MATHEMATICS 3Q

16 OCTOBER 2009 MIDTERM TEST

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DURATION OF EXAM: 50 MINUTES PLEASE SHOW ALL WORK

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1. Consider polynomial interpolation at equally spaced points on the interval $[-1/4, 3/4]$ of the function

$$f(x) = \frac{1}{1 + 16x^2}.$$

[4] (a) Will polynomial wiggle (i.e. Runge phenomenon) be a problem on this interval?

[2] (b) Over what sub-interval of $[-1/4, 3/4]$ centred at $1/4$ is polynomial wiggle *not* a problem?

[4] (c) How can one avoid polynomial wiggle entirely for *any* choice of interval?

[4] 2. (a) Calculate the rate of convergence of Newton's method in one dimension for the case of both simple and multiple roots.

[3] (b) How can one *accurately* estimate the rate and constant of convergence of an iterative method given the error at each iteration?

[4] 3. (a) Show that the absolute condition number for the problem of finding the minimum x^* of a function $f(x)$ is

$$C = \sqrt{\frac{2}{\Delta b |f''(x^*)|}},$$

where Δb is the absolute *backward* error.

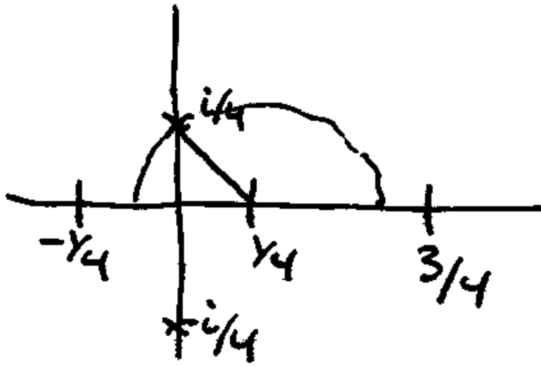
[4] (b) If we use a floating point number system with 16 digits of accuracy, to how many digits of accuracy can we find the minimum of a function? [Hint: use your result from (a).]

[Total: 25]

THE END

Math 3Q : Test (2009)

1. (a) poles of $f(z)$ are at $z = \pm i/4$



$\therefore$ Runge phenomenon is present outside interval $\frac{1}{4} \pm \frac{1}{2\sqrt{2}}$ or $[-0.1036, 0.6036]$

$\therefore$ Runge Phenomenon is a problem in $[-1/4, 3/4]$

(b) See (a) : $[-0.1036, 0.6036]$

(c) one can use a suitable distⁿ of nodes (eg. Chebyshev) and ~~ar~~ ~~as~~ a stable algorithm for computing the weights (eg. barycentric).

Equally spaced points are the worst choice!

2. (a) Consider fixed point iteration

$$x_{k+1} = g(x_k) \quad \text{where } g(x) = x - \frac{f(x)}{f'(x)}$$

$\to |x_{k+1} - x_*| = |g(x_k) - g(x_*)|$

Taylor series $\qquad \to g'(x) = \dfrac{f(x)\, f''(x)}{(f'(x))^2}$

$$\to \; = \begin{cases} |g'(\xi_k)|\,|x_k - x_*| & \text{if multiple root} \\ \frac{1}{2}|g''(\xi_k)|\,|x_k - x_*|^2 & \text{if \underline{simple} root i.e. } f(x)=0,\ f'(x)\neq 0 \end{cases}$$

~~since g(x*)~~

since for a simple root $g'(x_*) = 0$

but for a multiple root $f(x) = (x - x_*)^m \tilde{f}(x)$

$$\to g'(x_*) = \frac{m-1}{m} < 1 \neq 0$$

$\to e_{k+1} \le C e_k^1$ for ~~simple~~. multiple root

$\to$ linear convergence

$e_{k+1} \le D e_k^2$ for simple root

$\to$ quadratic convergence.

(b) constant C & rate p are defined by relation

$$e_{k+1} \le C e_k^p$$

∴ To find p plot $\log e_{k+1}$ vs. $\log e_k$ ~~on~~ & fit straight line to find p & c from $\log e_{k+1} = p \log e_k + \log c$.

or, To find C plot $\dfrac{e_{k+1}}{e_k^p}$ & fit line of slope zero.

3. (a) $f(x^* + h) = f(x^*) + \underbrace{f'(x^*)}_{= 0 \text{ since c.p.}} + \frac{1}{2} f''(x^*) h^2 + O(h^3)$

$\therefore \quad h = \text{forward error} \approx \sqrt{\dfrac{|f(\hat{x}) - f(x^*)|}{\frac{1}{2} |f''(x^*)|}}$

but $|f(\hat{x}) - f(x^*)| = \text{backward error} \equiv \Delta b$.

$\rightarrow \quad C = \dfrac{h}{\Delta b} = \sqrt{\dfrac{2}{\Delta b \, |f''(x^*)|}}$

(b) ~~The absolute condition~~
The minimum error is ~~therefore~~ when $\Delta b = \varepsilon_{mach}$

$\therefore$ min error $= C \varepsilon_{mach} = \sqrt{\dfrac{2 \varepsilon_{mach}}{|f''(x^*)|}}$

$\therefore$ if $|f''(x^*)| = O(1)$ we can find ~~so~~ min. to only half the digits accuracy of available from floating point system.

$\therefore$ we expect to find x^* to only 8 digits of accuracy.