TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 344, Number 2, August 1994

NONORIENTABLE 4-MANIFOLDS WITH FUNDAMENTAL GROUP OF ORDER 2

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ABSTRACT. In this paper we classify nonorientable topological closed 4-manifolds with fundamental group $\mathbb{Z}/2$ up to homeomorphism. Our results give a complete list of such manifolds, and show how they can be distinguished by explicit invariants including characteristic numbers and the η-invariant associated to a normal Pin^c-structure by the spectral asymmetry of a certain Dirac operator. In contrast to the oriented case, there exist homotopy equivalent nonorientable topological 4-manifolds which are stably homeomorphic (after connected sum with $S^2 \times S^2$) but not homeomorphic.

1. INTRODUCTION

In this paper we classify nonorientable topological closed 4-manifolds with fundamental group $\mathbb{Z}/2$ up to homeomorphism. Our main result is Theorem 3 which contains a complete list of such manifolds. We also give a simple set of invariants, namely the Euler characteristic, the Stiefel-Whitney number w_1^4, an Arf-invariant and the Kirby-Siebenmann obstruction, which classify these manifolds (Theorem 2). For smooth manifolds the Kirby-Siebenmann obstruction can be omitted and it turns out that the Arf-invariant has an analytic description as the η-invariant of a Dirac operator (Theorem 1).

In the oriented case the classification is contained in [9]. The result there is that the manifolds are classified by the intersection form on ordinary homology, the Kirby-Siebenmann obstruction and the three possibilities for existence of spin structures (universal covering not spin, manifold spin or manifold not spin but universal covering spin). Since the intersection form is not an interesting invariant for nonorientable manifolds, we expected a simpler classification in this case. This turned out to be true, although in carrying out the details we encountered additional invariants and a new phenomenon. In contrast to the oriented case, there exist homotopy equivalent nonorientable topological 4-manifolds which are stably homeomorphic (after connected sum with $S^2 \times S^2$) but not homeomorphic.

That new invariants exist in the nonoriented case was already known for smooth manifolds. All such manifolds admit a (normal) Pin^c-structure, where Pin^c is an appropriate 2-fold covering of $O \times U(1)$ (see for example [8, p. 256]) and a Pin^c-structure Φ is a specific reduction of the normal bundle to Pin^c.

Received by the editors February 15, 1993.
1991 *Mathematics Subject Classification*. Primary 57N13; Secondary 57S17.

Note that by composing with the projection to $U(1)$ we get a complex line bundle from a Pin^c-structure Φ and we denote its first Chern class by c_Φ. Choose such a structure Φ on M and a Riemannian metric g. Then there is a corresponding Dirac operator $D(M, g, \Phi)$. It was shown in [8] that the spectral asymmetry $\eta D(M, g, \Phi) \in \mathbb{R}/\mathbb{Z}$ is a Pin^c-bordism invariant. We will see in §5 that up to sign it is actually a homeomorphism invariant of homeomorphisms preserving c_Φ (but not a homotopy invariant) and denote it by $\pm\eta(M, c_\Phi)$. Recall that the two 2-fold coverings Pin^+ and Pin^- of O (compare §2) are subgroups of Pin^c and that a $Pin^\pm$-structure is a reduction of the normal bundle to these groups. We call a Pin^c-structure Φ *primitive* if either the structure comes from a $Pin^\pm$-structure or (in case the manifold does not admit a $Pin^\pm$-structure) c_Φ is a primitive cohomology class.

Theorem 1. *Two smooth closed nonorientable 4-manifolds M and M' with fundamental group $\mathbb{Z}/2$ are homeomorphic if and only if they both admit a normal Pin^+ or Pin^- or no such structure and*

$$e(M) = e(M'), \qquad w_1(M)^4 = w_1(M')^4, \qquad \eta(M, c_\Phi) = \pm\eta(M', c_{\Phi'})$$

for some primitive Pin^c-structure Φ and Φ'.

