Math 3D03
Short answers to assignment #4

1. *Problem 30.26 on page 1216 in the textbook*

Suppose you have x dollars in your hand. If you play one more time your expected wealth after that is $0.2 \times 0 + 0.5 \times (x+1) + 0.3 \times (x+2) = 1.1 + 0.8\,x$ The optimal strategy therefore is to stop playing when $x \geq 1.1 + 0.8\,x$, i.e. when you have more than 5.5 dollars in your hand and continue otherwise. Let $v(x)$ be the pay-off using this strategy. Then:
$v(5) = 0.5\,v(6) + 0.3\,v(7) = 5.1\,,\;\; v(4) = 0.5\,v(5) + 0.3\,v(6) = 4.35\,,\;\; v(3) = 0.5\,v(4) + 0.3\,v(5) = 3.705$
$v(2) = 0.5\,v(3) + 0.3\,v(4) = 3.1575\,,\;\; v(1) = 0.5\,v(2) + 0.3\,v(3) = 2.68025$
and finally $v(0) = 0.5\,v(1) + 0.3\,v(2) = 2.587375$ which is the value of the game if you would play the optimal strategy.
To fix the number of draws at the beginning is not the optimal strategy, since you would not be using the information that becomes available. However to answer the question in the book, if you draw n times, the probability that you can do all the n draws without getting the blackball is $(0.8)^n$ and hence the expected winnings would be $u(n) = (0.8)^n \times n \times \frac{0.5\times 1 + 0.3\times 2}{0.5+0.3} = 1.1 \times n \times (0.8)^{n-1}$. The maximum value of $u(n)$ is attained at $u(4) = 11 \times 4 \times (0.8)^3 = u(5) = 11 \times 5 \times (0.8)^4 = 2.2528$ which is strictly less than the expected winnings if you are using the optimal strategy, but both values are strictly less than 3, so don't play the game for that fee.

2. *Problem 30.36 on page 1218 of the textbook*

Let $\tilde{X} = X - \mu$. Then $\tilde{X}$ has mean 0 and so:

$$Cov(X,Y) = Cov(X, \tilde{X}^2) = \mathbb{E}[\tilde{X}^3] = \sum_{n=0}^{N} (n-\mu)^3 p_n$$

$$= \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 3\mu^2 \sum_{n=0}^{N} n p_n - \mu^3 \sum_{n=0}^{N} p_n$$

$$= \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 2\mu^3$$

3. *A point starts at the origin on the real line and takes steps of length δ with probability $p > 0$ to the right and with probability $q = 1-p$ to the left. Assuming that the steps are independent find the expected value of the* **squared** *distance from the origin after n steps.*

As I explained in class, an elegant way to solve this question is to write X as a sum $X = X_1 + \cdots + X_n$, where X_i is the ith step. Then $\mathbb{E}[(X_1 + \cdots + X_n)^2] = (n + n(n-1)(p-q)^2)\delta^2 = (4npq + n^2(p-q)^2)\delta^2$, using the easy facts that $\mathbb{E}[X_i^2] = \delta^2$ for all i and $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = (p-q)^2\delta^2$, for $i \neq j$.
A somewhat more cumbersome method is as follows:
Let X denote the distance from the origin after n steps. $\mathbb{P}(X = (n-2k)\delta = \binom{n}{k} p^{n-k} q^k$, where $k = 0, \ldots, n$, is the number of steps to the left. The expected value of the squared distance from the origin after n steps is $\delta^2 \sum_{k=0}^{n} (n-2k)^2 \binom{n}{k} p^{n-k} q^k = \delta^2 \sum_{k=0}^{n} (n^2 - 4nk + 4k^2) \binom{n}{k} p^{n-k} q^k$
$= \delta^2 (n^2 - 4n^2 q + 4(npq + n^2 q^2)) = (n^2(p-q)^2 + 4npq)\delta^2$.

4. *A model for the movement of a stock price supposes that if the present price is S then after one period, it will either go up to uS with probability p or go down to dS with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock price will be up by at least 10% after the next 1000 periods for $u = 1.02$, $d = 0.95$ and $p = 0.6$*

The stock price will be up by at least 5% after the next 1000 periods only if there are at least 723 periods where the stock goes up (out of the 1000 periods). This is determined by solving the inequality:

$(1.02)^n (0.95)^{1000-n} \geq 1.1 \iff n \geq \frac{\log(1.1) - 1000\log(0.95)}{\log(1.02) - \log(0.95)} \approx 722.8$

Let X be the number of periods where the stock is going up. Then $X \sim \text{binomial}(1000, 0.6)$.
$P(\text{the stock price will be up by at least } 10\%) = P(X \geq)$. By using the normal approximation,

$$P(X \geq 723) \approx P\left(Z \geq \frac{723 - 600}{\sqrt{1000(0.6)(0.4)}}\right) \approx 10^{-15}$$

Fat Chance!

