for the (dense) set of complex exponential functions of the form $H^U(\tau) = e^{-iw\tau}$, $w \in \mathbb{R}$ with the result extending by linearity to general bounded measurable H supported on \mathbb{R}_+ . By (7) and the Fourier inversion theorem, we have

$$\begin{aligned} \int_0^T E_{0, \mathbf{z}_0} \left[e^{-iw\tau t} e^{-\int_0^t r_s ds} \right] dt &= \int_0^T G^Q(t, \mathbf{Z}_0; \mathbf{m}^{(r)} + iw\mathbf{m}^{(\tau)}, \mathbf{0}, iw\mathbf{m}^{(3)}) dt \\ &= \int_{-\infty}^{\infty} e^{-iw\tau} F^P(\tau; \mathbf{Z}_0) d\tau \end{aligned}$$

which completes the proof. \square

Remark 2. 1. In practise, the bulk of computational intensity will come in calculating what we shall call the *conditional tranche function* H^U . Here this is a function of the stochastic time variable, but in general will be a function of several conditioning variables. In comparison, a similar situation holds in factor models, but the time variable will always count as one extra integration dimension, leading to an intrinsically slower algorithm.

2. To compute the function $F^P(\tau; \mathbf{z})$ given by equation (16) it is sufficient to evaluate the integral in t numerically and then perform the inverse Fourier transform in w .
3. An important advantage of our formula is that it separates the effects of the stochastic time change (hidden in $F^P(\tau; \mathbf{z})$) from all information about the Markov chains $\tilde{\mathbf{Y}}$, the loss process \tilde{L}_t and the payoff function U (hidden in $H^U(\tau; \mathbf{y}_0)$).
4. Thus, in particular, computing the function F^P for n choices of parameters and H^U for m parameter choices leads immediately to $n \times m$ CDO prices.

3.2 Credit insurance

The insurer pays the insured a tranche of the losses by default of firms in the basket portfolio. A general claim of this type is defined by

$$W^S = E_{0, \mathbf{z}_0} \left[\int_0^T e^{-\int_0^t r_s ds} dS_t \right] \quad (18)$$

where $S_t = S(L_t)$ for some nondecreasing deterministic function $S(L)$ with $S(0) = 0$. The usual choice defines the default leg of a CDO tranche with range $[\underline{x}, \bar{x}]$:

$$S(x) = \frac{1}{\bar{x} - \underline{x}} [(x - \underline{x})^+ - (x - \bar{x})^+] = 1 - U(x) \quad (19)$$

We can simplify the expression for W^S by integrating by parts ($S(t)$ is of finite variation)

$$W^S = E_{0, \mathbf{Z}_0, \mathbf{y}_0}^Q \left[e^{-\int_0^T r_s ds} S_T + \int_0^T r_t e^{-\int_0^t r_s ds} S_t dt \right]. \quad (20)$$

The analogue of Theorem 2 for the price of the insurance leg is proved in exactly the same way:

Theorem 3. *If S is bounded and measurable with $S(0) = 0$ then*

$$W^S = \int_0^\infty H^S(\tau; \mathbf{y}_0) F^I(\tau, \mathbf{Z}_0) d\tau, \quad (21)$$

where $H^S(\tau; \mathbf{y}_0)$ is given by $H^S(\tau; \mathbf{y}_0) = E_{0, \mathbf{y}_0}^Q[S(\tilde{L}_\tau)] = 1 - H^U(\tau; \mathbf{y}_0)$. The function $F^I(\tau; \mathbf{Z}_0)$ depends only on the parameters of the interest rate and time change processes and is given by

$$F^I(\tau; \mathbf{Z}_0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iw\tau} \left\{ G^Q(T, \mathbf{Z}_0; \mathbf{m}^{(r)} + iw\mathbf{m}^{(\tau)}, \mathbf{0}, iw\mathbf{m}^{(3)}) \right. \\ \left. + \int_0^T (\mathbf{m}^{(r)} \cdot \mathbf{D}_v) G^Q(t, \mathbf{Z}_0; \mathbf{m}^{(r)} + iw\mathbf{m}^{(\tau)}, \mathbf{0}, iw\mathbf{m}^{(3)}) dt \right\} dw. \quad (22)$$

Here $\mathbf{D}_v G^Q$ denotes the derivative of G^Q in the \mathbf{v} variable.

