Math 3D03
Short solutions to assignment #2

1. Evaluate the following definite (real-valued) integrals:

$$(i)\ \int_0^{2\pi} (\sin\theta)^n \, d\theta \qquad \text{for } n \in \mathbb{N}. \qquad \text{What happens when } n \to \infty\,?$$

$$(ii)\ \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx \qquad \text{for } 0 < a < 1$$

$$(iii)\ \int_0^{\infty} \frac{dx}{1+x^n} \qquad \text{where } n \geq 2 \text{ is an integer}$$

(i) The integral is obviously zero for odd n, since $\sin(\theta) = -\sin(2\pi - \theta)$. For even n, we have using the binomial formula:

$$\int_0^{2\pi} (\sin\theta)^{2n}\, d\theta = \oint_{|z|=1} \left(\frac{z - z^{-1}}{2i}\right)^{2n} \frac{dz}{iz} = \frac{1}{(2i)^n} \oint_{|z|=1} \frac{z^{2n}}{iz} (1 - z^{-2})^{2n} dz$$

$$= \frac{1}{2^{2n} i^{2n+1}} \sum_{k=0}^{2n} \oint_{|z|=1} (-1)^k \binom{2n}{k} z^{2n-2k-1}\, dz = \frac{2\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}$$

As $n \to \infty$, the integral goes to zero. This can be seen, for example, by using Stirling's formula: $\lim_{N\to\infty} \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \frac{1}{N!} = 1$ or by looking at the graph of the function $(\sin\theta)^{2n}$ for large n .

(ii) Use a long horizontal strip $[-R, +R] \times [0, 2\pi]$ above the $x-$axis as your contour The integral on the two vertical lines $\to 0$, as $R \to \infty$ since the integrand is bounded from above in amplitude by $\frac{e^{aR}}{e^R}$ (on the right vertical line) and by e^{-aR} (on the left vertical) and $0 < a < 1$. On the upper horizontal line the integral is a phase shift by $e^{ia2\pi}$ of the integral on the $x-$ axis (in the opposite direction). There is exactly one simple pole at $i\pi$ within the strip with residue $= \frac{e^{ia\pi}}{e^{i\pi}} = -e^{ia\pi}$. Therefore $(1 - e^{i2a\pi}) \int_{-\infty}^{\infty} \frac{e^{ax}dx}{1+e^x} = -2\pi i e^{ia\pi}$ and hence the answer is

$$\int_{-\infty}^{\infty} \frac{e^{ax}dx}{1+e^x} = \frac{\pi}{\sin(a\pi)}$$

(iii) This is problem 24.18 in the textbook but here is how you can do it as I showed you in class: $\oint_C \frac{1}{1+z^n} dz$, where the contour C is the $\frac{2\pi}{n}-$sector (of radius $R \to \infty$) in the first quadrant. There is a single simple pole at $e^{i\frac{\pi}{n}}$ inside C with residue $= \frac{1}{n} e^{-i\frac{(n-1)\pi}{n}} = -\frac{1}{n} e^{i\frac{\pi}{n}}$. The integral along the ray $z = re^{i\frac{2\pi}{n}}$ is a phase shift by $e^{i\frac{2\pi}{n}}$ of the integral on the $x-$axis (in the opposite direction). The integral on the circular arc tends to zero as $R \to \infty$, since $n \geq 2$. Therefore $(1 - e^{i\frac{2\pi}{n}}) \int_0^{\infty} \frac{dx}{1+x^n} = -\frac{2\pi i}{n} e^{i\frac{\pi}{n}}$ and hence

$$\int_0^{\infty} \frac{dx}{1+x^n} dx = \frac{\pi}{n} \csc\left(\frac{\pi}{n}\right)$$

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2. Do problems 24.20 and 24.21 on page 869 in the text book.

Problem 24.20. Use a key hole contour around the origin with a cut along the positive real axis. There is exactly one simple pole at $z = \exp(i\pi)$ with residue $= i\pi\ \exp(-i\frac{3\pi}{4})$.
Both circular integrals (around the little circle around zero and the big circle around ∞) go to 0 when you let the radii go to zero and ∞ respectively. The integral along the cut (the positive real axis) undergoes a phase shift when it comes back from ∞:

$$\int_{\infty}^{0} \frac{log(x) + 2\pi i}{\exp(i\frac{3\pi}{2})\, x^{\frac{3}{4}}(1+x)} dx = -i \int_{0}^{\infty} \frac{log(x)}{x^{\frac{3}{4}}(1+x)} dx + 2\pi \int_{0}^{\infty} \frac{dx}{x^{\frac{3}{4}}(1+x)}$$

Hence

$$(1-i)\int_{0}^{\infty} \frac{log(x)}{x^{\frac{3}{4}}(1+x)} dx + 2\pi \int_{0}^{\infty} \frac{dx}{x^{\frac{3}{4}}(1+x)} = -2\pi^2\ \exp(-i\frac{3\pi}{4}) = \pi^2\sqrt{2}(1+i)$$