5. *Let X be a random variable having the χ^2 distribution with n degrees of freedom and let Z be a standard normal variate. Assuming X and Z are independent, compute the probability density function of*

$$T = \frac{Z}{\sqrt{\frac{X}{n}}}$$

$X \sim Gamma(\frac{n}{2}, \frac{1}{2}), Z \sim N(0,1)$ and $T = \frac{Z\sqrt{n}}{\sqrt{X}}$. X and Z are independent so their joint p.d.f. is

$$f_{X,Z}(x,z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{2^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})} x^{\frac{n-2}{2}} e^{-x/2}\, dxdz \qquad x \geq 0,\ z \in \mathbb{R}$$

We now make the change of variables: $u = x,\ t = \frac{z\sqrt{n}}{\sqrt{x}}$, or $x = u,\ z = \frac{t\sqrt{u}}{\sqrt{n}}$ with Jacobian $J = \frac{\partial(x,z)}{\partial(u,t)} = \begin{pmatrix} 1 & 0 \\ \frac{t}{2\sqrt{u}\sqrt{n}} & \sqrt{\frac{u}{n}} \end{pmatrix}$ with determinant $|detJ| = \sqrt{\frac{u}{n}}$, so $dxdz = \sqrt{\frac{u}{n}} dudt$ and
It follows that the joint pdf of X and T is

$$f_{X,T}(x,t) = \frac{1}{\Gamma(\frac{n}{2})\sqrt{2^{n+1}\,n\pi}} e^{-\frac{t^2 u}{2n}} u^{\frac{n-2}{2}} e^{-\frac{u}{2}} u^{\frac{1}{2}}\, dudt = \frac{1}{\Gamma(\frac{n}{2})\sqrt{2^{n+1}\,n\pi}} u^{\frac{n-1}{2}} e^{-\frac{u}{2}(1+\frac{t^2}{n})}\, dudt$$

Integrating out the u variable by using the substitution $v = \frac{u}{2}(1 + \frac{t^2}{n})$, we get:

$$f_T(t) = \frac{1}{\Gamma(\frac{n}{2})\sqrt{2^{n+1}\,n\pi}} \int_0^\infty u^{\frac{n-1}{2}} e^{-\frac{u}{2}(1+\frac{t^2}{n})}\, du = \frac{(1+\frac{t^2}{n})^{-\frac{n+1}{2}}}{\Gamma(\frac{n}{2})\sqrt{n\pi}} \int_0^\infty v^{\frac{n-1}{2}} e^{-v}\, dv = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

which is known as the Student's t-distribution. Note that for $n = 1$, this is the Cauchy distribution.

6. *(bonus question)*
Let r (*the interest rate*), σ (*the volatility*), K (*strike price*) and T (*time to expiration*) be positive constants. Suppose that the price of a stock at time t can be expressed as

$$S(t) = S(0)\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\,Z\right)$$

where $S(0)$ (*present value of the stock*) is a positive number and Z is the standard normal.

Let I be the indicator variable for the event that $S(T) \geq K$ (*to be in the money*), i.e. $I = 1$ if $S(T) \geq K$ and $I = 0$ otherwise.

(i) Show that

$$\mathbb{E}[I] = \Phi(\omega - \sigma\sqrt{T}) \qquad \text{and} \qquad \mathbb{E}[IS(T)] = S(0)e^{rT}\Phi(\omega)$$

where

$$\omega = \frac{rT + \frac{1}{2}\sigma^2 T - log(\frac{K}{S(0)})}{\sigma\sqrt{T}}$$

(ii) Compute (*Black-Scholes formula for a simple European Call Option*):

$$C(S(0), T, K, r, \sigma) = e^{-rT}\,\mathbb{E}[(S(T) - K)^+] = S(0)\,\Phi(\omega) - e^{-rT}\,K\,\Phi(\omega - \sigma\sqrt{T})$$

where $x^+ = max(x, 0) = \frac{1}{2}(|x| + x)$ for any real number x.

$$S(T) \geq K \Longleftrightarrow Z \geq \frac{\log\frac{K}{S(0)} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \Longleftrightarrow Z \geq \sigma\sqrt{T} - \omega$$

so $\mathbb{E}[I] = \Phi(\omega - \sigma\sqrt{T})$

$$\begin{aligned}
\mathbb{E}[IS(T)] &= \frac{1}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}-\omega}^{\infty} S(T)\exp(-\frac{1}{2}z^2)dz \\
&= \frac{S(0)}{\sqrt{2\pi}}e^{rT}\int_{\sigma\sqrt{T}-\omega}^{\infty} \exp(-\frac{1}{2}(\sigma^2 T - 2\sigma\sqrt{T}\,z + z^2))\,dz \\
&= \frac{S(0)}{\sqrt{2\pi}}e^{rT}\int_{\sigma\sqrt{T}-\omega}^{\infty} \exp(-\frac{1}{2}(z - \sigma\sqrt{T})^2)\,dz \\
&= S(0)e^{rT}\Phi(\omega)
\end{aligned}$$

and hence

$$e^{-rT}\,\mathbb{E}[(S(T)-K)^+] = e^{-rT}\mathbb{E}[I(S(T)-K)] = e^{-rT}\mathbb{E}[IS(T)] - K\mathbb{E}[I] = S(0)\,\Phi(\omega) - e^{-rT}\,K\,\Phi(\omega - \sigma\sqrt{T})$$