Conformal Maps

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February 4, 2014

Outline

The Unit Disc and the Half Plane

The unit disc and the upper half–plane are defined as:

$$\begin{aligned} \mathbb{D} &= \{z \in \mathbb{C} \colon |z| < 1\} \quad \text{and} \\ \mathbb{H} &= \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}\,. \end{aligned}$$

Example

Consider F and G defined as:

$$F(z) = \frac{i - z}{i + z} \qquad G(w) = i\,\frac{1 - w}{1 + w}.$$

The map $F : \mathbb{H} \to \mathbb{D}$ is conformal, with inverse $F^{-1} = G : \mathbb{D} \to \mathbb{H}$.

Unit Disc and the Half Plane

Notice that $F(0) = 1$ and

$$|F(z)|^2 = \frac{|i-z|^2}{|i+z|^2} = \frac{1+|z|^2 - 2\mathrm{Im}(z)}{1+|z|^2+2\mathrm{Im}(z)} < 1 \quad \text{for} \quad \mathrm{Im}(z) > 0$$

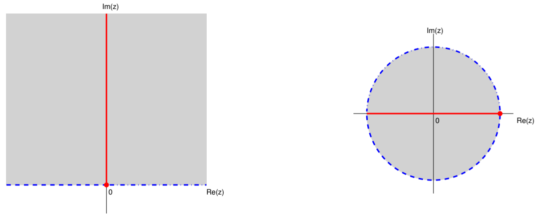
$$\mathrm{Im}\,(G(w)) = \mathrm{Re}\left(\frac{1-w}{1+w}\right) = \frac{1-|w|^2}{1+|w|^2} > 0 \quad \text{for} \quad |w|^2 < 1.$$

$$F(x+iy) = \frac{1-x^2-y^2}{x^2+(1+y)^2} + i\frac{2x}{x^2+(1+y)^2} \Rightarrow$$

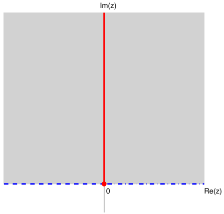
$$F(iy) = \frac{1-y^2}{(1+y)^2} \Rightarrow \quad \text{y axis} \quad \mapsto \quad (-1,1) \quad \text{and}$$

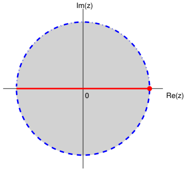
$$|F(x)| = 1 \Rightarrow \quad \text{x axis} \quad \mapsto \quad \text{boundary of unit circle.}$$

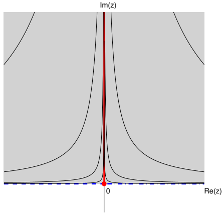
Unit Disc and Half Plane

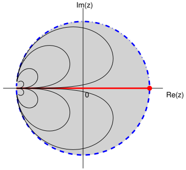


Unit Disc and Half Plane



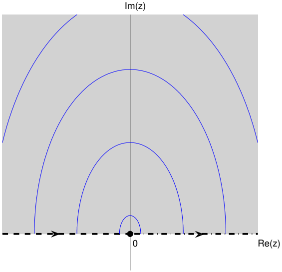


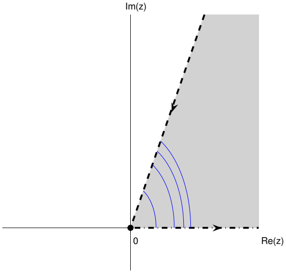




Example 1

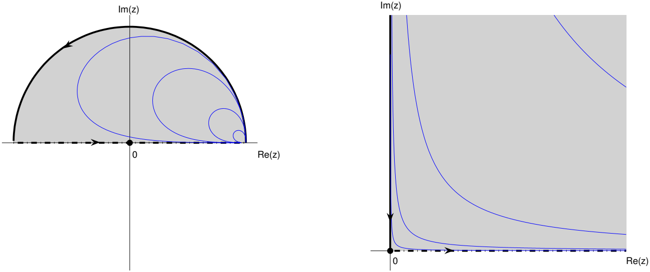
For $\alpha \in (0,2)$, the map $f(z) = z^{\alpha}$ takes the upper half–plane $\mathbb{H}$ into the sector $S = \{w \in \mathbb{C} \colon 0 < \arg(w) < \alpha\pi\}$.





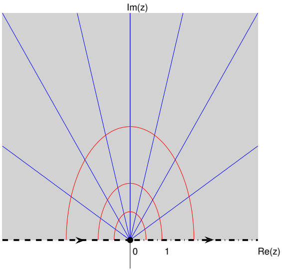
Example 2

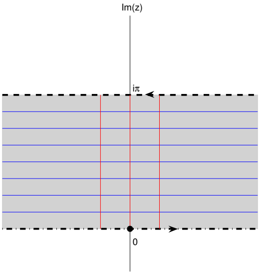
The map $f(z) = (1+z)/(1-z)$ takes $\mathbb{D}_+ = \{z = x + iy : |z| < 1, y > 0\}$ to $\mathrm{QI} = \{w = u + iv : u > 0, v > 0\}$.