Therefore:

$$\int_{0}^{\infty} \frac{log(x)}{x^{\frac{3}{4}}(1+x)} dx = -\pi^2\sqrt{2} \qquad \int_{0}^{\infty} \frac{dx}{x^{\frac{3}{4}}(1+x)} = \pi\sqrt{2}$$

Problem 24.21. Use a large semicircle of radius R in the upper half plane and the real line with a small semicircular dent of radius ϵ around the origin. Log is well-defined there. There is a simple pole at $z = i$ inside ythe contour with residue $\frac{(\log(i))^2}{2i} = i\frac{\pi^2}{8}$. The integrals on the semicircular pieces go to zero when $R \to \infty$ and $\epsilon \to 0$. On the left part of the real axis, the log is phase-shifted: $(\log(xe^{i\pi}))^2 = (\log x)^2 + 2i\pi \log x - \pi^2$, so we get:

$$2\int_{0}^{\infty} \frac{(\log(x))^2}{1+x^2}\, dx + 2\pi i \int_{0}^{\infty} \frac{\log(x)}{1+x^2}\, dx - \pi^2 \int_{0}^{\infty} \frac{dx}{1+x^2} = -\frac{\pi^3}{4}$$

Equating real and imaginary parts we get:

$$\int_{0}^{\infty} \frac{(\log(x))^2}{1+x^2}\, dx = \frac{\pi^3}{8} \qquad \int_{0}^{\infty} \frac{\log(x)}{1+x^2}\, dx = 0$$

3. Sum the following infinite series:

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{n^2+9} \qquad (b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \qquad (c)\ \sum_{n=-\infty}^{\infty} \frac{n^2}{n^4-\pi^4}$$

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+9} = -\frac{1}{2}\left(\text{sum of residues}\left(\frac{\pi\cot(\pi z)dz}{z^2+9}; \pm 3i\right) + \frac{1}{9}\right) = \frac{\pi}{6}\coth(3\pi) - \frac{1}{18}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = -\frac{1}{2}\text{Res}\left(\frac{\pi\csc(\pi z)dz}{z^4}; 0\right) = \frac{1}{2}\frac{(-1)^3 2(2^3-1)\pi^4}{4!}B_4 = \frac{7\pi^4}{720}$

(c) $\sum_{n=-\infty}^{\infty} \frac{n^2}{n^4-\pi^4} = \text{sum of residues}\left(\frac{\pi\cot(\pi z)z^2dz}{z^4-\pi^4}; \pm\pi, \pm i\pi\right) = \frac{1}{2}(\coth(\pi^2) - \cot(\pi^2))$

4. How many zeros of the polynomial $z^4 - 5z + 1$ lie in the annulus $1 \le |z| \le 2$?

$|-5z+1| \le 5|z| + 1 = 11 < 16 = |z^4|$ on the outer circle $|z| = 2$ and z^4 has a quadruple zero at $z = 0$. On the other hand, $|-5z+1| \ge 5|z| - 1 = 4 > 1 = |z^4|$ on the inner circle $|z| = 1$ and $-5z+1$ has exactly one zero at $z = \frac{1}{5}$ inside the inner circle.

Therefore there are 3 roots of the given quartic inside the given annulus.

5.

(i) Suppose that $f(z)$ is a non-constant analytic function defined for all $z \in \mathbb{C}$. Show that for every $R > 0$ and for every $M > 0$ there exists a z such that $|z| > R$ and $|f(z)| > M$.

(ii) Suppose that $f(z)$ is a non-constant polynomial. Show that for every $M > 0$ there exists an $R > 0$, such that $|f(z)| > M$ for all $|z| > R$.

(iii) Show that there exists an $M > 0$, such that for every $R > 0$, there exists a z satisfying $|z| > R$ and $|e^z| \le M$.

(i) Arguing by contradiction, let us assume that there exists $R > 0$ and $M > 0$ such that $f(z) \le M$ for every $|z| > R$. Let $a \in \mathbb{C}$. By the Cauchy integral formula : $f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$, where we choose C to be a circle of very large radius r (say $r = 100(R + |a|)$). The integral is bounded from above in absolute value by the product of the length of that circle and the maximum absolute value of the integrand which is $\le 2\pi r \frac{M}{r^2} = \frac{2\pi M}{r}$ and so $|f'(a)| \le \frac{M}{r}$ for any r sufficiently large. This proves that $f'(a) = 0$ for any a and hence f is a constant function.

(ii) By factoring out the top coefficient, we may assume that $p(z) = z^n + q(z)$, where $q(z)$ is a polynomial of degree $\le n-1$ $(n \ge 1)$. Since $\lim_{z \to \infty} \frac{q(z)}{z^n} = 0$, we see that for $|z|$ sufficiently large $|q(z)| < 0.1|z^n|$ and so $|p(z)| \ge 0.9|z|^n \ge 0.9R^n$ for $|z| \ge R$.
So for any given $M > 0$, we can find R such that $|p(z)| > M$ for every z with $|z| > R$.

(iii) For every $R > 0$, $z = 2iR$ satisfies $|z| > R$ and $|e^z| \le 1$.