If $\eta(M, c_\Phi)$ is a homeomorphism invariant there should be a topological formula for it. This is the case and leads to the invariants needed for classifying all topological 4-manifolds with fundamental group $\mathbb{Z}/2$. First note that in analogy to the smooth category we have a 2-fold covering group $TopPin^c$ over $Top \times U(1)$, where $Top = \lim Top(n)$, and $Top(n)$ is the group of homeomorphisms of $\mathbb{R}^n$ fixing 0. Similarly, we have subgroups $TopPin^\pm$. As before, a topological manifold as in the title admits a normal $TopPin^c$-structure. Such a structure Φ determines a class c_Φ and we define primitive $TopPin^c$-structures as in the smooth case. The complex line bundle associated to c_Φ is classified by a map to $\mathbb{CP}^N$ for some N. The transverse preimage of $\mathbb{CP}^{N-1}$ has a canonical Pin^+-structure on the normal bundle and for surfaces with such structure there is a $\mathbb{Z}/8$-valued Brown-Arf invariant denoted by $\operatorname{arf}(M, c_\Phi)$, compare [12]. We will show in §5 that for smooth manifolds

$$8\eta(M, c_\phi) = \pm \operatorname{arf}(M, c_\Phi)$$

and thus Theorem 1 follows directly from the following result. (Here KS denotes the Kirby-Sibenmann invariant.)

Theorem 2. *Two topological closed nonorientable 4-manifolds M and M' with fundamental group $\mathbb{Z}/2$ are homeomorphic if and only if they both admit a normal $TopPin^+$ or $TopPin^-$ or no such structure and*

$$e(M) = e(M'), \quad w_1(M)^4 = w_1(M')^4,$$
$$\operatorname{arf}(M, c_\Phi) = \pm \operatorname{arf}(M', c_{\Phi'}), \quad KS(M) = KS(M')$$

for some primitive $TopPin^c$-structures Φ and Φ'.

We now give a complete list of all closed nonorientable topological 4-manifolds with fundamental group $\mathbb{Z}/2$. To give such a list we need two operations. Let M be a nonorientable closed 4-manifolds and E_8 the simply connected manifold with intersection form E_8 from [6]. There is a degree 1 normal map $E_8 \to S^4$ and taking connected sum we obtain a degree 1 normal map

$M \# E_8 \to M$. Since the map $L_4(1) \to L_4(\mathbb{Z}/2, -)$ vanishes [15, §13A], by the exactness of the topological surgery sequence [6] $M \# E_8$ is homeomorphic to $X \# 4(S^2 \times S^2)$ where X is homotopy equivalent to M. By construction, $KS(M) \neq KS(X)$ and in analogy with [7] we denote any such manifold X by $*M$. We note that Theorem 3 will imply that except in the case where the intersection form on the universal covering is odd and $e(M) = 2$, $*M$ is unique up to homeomorphism.

For the second operation let M_1 and M_2 be 4-dimensional Pin^--manifolds with fundamental group $\mathbb{Z}/2$. Denote the nontrivial 3-dimensional real disc bundle over S^1 by E. Fix a Pin^--structure on E. Choose embeddings of E into M_i representing the nontrivial element in π_1, such that the first embedding preserves the Pin^--structure and the second reverses it. Then we define

$$M_1 \underset{S^1}{\#} M_2 := (M_1 \backslash E) \cup (M_2 \backslash E).$$

Denote the Chern manifolds $*\mathbb{CP}^2$ from [6] by CH and the real Hopf bundle over $\mathbb{RP}^2$ by γ.

Theorem 3. *Every closed nonorientable topological* 4*-manifold with fundamental group* $\mathbb{Z}/2$ *is homeomorphic to exactly one manifold in the following list:*

(I) $\mathbb{RP}^4 \# k \cdot \mathbb{CP}^2$, $\mathbb{RP}^2 \times S^2 \# k \cdot \mathbb{CP}^2$, $*\mathbb{RP}^4 \# CH$, $*\mathbb{RP}^4 \# k \cdot \mathbb{CP}^2$, $*(\mathbb{RP}^2 \times S^2) \# k \cdot \mathbb{CP}^2$, $\mathbb{RP}^4 \# CH$, $k \geq 1$.
(II) $\mathbb{RP}^2 \times S^2 \# k \cdot (S^2 \times S^2)$, $*(\mathbb{RP}^2 \times S^2) \# k \cdot (S^2 \times S^2)$, $k \geq 0$.
(III) $S(2\gamma \oplus \mathbb{R}) \# k \cdot (S^2 \times S^2)$, $\#_{S^1} r \cdot \mathbb{RP}^4 \# k \cdot (S^2 \times S^2)$, $*S(2\gamma \oplus \mathbb{R}) \# k \cdot (S^2 \times S^2)$, $*(\#_{S^1} r \cdot \mathbb{RP}^4) \# k \cdot (S^2 \times S^2)$, $1 \leq r \leq 4$, $k \geq 0$.