Figure 2 shows a typical shape of the function F^I and six tranches of the function H^S when $M = 100$. The value of the insurance leg is obtained by integrating F^I against the appropriate H^S .

3.3 CDO prices and tranche spreads

A simple CDO tranche $[\underline{x}, \bar{x}] \subset [0, 1]$ is a swap (a contract with zero value at time 0) of a multiple p (called the CDO spread, and measured in basis points) of the premium leg V^U with U given by (12) for a default leg W^S with S given by (19). Thus $p = W^S/V^U$ is selected to balance the two legs. At future times the CDO price $W^S - pV^U$ evolves stochastically.

4 Approximating the conditional tranche function

In computing the expectation defining the conditional tranche function $H^U = 1 - H^S$, the process \tilde{L}_t is the sum of M independent Bernoulli random variables which is itself independent of the affine processes Z . Of course for large M , the distribution functions of \tilde{L}_t are in general hard to compute explicitly, so in this section we propose some methods for high speed approximations.

4.1 A normal approximation

Since all the random variables $I\{\tilde{Y}_t^i = 1\}$ are independent, one approach is to use the central limit theorem to approximate the distribution of \tilde{L}_t as

$$\tilde{L}_t \stackrel{d}{\approx} L(t, \xi) = m(t) + \xi\sigma(t), \quad (23)$$

where ξ is Gaussian $N(0, 1)$. The mean $m(t)$ and variance $\sigma^2(t)$ of \tilde{L}_t can easily be computed

$$\begin{aligned} m(t) &= \sum_{k=1}^K \alpha_k q_{k0}(t), & \alpha_k &= \sum_{i=1}^M (1 - R_0) I\{\tilde{Y}_0^i = k\} \frac{N_i}{N} \\ \sigma^2(t) &= \sum_{k=1}^K \beta_k q_{k0}(t)(1 - q_{k0}(t)), & \beta_k &= \sum_{i=1}^M (1 - R_0)^2 I\{\tilde{Y}_0^i = k\} \frac{N_i^2}{N^2} \end{aligned}$$

Provided the sequence of notionals N_i is bounded above and below uniformly in i , we note that $m(t) = O(1)$, $\sigma(t) = O(M^{-\frac{1}{2}})$ and the approximation is accurate to $O(M^{-1})$.

The distribution of real time loss process thus is approximated by

$$L_t = \tilde{L}_{\tau_t} \stackrel{d}{\approx} L(\tau_t, \xi) = m(\tau_t) + \xi\sigma(\tau_t). \quad (24)$$

Then the explicit formula for H^U , H^S in the normal approximation is

$$H^U(\tau) = 1 - H^S(\tau) = \frac{\sigma(\tau)}{\bar{x} - \underline{x}} \left[\tilde{\Phi} \left(\frac{\bar{x} - m(\tau)}{\sigma(\tau)} \right) - \tilde{\Phi} \left(\frac{\underline{x} - m(\tau)}{\sigma(\tau)} \right) \right] \quad (25)$$

where

$$\tilde{\Phi}(x) = \int_{-\infty}^x \Phi(y) dy = x\Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad (26)$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ is the cumulative distribution function of $N(0, 1)$.

4.2 A no default correction

The above continuous approximation to the conditional loss distribution of \tilde{L}_τ makes one systematic error which becomes most important for small times and small default probability, namely they neglect the point mass corresponding to no default. To make a simple correction, we write $\tilde{L}_\tau = \tilde{L}_{\tau, > 0} X(\tau)$ where

$$\tilde{L}_{\tau, > 0} := \tilde{L}_\tau |_{\tilde{L}(\tau) > 0}, \quad X(\tau) = I(\tilde{L}_\tau > 0). \quad (27)$$

Then, rather than approximating \tilde{L} , we instead approximate $\tilde{L}_{\tau, > 0}$ by a normal distribution with matched moments. Moments of $\tilde{L}_{\tau, > 0}$ are easily computed using the relation between moment generating functions

$$\Phi_{\tilde{L}}(u) = q + (1 - q)\Phi_{\tilde{L}_{> 0}}(u)$$

where $q = q(\tau) = E^Q[I(\tilde{L}_\tau = 0)]$.

