Math 3D03
Short solutions to assignment #5

1. If X_1, X_2, X_3 are independent and identically distributed exponential random variables with the same parameter $\lambda > 0$, compute the probability

$$\mathbb{P}\{max(X_1 + X_2\,,\, X_3) \leq 2\}$$

$X_1 + X_2$ is a Gamma distribution with $\alpha = 2, \lambda = \lambda$. so the answer is

$$\mathbb{P}\{X_1 + X_2 \leq 2\} \times \mathbb{P}\{X_3 \leq 2\} = \int_0^2 \lambda t\, e^{-\lambda t} \lambda dt \int_0^2 e^{-\lambda t} \lambda dt = (1 - (2\lambda + 1)e^{2\lambda})(1 - e^{-2\lambda})$$

2. Let X be a random variable having the χ^2 distribution with n degrees of freedom and let Z be a standard normal variate. Assuming X and Z are independent, compute the probability density function of

$$T = \frac{Z}{\sqrt{\frac{X}{n}}}$$

medskip The joint pdf of X and Z is $f_{X,Z}(x,z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{1}{\Gamma(n/2)2^{(n/2)}} x^{(n-2)/2} e^{-x/2}$. Since $Z = \frac{T\sqrt{X}}{\sqrt{n}}$, we have by the change of variables formula:

$$dx\,dz = \left|\frac{\partial(x,z)}{\partial(x,t)}\right| dx\,dt = \frac{\sqrt{X}}{\sqrt{n}} dx\,dt$$

It follows that the joint pdf of X and T is

$$\begin{aligned} f_{X,T}(x,t) &= \frac{1}{\sqrt{2\pi}} e^{-t^2 x/2n} \frac{1}{\Gamma(n/2)2^{(n/2)}} x^{(n-2)/2} e^{-x/2} \frac{\sqrt{x}}{\sqrt{n}} \\ &= \frac{1}{\sqrt{n\pi}\Gamma(n/2)2^{(n+1)/2}} x^{(n-1)/2} e^{-x(n+t^2)/2n} \end{aligned}$$

Therefore

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{X,T}(x,t)dx \\ &= \frac{1}{\sqrt{n\pi}\Gamma(n/2)2^{(n+1)/2}} \int_0^\infty x^{\frac{n+1}{2}-1} e^{-x(n+t^2)/2n} dx \\ &= \frac{1}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \int_0^\infty y^{\frac{n+1}{2}-1} e^{-y} dy \qquad \text{where } y = (1 + \frac{t^2}{n})\frac{x}{2} \\ &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \end{aligned}$$

which is the student's t distribution.

3. For a positive parameter α, the Rayleigh distribution is defined by

$$p(x;\alpha) = \frac{x}{\alpha}\, e^{-\frac{x^2}{2\alpha}} \qquad \text{for } x \geq 0$$

and 0 otherwise.

(i) Compute the mean: $\mu = \sqrt{\frac{\pi\alpha}{2}}$

(ii) Given a sample $x_1, \ldots, x_N$, compute the maximum likelihood estimator: $\widehat{\alpha} = \frac{1}{2N}\sum_{i=1}^{N} x_i^2$

(iii) Show that the log-likelihood function has a local maximum at $\widehat{\alpha}$: $\frac{\partial^2 l}{\partial \alpha^2} == \frac{N}{\alpha^2} < 0$ at $\alpha = \widehat{\alpha}$.

4. Do problem 31.4 on page 1298 in the textbook. JUST DO IT

5. Do problem 31.15 on page 1301 in the textbook. JUST DO IT

6. (*bonus question*) Let r (*the interest rate*), σ (*the volatility*), K (*strike price*) and T (*time to expiration*) be positive constants. Suppose that the price of a stock at time t can be expressed as

$$S(t) = S(0)\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z\right)$$

where $S(0)$ (*present value of the stock*) is a positive number and Z is the standard normal.

Show that (*Black-Scholes formula for a simple European Call Option*):

$$e^{-rT}\, \mathbb{E}[(S(T) - K)^+] = S(0)\,\Phi(\omega) - e^{-rT}\, K\,\Phi(\omega - \sigma\sqrt{T})$$

where $x^+ = max(x, 0) = \frac{1}{2}(|x| + x)$ for any real number x.

Let I be the indicator variable for the event that $S(T) \geq K$ (*to be in the money*), i.e. $I = 1$ if $S(T) \geq K$ and $I = 0$ otherwise.

$$S(T) \geq K \Longleftrightarrow Z \geq \frac{\log\frac{K}{S(0)} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \Longleftrightarrow Z \geq \sigma\sqrt{T} - \omega$$

so $\mathbb{E}[I] = \Phi(\omega - \sigma\sqrt{T})$

$$\begin{aligned}
\mathbb{E}[IS(T)] &= \frac{1}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}-\omega}^{\infty} S(T)\exp(-\frac{1}{2}z^2)dz \\
&= \frac{S(0)}{\sqrt{2\pi}}e^{rT}\int_{\sigma\sqrt{T}-\omega}^{\infty} \exp(-\frac{1}{2}(\sigma^2T - 2\sigma\sqrt{T}\,z + z^2))\,dz \\
&= \frac{S(0)}{\sqrt{2\pi}}e^{rT}\int_{\sigma\sqrt{T}-\omega}^{\infty} \exp(-\frac{1}{2}(z - \sigma\sqrt{T})^2)\,dz \\
&= S(0)e^{rT}\Phi(\omega)
\end{aligned}$$

and hence

$$e^{-rT}\,\mathbb{E}[(S(T)-K)^+] = e^{-rT}\mathbb{E}[I(S(T)-K)] = e^{-rT}\mathbb{E}[IS(T)] - K\mathbb{E}[I] = S(0)\,\Phi(\omega) - e^{-rT}\,K\,\Phi(\omega - \sigma\sqrt{T})$$