The noncancellation examples mentioned in the beginning are $\mathbb{RP}^4 \# \mathbb{CP}^2$ and $*\mathbb{RP}^4 \# CH$. The three cases I, II and III correspond to admitting no $TopPin^\pm$-structure, admitting a $TopPin^+$-structure and admitting a $TopPin^-$-structure.

From the list above we see that all manifolds with $KS = 0$ admit a smooth structure except perhaps $*\mathbb{RP}^4 \# CH$. On the other hand, we see from Theorem 2 that $*\mathbb{RP}^4 \# CH \# S^2 \times S^2$ is homeomorphic to $\mathbb{RP}^4 \# \mathbb{CP}^2 \# S^2 \times S^2$ and thus admits a smooth structure. The problem whether $*\mathbb{RP}^4 \# CH$ admits a smooth structure is open. We would like to point out that this is analogous to $E_8 \# E_8$ which stably admits a smooth structure but unstably does not [5].

From our homeomorphism classification one can derive a homotopy classification. For this we first note that the existence of no normal $TopPin^\pm$-structure, of a $TopPin^+$-structure or of a $TopPin^-$-structure is equivalent to $w_2(\nu\widetilde{M}) \neq 0$, $w_2(\nu M) = 0$ or $w_2(\nu M) = w_1(\nu M)^2$ resp. (see Lemma 2). Thus this is a homotopy condition, and we refer to the three different possibilities by saying that M has w_2-type I, II, or III. We will see in §2 that for $TopPin^-$-manifolds, $\pm \operatorname{arf}(M, c_\Phi)$ denoted by $\operatorname{arf}(M)$ is a Brown-Arf invariant and thus a homotopy invariant. Theorem 3 implies that $\operatorname{arf}(M)$ is not a homotopy invariant for manifolds with w_2-type I. Thus we obtain

Corollary 1. *Let M and M' be closed, nonorientable topological* 4*-manifolds with fundamental group* $\mathbb{Z}/2$*. Then M and M' are homotopy equivalent if and only if*

(i) *M and M' have the same w_2-type.*
(ii) $e(M) = e(M')$.

(iii) $w_1(M)^4 = w_1(M')^4$.
(iv) $\operatorname{arf}(M) = \pm \operatorname{arf}(M')$ *in w_2-type* III.

In [11], the authors studied the homotopy classification and obtained some partial results. In particular they used a quadratic refinement $q\colon \pi_2(M) \to \mathbb{Z}/4$ of the mod 2 intersection pairing on spherical classes in $H_2(M)$ to distinguish some of the 4-manifolds in our list. We would like to mention that using this list one can also prove that the following is a complete set of homotopy invariants for nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2$: π_2 (as π_1-module), S, q. Here S denotes the equivariant intersection form on π_2 and q carries the same additional information as Wall's self-intersection form at the nontrivial group element. This result was conjectured in [11] but it is clear that the invariants from Corollary 1 are much simpler to compute than the triple (π_2, S, q).

2. A digression on Pin-structures

Recall that by definition $Pin^\pm := \lim Pin^\pm(n)$ and the groups $Pin^\pm(n)$ are central extensions

$$0 \to \mathbb{Z}/2 \to Pin^\pm(n) \to O(n) \to 0$$

classified by w_2 for Pin^+ respectively $w_2 + w_1^2$ for Pin^-. We obtain fibrations $p^\pm\colon BPin^\pm \to BO$. Also $Pin^c := \lim Pin^c(n)$ with $Pin^c(n) := Pin^+(n) \times U(1)/\langle(-1, -1)\rangle$ being a central extension $0 \to U(1) \to Pin^c(n) \to O(n) \to 0$. This gives a fibration $p^c\colon BPin^c \to BO$.