4.3 The equal notional case

The special case when all obligors have equal notional amounts $N_i = N/M$ can be computed exactly relatively efficiently by the recursive algorithm (see eg [Hull and White (2004)]) and provides us with a range of cases to test these approximation schemes. Because the loss given default of any single firm is $\Delta L = (1 - R_0)N/M$, the loss distribution $Q_{i,k} = E^Q[I(\tilde{L}_\tau^{\leq i} = k\Delta L)]$ is explicitly determined by the following recursion:

$$Q_{i,k} = Q_{i-1,k}q_i + Q_{i-1,k-1}(1 - q_i)$$

where $q_i = E^Q[\tilde{Y}_\tau^i = 0]$ and $L_\tau^{\leq i} = \sum_{j=1}^i I(\tilde{Y}_\tau^j = 0)\Delta L$.

5 Comparison with the normal copula model

To compare our model with the one factor normal copula model (see [Hull and White (2004)]) we use the following modeling assumptions: the Markov chains \tilde{Y}_t^i are just two-state ($K = 1$) processes with Markov generator \mathcal{L}_Y given by

$$\mathcal{L}_Y = \begin{pmatrix} 0 & 0 \\ \lambda & -\lambda \end{pmatrix},$$

where the default intensity has the value $\lambda = 0.01$. Other parameters are: a constant recovery rate $R_0 = 0.4$; a constant interest rate $r_t = 0.05$.

[ALEXEY: PLEASE FIX THIS UP] Take: $\mathbf{m}^{(\tau)} = (1/3, 1/3)$; $M^{(3)} = 1/3$, $(Z_0^{(1)}, a, c) = (1, 1, 1)$, ????. The important difference between the two models is the correlation structure. While the copula model has a unique parameter ρ which explains all correlations between default events, in our model we have several parameters which are responsible for the default correlations (the most important ones are h_2, h_3, λ_2). In order to compare the correlation structures of the two models we look at the term structure of default correlations given by the following function

$$\text{corr}_{ij}(t) = \text{corr}(I\{t_i^* < t\}, I\{t_j^* < t\}) \quad (28)$$

where ‘‘corr’’ is the usual correlation between random variables. To match the correlation term structures we simply choose parameters of the time change process which give a reasonable fit. For a detailed discussion of the correlation issues see [Hurd and Kuznetsov (2006)].

We observe that although these two models are certainly different, they produce qualitatively similar results across all tranches.

6 CDOs in the credit migration model

In this section we implement two parameter specifications, Model A and Model B, of the AMC framework with $K = 7$ rating classes (these can be interpreted as

tranche	$\rho = .1, \text{ HW}$	$\rho = .1, \text{ AMC}$	$\rho = .3, \text{ HW}$	$\rho = .3, \text{ AMC}$
[0, 0.03]	1740.6	1731.7	1206.6	1112.4
[0.03, 0.07]	251.1	268.7	323.5	344.2
[0.07, 0.10]	36.9	26.9	131.5	187.7
[0.10, 0.15]	5.33	1.79	57.3	64.6
[0.15, 0.30]	0.13	0.03	10.6	4.3
[0.30, 1.00]	1.5e-6	3.0e-3	10.0e-3	4.0e-3

Table 1: WRITE CAPTION

the Moody's or Standard and Poor's system) and price the six standard five year CDO tranches written on $M = 20, 100$ and 400 firms, divided equally into the four investment grade ratings classes (BBB, A, AA, AAA). We consider only CDOs with equal notionals so that the accuracy of our approximation methods can be compared to the exact answers. When notionals are not equal, but the nonhomogeneity is not too wild, we expect the approximate methods to perform with similar accuracy.

Model A is identical to Model A of [Hurd and Kuznetsov (2006)], and has an average credit risk premium of 2 with zero $Z^{(3)}$ component, and will be interpreted as a model with low (but not zero) default correlation. Model B shares the same historical Markov chain generator, has the same average credit risk premium but with a large $Z^{(3)}$ component: it will be interpreted as a model with high default correlation. For both models, we take the historical transition generator

$$\mathcal{L}_Y = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 & 0.0000 \\ 0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 & 0.0000 \\ 0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 & 0.0005 \\ 0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 & 0.0007 \\ 0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 & 0.0010 \\ 0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 & 0.0091 \\ 0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 & -0.1153 \end{pmatrix}$$

which is simply an approximate logarithm of the historical one-year credit migration generator published in [Jarrow et al. (1997)].