Example 3

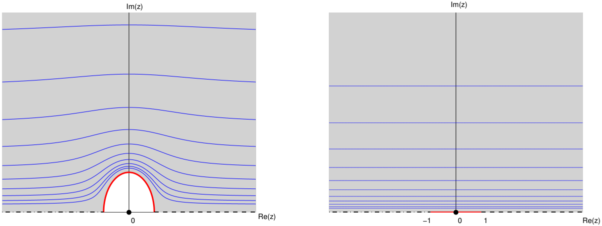
The map $f(z) = \log_\delta z$, $\delta = \pi/2$, takes the upper half–plane $\mathbb{H}$ to the strip $\{w = u + iv : \ u \in \mathbb{R}, v \in (0, \pi)\}$.





Example 4

The function $f(z) = 1/2(z + 1/z)$ maps the exterior of $\mathbb{D}$ to $\mathbb{C} \setminus [-1, 1]$.



Flow Past an Airfoil

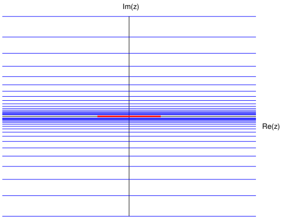
Velocity field $\mathbf{u} = (u_x, u_y)^T$ is tangent to the level curves of the stream function ψ. For incompressible ($\nabla \cdot \mathbf{u} = 0$), irrotational ($\nabla \times \mathbf{u} = 0$), ideal (no viscosity), steady (no time-dependance) flows, ψ is the imaginary part of the complex velocity potential, and it satisfies $\Delta\psi = 0$. The velocity is recovered from

$$\mathbf{u} = (\partial_y \psi, -\partial_x \psi)^T$$

Example

The stream function $\psi(x, y) = y$ satisfies Laplace's equation in $\mathbb{C}$. The level curves are horizontal lines and $u_x = \partial_y \psi = 1$.

Flow Past an Airfoil



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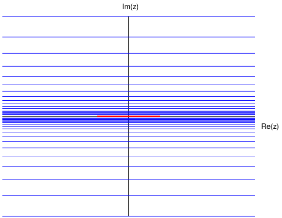

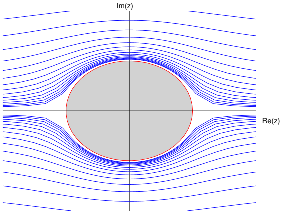

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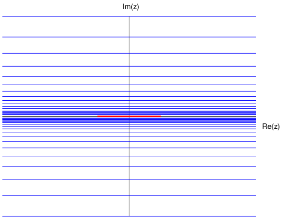

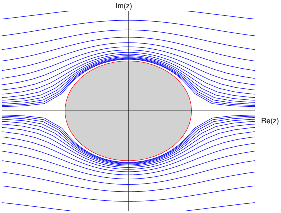

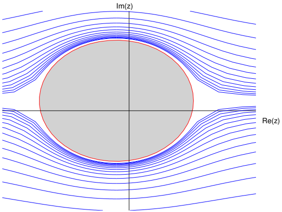

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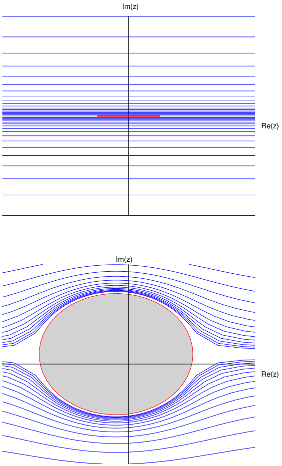


Dirichlet Problem on a Strip

Proposition

Let $F : V \to U$ a holomorphic function. If $u : U \to \mathbb{R}$ is a harmonic function, then $u \circ F$ is also a harmonic function on V.

Consider, for $\Omega = \{z = x + iy \colon \ x \in \mathbb{R}, 0 < y < 1\}$, the Dirichlet problem:

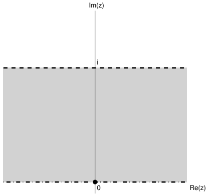
$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

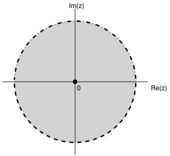
The map $G : \Omega \to \mathbb{D}$ is conformal with inverse $G^{-1} = F : \mathbb{D} \to \Omega$.

$$G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}} \qquad F(w) = \frac{1}{\pi} \log\left(i \frac{1 - w}{1 + w} \right).$$

Dirichlet Problem on a Strip

Let $f(x+i0) = e^{-\alpha x^2}$ and $f(x+i1) = e^{-\alpha x^2}$.





Dirichlet Problem on a Strip

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