This assignment is due in the locker in the basement of the Burke Science Building by 15:00 on Friday 7 April 2000.

1. Consider the heat equation with initial and boundary conditions

$$\begin{aligned} \partial u/\partial t &= \partial^2 u/\partial x^2, & 0 < x < 1, & \\ u(x,0) &= f(x), & 0 \le x \le 1, & \\ u(0,t) &= g_0(t), & u(1,t) = g_1(t), & \quad t > 0 \end{aligned} \tag{1}$$

We divide the spatial domain $[0,1]$ into J equispaced subintervals and let $h = 1/J$ be the meshsize and $x_j = jh$, $0 \le j \le J$ be the meshpoints. By applying the method of lines, the solution values $u(x_j,t)$ can be approximated by $u_j(t)$, which satisfies the system of ODEs:

$$\begin{aligned} U'(t) &= AU(t) + S(t), \\ U(0) &= (f(x_1), f(x_2), \ldots, f(x_{J-1})) \end{aligned} \tag{2}$$

where $U(t) = (u_1(t), u_2(t), \ldots, u_{J-1}(t))^T$, $S(t) = (1/h^2)(g_0(t), 0, \ldots, 0, g_1(t))^T$ and $A = (1/h^2)(C - 2I)$ is a $(J-1) \times (J-1)$ matrix with

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

and where I is the identity matrix.

[5] (a) What is the maximum stepsize k that can be used to solve (2) by Euler's method? (You may refer to results from the class notes.)

[10] (b) Let $f(x) = 2x$ on $[0, 0.5]$ and $f(x) = 2 - 2x$ on $(0.5, 1]$ and $g_0(t) = g_1(t) = 0$. Use the method of lines with $h = 0.1$ to obtain approximate values $u_5(t)$ of $u(0.5,t)$ for $0 \le t \le 0.6$. In order to solve system (2), use Euler's method with time stepsize $k = 0.001, 0.005, 0.0052, 0.0056$, respectively. In each case plot your numerical solution over the time interval $[0, 0.6]$. Note that the analytic solution is

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\sin(n\pi x) e^{-n^2\pi^2 t}}{n^2}.$$

Explain the anomalies that you may observe in terms of stability.

2. An equation frequently used as a model for fluid mechanics is Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

[5] (a) Give an explicit second-order in space, first-order in time finite difference method for solving this equation.

[10] (b) Given the initial condition

$$u(x,0) = \sin \pi x \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0$$

solve the problem for $\nu = 0.1, 0.01, 0.001$ with space stepsize $h = 0.02$, time stepsize $k = 0.001$. Plot the solution for $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ up to $t = 1$, and explain the results.

3. Consider the one-dimensional linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < t < 2$$

with the following boundary and initial conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0$$
$$u(x,0) = f(x) \quad u_t(x,0) = 0$$

[10] (a) Write a **Matlab** function to solve the above wave equation using the explicit finite difference method (where both the time and space derivatives are approximated using second-order central differences). State the stability criterion.

[5] (b) Use the above **Matlab** function to solve the wave equation with initial condition $f(x) = \sin 2\pi x$ with space stepsize $h = 0.01$ and time stepsize $k = 0.005$. What is the exact solution? What is special about the solution at the time $t = 2/c = 2$? Plot the numerical solution at $t = 0, 0.5, 1.0, 1.5, 2.0$. Comment on the accuracy of the method.

[5] (c) Now consider the following initial condition:

$$f(x) = \begin{cases} 5(x-0.3) & 0.3 < x < 0.5 \\ 5(0.7-x) & 0.5 < x < 0.7 \\ 0 & 0 < x < 0.3 \text{ and } 0.7 < x < 1 \end{cases} \tag{3}$$

Plot the numerical solution at the same times as in (b). Comment on the accuracy of the method. What is the problem? [Hint: the initial condition can be thought of as being made up of waves of different amplitudes. What is happening to these waves?]

[Total: 50]