[ALEXEY: I THINK THESE ARE MOREORLESS THE RIGHT PARAMETERS] Both models A and B are built from Z processes with the following parameters: $Z_t^{(1)}$ has parameters $a_1 = 0.3790, c_1 = 0.3486$ and initial value $Z_0^{(1)} = 1$; $Z_t^{(2)}$ has parameters $a_2 = 1, c_2 = 1/3$ and initial value $Z_0^{(2)} = 1$; $Z_t^{(3)}$ has parameter $c_3 = 1/10$ and initial value $Z_0^{(3)} = 0$; The "low-correlation model" (Model A) is defined to have the coefficients

$$\mathbf{m}^{(r)} = (0.0365, 0), \quad \mathbf{m}^{(\tau)} = (1.0, 1.0), \quad m^{(3)} = 0.$$

The "high-correlation model" (Model B) is defined to have the coefficients

$$\mathbf{m}^{(r)} = (0.0365, 0), \quad \mathbf{m}^{(\tau)} = (0.5, 1.0), \quad m^{(3)} = 0.5.$$

These two models generate similar, but not identical marginal default time distributions. In both models we take the fractional recovery to have the fixed value $R_0 = 0.4$.

These two models for individual firms lead to the following table of CDO tranche spreads (in basis points), computed in the normal approximation and exactly:

M	Tranche	Model A Lo Corr Normal	Model A Lo Corr Exact	Model B Hi Corr Normal	Model B Hi Corr Exact
20	[0,0.03]	1703.5	1926.2	1268.5	1384.8
	[0.03,0.07]	819.5	677.1	652.8	578.0
	[0.07,0.10]	288.2	287.7	380.2	384.7
	[0.10,0.15]	87.0	100.3	259.5	263.8
	[0.15,0.30]	6.31	7.62	80.1	79.7
	[0.30,1.00]	0.0	0.0	1.4	1.4
100	[0,0.03]	2487.7	2483.5	1633.3	1625.9
	[0.03,0.07]	666.0	661.9	533.7	534.3
	[0.07,0.10]	180.0	181.1	386.1	386.7
	[0.10,0.15]	38.4	39.1	276.1	276.2
	[0.15,0.30]	1.28	1.31	61.7	61.7
	[0.30,1.00]	0.0	0.0	0.8	0.8
400	[0,0.03]	2642.5	2641.6	1683.4	1682.0
	[0.03,0.07]	645.8	645.5	517.0	517.0
	[0.07,0.10]	155.3	155.4	388.9	388.9
	[0.10,0.15]	28.4	28.4	287.5	287.5
	[0.15,0.30]	0.68	0.69	56.4	56.4
	[0.30,1.00]	0.0	0.0	0.6	0.6

Table 2: CDO Spreads in bps for different tranches and $M = 20, 100, 400$, computed in two models, both exactly and in the normal approximation.

The computations on this table took 8.4 seconds when implemented in MATLAB on a 2.4GHz laptop.

Figure 3 shows credit default spread curves for each rating class as computed in [Hurd and Kuznetsov (2006)].

Figures 4,5,6,7 graph the dependencies of CDO tranches in the $\rho = 0.1$ model on the underlying parameters Z_0^1, Z_0^2, h_2, h_3 with remaining parameters fixed at their default values.

Figure 8 shows a simulation of the values of the six tranches in the $\rho = 0.1$ model over the entire duration $[0, 5]$ of the contract. In the particular sample path shown, the following sequence of credit migrations and defaults occurred:

7 Sensitivity

One good thing about an analytical or semi-analytical treatment of securities such as ours compared to Monte Carlo based methods is that sensitivity analysis is both conceptually and computationally straightforward. In our model, security prices are sensitive to the underlying dynamic risk factors \tilde{Y}, Z^1, Z^2, Z^3 : these evolve in time, so they are the most important factors for risk management. Next in importance are the model parameters which are taken to be constant in time, but are subject to calibration error: these are $(\mathcal{L}_Y, a, c, \lambda_2, h_2, h_3, \mathbf{m}^{(\tau)}, \mathbf{m}^{(\tau)}, m_\tau)$. In the present discussion we focus on hedging the sensitivities to the dynamic factors, and leave parameter hedging for future study.

