**Assignment 3** **MATH4Q3** 4 February 2000

This assignment is due in the locker in the basement of the Burke Science Building by 15:00 on Friday 18 February 2000.

1. Milne's fourth-order predictor–corrector method for the initial value problem $y' = f(x,y)$, $y(0) = y_0$ is given by the formulas

$$\begin{aligned} y_{n+1}^{(P)} &= y_{n-3} + \frac{4h}{3}[2f_n - f_{n-1} + 2f_{n-2}] + \frac{14}{45}h^5 y^{(5)}(\xi) \\ y_{n+1}^{(C)} &= y_{n-1} + \frac{h}{3}[f(x_{n+1}, y_{n+1}^{(P)}) + 4f_n + f_{n-1}] - \frac{1}{90}h^5 y^{(5)}(\xi) \end{aligned}$$

where the terms proportional to $h^5$ are the local truncation errors.

[2] (a) By applying the corrector of Milne's method to the model problem $y' = \lambda y$, show that it leads to a differential equation of the form

$$a y_{n+1} + b y_n + c y_{n-1} = 0 \tag{1}$$

where $a, b, c$ are constants depending on $\lambda$ and $h$.

[6] (b) Assume that the solution of (1) has the form $y_n = \theta^n$ where $\theta \neq 0$ is a constant, and determine the two roots of (1), $\theta_1$, $\theta_2$. The stability region for Milne's method is determined by $|\theta_1| \leq 1$ and $|\theta_2| \leq 1$. What happens to the stability of the method when $h\lambda \to 0$?

[4] (c) Assuming that the two fifth-order derivatives in the error terms are the same, find an estimate of the local truncation error $e_{n+1}$ for Milne's fourth-order method in terms of $y_{n+1}^{(P)}$ and $y_{n+1}^{(C)}$. How could this estimate be used in practice?

[8] 2. (a) Write `Matlab` functions to integrate the initial value problem

$$y' = f(x,y), \quad y(a) = y_0,$$

on an interval $[a, b]$ using:

- Euler's method
- Modified Euler
- Improved Euler
- RK4

It is suggested that you implement, for example, Improved Euler as $[x, y] = eulerimp('f', a, y0, b, stepsize)$, where $(x, y) = (x_n, y_n)$ is the computed solution.

[5] (b) Use the four `Matlab` function you wrote in (a) with stepsize $h = 1/40$ and $1/80$ to solve the following ODE:

$$y' = -y^3/2, \quad y(0) = 1 \quad \text{with} \quad y_{exact}(x) = \frac{1}{\sqrt{x+1}}. \tag{2}$$

Calculate the absolute errors (i.e. $|y_{exact}(x) - y_{computed}(x)|$) at $x = 1$ to see if they are roughly reduced by a half for Euler's method, by a quarter for Modified Euler and

Improved Euler, and by $1/16$ for RK4, as the stepsize $h$ is halved from $1/40$ to $1/80$. Discuss the performance of the methods by comparing the computation time used by each method to calculate $y(1)$ with stepsize $h = 1/40$ and $1/80$ (use the `Matlab` functions `tic` and `toc` to do the timing).

[5] 3. (a) Let $g(z)$ be a user-defined function with possibly complex-valued coefficients and variables. Then the set of all such $z$ in the complex plane that satisfy $|g(z)| = 1$ is in general a curve or consists of several curves. Write a `Matlab` script to plot the curves(s) (in the complex plane) determined by the equation $|g(z)| = 1$, where $g(z) = \sum_{k=1}^{10} z^k/k!$.

[5] (b) Recall that one of the RK3 methods is as follows:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 4k_2 + k_3), \quad \text{where} \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\ k_3 &= hf(x_n + h, y_n - k_1 + 2k_2). \end{aligned}$$

Show that the stability region of the above RK3 method is determined by $|E(\lambda h)| \le 1$ where

$$E(\lambda h) = 1 + \lambda h + \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!}.$$

[5] (c) In general, the stability region of a $p-$th order Runge–Kutta method is determined by $|E(\lambda h)| \le 1$ with $E(\lambda h) = \sum_{k=0}^{p} \frac{(\lambda h)^k}{k!}$. Write `Matlab` scripts (similar to that used in (a)) to plot the stability regions for $p = 1, 2, 3, 4, 5$ by letting $\lambda h = z$. Are the stability regions larger for larger $p$?

[5] 4. (a) Modify your Runge–Kutta program of question 2(a) to solve a second-order ordinary differential equation.

[5] (b) Duffing's equation

$$\frac{d^2y}{dt^2} + y - \epsilon y^3 = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0,$$

describes an oscillator with a slightly nonlinear restoring force. The size of the nonlinear part of the restoring force is given by the parameter $\epsilon$. Use your program from (a) with stepsize $h = 0.1$ to solve Duffing's equation with $\epsilon = 0.05$ from $t = 0$ to $t = 6\pi$. Plot the results compared with the linear oscillator ($\epsilon = 0$). Can the small term $\epsilon y^3$ be neglected? What is the effect of the nonlinearity?