Since Z^1, Z^2 control the overall shape of credit spread and correlation curves, hedging these factors may be thought of as hedging general market risk. The most important hedge is thus to create delta-neutral combinations with respect to these risk factors. Fortunately, the requisite derivatives of both the premium and insurance legs are explicitly computable:

$$(\Delta_{V,1}, \Delta_{V,2}) = \partial_{\mathbf{z}} V^U = \int_0^\infty H^U(\tau) \partial_{\mathbf{z}} F^P(\tau, \mathbf{z}) d\tau, \quad (29)$$

$$(\Delta_{W,1}, \Delta_{W,2}) = \partial_{\mathbf{z}} W^S = \int_0^\infty H^S(\tau) \partial_{\mathbf{z}} F^I(\tau, \mathbf{z}) d\tau, \quad (30)$$

where $\partial_{\mathbf{z}} F^P, \partial_{\mathbf{z}} F^I$ are explicit in terms of the building blocks defined so far. Thus hedging general market risk is a tractable problem in our model. Figures 9,10,11,12 show graphs of V^U, W^S against Z_0^1, Z_0^2 for the contract A tranches in the $\rho = 0.1$ model with all remaining parameters chosen as in section 5.

Next we consider hedging the risk factors Y . This amounts to protecting against the risk of any individual downgrade, upgrade or default, and such firm specific risks can only be delta hedged by holding additional credit securities on each name of the basket. Similarly hedging for jumps of Z^3 involves firm specific risks. For large scale baskets this type of hedging is of secondary importance.

It is interesting to observe from Figures 5-8 that the third tranche in contract A is highly sensitive to all parameters. This can be understood graphically because the degree of overlap of the functions F^I, F^P with H^S, H^U is intermediate, and varies rapidly with the parameters.

8 Conclusion

The AMC framework gives dynamical models of multifirm credit migration and default which fall in the class of reduced form or doubly stochastic models, yet it builds in some of the features of the structural models. The particular way of combining a continuous time Markov chain with an independent set of affine processes yields a flexible framework within which computations are very efficient.

In this paper and its companion [Hurd and Kuznetsov (2006)] we have demonstrated methods for computing credit spreads, correlation curves and CDO tranches, all of which pass visual inspection to be a plausible representation of real markets. In contrast to typical static copula models for large scale basket derivatives, our approach attempts to capture a resemblance to real market dynamics. This suggests using the AMC framework as the basis for scenario generation and stress testing of other pricing methodologies.

The increased realism of our framework does not lead to slower computation times. In fact, in our specification, the computation times are not particularly longer than those achievable in a simple one factor normal copula CDO model. In common with such models, the loss process is a sum over firms of conditionally independent random variables. In contrast, however, the conditioning variable in our modeling is a stochastic process τ_t rather than a single random variable. Nonetheless, the computation speeds are similar because the affine structure of the conditioning process τ_t reduces critical computations to one-dimensional integrals.

The version we have presented appears to be an example of a rather general dynamic approach to credit risk, one which can be extended in several distinct directions. We mention here two important types of improvements which are easy to add. The first is to replace the conditioning process τ_t by m -dimensional processes which might include for example the stochastic recovery or a variety of time changes for different sectors of the economy. It appears that our CDO pricing framework extends easily to this setting and would yield formulas for CDOs involving m -dimensional integrals and Fourier transforms. Another type of extension would be to include idiosyncratic factors which are specific to individual firms: as long as these factors are deterministic this would avoid a curse of dimensionality of integrals in M dimensions.

We have shown that in the normal approximation, CDO prices have M dependence consistent with an $O(1/M)$ error. Another direction for improvement of the method would be to improve the large M asymptotics of the errors $|V^U - V^{U*}|, |W^S - W^{S*}|$ by including further correction terms. [?] derives a very general asymptotic expansion which can perform this task.

Our aim here has been to demonstrate that the AMC framework is flexible enough in principle to fit defaultable bond data and credit derivatives such as CDOs. A detailed study of the validity of the approach for modeling real data sets is clearly justified.

References

- [Andersen et al. (2003)] L. Andersen, J. Sidenius, and S. Basu. All your hedges in one basket. *Risk*, 16:67–72, 2003.
- [Arvanitis et al. (1999)] A. Arvanitis, J. Gregory, and J.-P. Laurent. Building models for credit spreads. *Journal of Derivatives*, 6:27–43, 1999.
- [Chen and Filipović (2004)] L. Chen and D. Filipović. Credit derivatives in an affine framework. working paper, 2004.
- [Hull and White (2004)] J. Hull and A. White. Valuation of a CDO and an n th to default CDS without Monte Carlo simulation. *Jour. of Derivatives*, pages 8–23, 2004.
- [Hurd and Kuznetsov (2006)] T. Hurd and A. Kuznetsov. Affine Markov chain models of multifirm credit migration. Preprint available at http://www.defaultrisk.com/pp_other_32.htm, 2006.
- [Jarrow et al. (1997)] R. Jarrow, D. Lando, and S. Turnbull. A Markov model for the term structure of credit risk spreads. *Review of Financial Studies*, pages 481–523, 1997.
- [Lando (1998)] D. Lando. On Cox processes and risky bonds. *Review of Derivatives Research*, 2:99–120, 1998.
- [Laurent and Gregory (2003)] J.-P. Laurent and J. Gregory. Basket default swaps, CDOs and factor copulas. *Jour. of Risk*, page 21, 2003.
- [Li (2000)] D. X. Li. On default correlation: A copula function approach. *Jour. of Fixed Income*, 6:43–54, 2000. .
- [Madan et al. (1999)] D. Madan, P. Carr, and E. Chang. The variance gamma model and option pricing. *European Finance Review*, 3(1), 1999.

Figure 1: Simulation of the time change factors. The first graph shows typical sample paths of the diffusion process Z^1 , the jump process Z^2 and the jump process Z^3 as well as the integrated time change generated by Z^1 alone. The second graph shows the total integrated time change resulting from the three sample paths combined together.

!9cmtimechange1 !9cmtimechange2

Figure 2: Simulation of the time change factors. The first graph shows typical sample paths of the diffusion process Z^1 , the jump process Z^2 and the jump process Z^3 as well as the integrated time change generated by Z^1 alone. The second graph shows the total integrated time change resulting from the three sample paths combined together.

!9cmH_and_F

Figure 3: Computing the insurance leg: The heavy curve shows a typical graph of the function $F^I(\tau)$; the light curves show the $H^S(\tau)$ functions for the junior (leftmost) to senior (rightmost) tranches. Integrating $H^S(\tau)$ against $F^I(\tau)$ yields the expected value of the insurance leg.

!9cmspread

Figure 4: Hazard rates: These show typical hazard rate curves over a 10 year period for firms of all rating classes. Parameter values are the default values from section 6.

!8cmCDO_z10

Figure 5: Dependence of CDO spreads on the initial value Z_0^1 . Parameter values are the default values from section 6

!8cmCDO_z20

Figure 6: Dependence of CDO spreads on the initial value Z_0^2 . Parameter values are the default values from section 6

!8cmcdoprice_{h2}

Figure 7: Dependence of CDO spreads on the jumpsize h_2 . Parameter values are the default values from section 6

!8cmcdoprice_{h3}

Figure 8: Dependence of CDO spreads on the jumpsize h_3 . Parameter values are the default values from section 6

!8cmpremiumleg_{z10}

Figure 9: Dependence of premium leg on the initial value Z_0^1 . Parameter values are the default values from section 6

!8cmpremiumleg_{z20}

Figure 10: Dependence of premium leg on the initial value Z_0^2 . Parameter values are the default values from section 6

!8cminsuranceleg_{z10}

Figure 11: Dependence of insurance leg on the initial value Z_0^1 . Parameter values are the default values from section 6

!8cminsuranceleg_{z20}

Figure 12: Dependence of insurance leg on the initial value Z_0^2 . Parameter values are the default values from section 6