Let $\dagger \in \{c, +, -\}$. Since a $Pin^\dagger$-structure on a (stable) vector bundle E over a space X is the same as a fibre homotopy class of lifts of a classifying map $c_E\colon X \to BO$ over $p^\dagger\colon BPin^\dagger \to BO$, we can conclude the following lemma, compare [12, §1] for $\dagger = \pm$.

Lemma 1.

(i) *A vector bundle E over X admits a $Pin^\dagger$-structure if and only if*

$$\begin{aligned} \beta(w_2(E)) &= 0 && \textit{for } \dagger = c, \\ w_2(E) &= 0 && \textit{for } \dagger = +, \\ w_2(E) &= w_1(E)^2 && \textit{for } \dagger = -, \end{aligned}$$

where $\beta\colon H^2(X; \mathbb{Z}/2) \to H^3(X; \mathbb{Z})$ is the Bockstein operator induced from the exact coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$.

(ii) *$Pin^\pm$-structures are* 1-1 *correspondence with $H^1(X; \mathbb{Z}/2)$ and Pin^c-structures are in* 1-1 *correspondence with $H^2(X; \mathbb{Z})$.*

Proof. Since the above sequences mapping $Pin^\dagger$ onto O are central extensions, we get group homomorphisms $\mathbb{Z}/2 \times Pin^\pm \to Pin^\pm$ respectively $U(1) \times Pin^c \to Pin^c$ over O. Thus there are induced maps $B\mathbb{Z}/2 \times BPin^\pm \to BPin^\pm$ respectively $BU(1) \times BPin^c \to BPin^c$ over BO which shows that the projections $p^\dagger$ are principal fibrations with fibre

$$\begin{aligned} BU(1) &= K(\mathbb{Z}, 2) && \text{for } \dagger = c, \\ B\mathbb{Z}/2 &= K(\mathbb{Z}/2, 1) && \text{for } \dagger = \pm. \end{aligned}$$

Now the result follows from [2, Theorem (1.3.8)]. □

By a $Pin^{\dagger}$-structure on a smooth manifold M we mean a $Pin^{\dagger}$-structure on its (stable) normal bundle $\nu\colon M \to BO$. Now let M be nonorientable with fundamental group $\mathbb{Z}/2$. Then the above lemma and the exact sequence

$$0 \to H^2(\pi_1 M; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \to H^2(\widetilde{M}; \mathbb{Z}/2)$$

shows that M admits a $Pin^{\pm}$-structure if and only if $w_2\widetilde{M} = 0$ because $H^2(\mathbb{Z}/2; \mathbb{Z}/2)$ is generated by the square of the 1-dimensional class corresponding to $w_1 M$. This also proves that for $w_2\widetilde{M} = 0$ one has either $w_2(\nu M) = 0$ or $w_2(\nu M) = w_1(M)^2$. We thus have proved the following

Lemma 2. *Let M^n be a nonorientable manifold with fundamental group $\mathbb{Z}/2$. Then M has w_2-type* I *if and only if M does not admit a $Pin^{\pm}$-structure. Furthermore, M has w_2-type* II *if and only if M admits a Pin^{+}-structure and M has w_2-type* III *if and only if M admits a Pin^{-}-structure.*

Note that a $Pin^{\pm}$-structure $\overline{\nu}\colon M \to BPin^{\pm}$ is automatically a 2-equivalence because $\pi_2(BPin^{\pm}) = 0$ and $\overline{\nu}$ induces an isomorphism on π_1 since $p^{\pm}$ and $\nu\colon M \to BO$ both do. Also, there are exactly two $Pin^{\pm}$-structures if $\overline{\nu}$ exists, namely $\overline{\nu}$ and $\overline{\nu}+w_1 M$. Here the addition denotes the action of $H^1(M; \mathbb{Z}/2)$ on its affine space of $Pin^{\pm}$-structures.

We now turn to Pin^c-structures. We defined $Pin^c = Pin^+ \times U(1)/\langle(-1, -1)\rangle$, so there is a natural projection $\pi^+\colon Pin^c \to U(1)$ with kernel Pin^+. It is easy to see that $Pin^c \cong Pin^- \times U(1)/\langle(-1, -1)\rangle$ which gives a projection $\pi^-\colon Pin^c \to U(1)$ with kernel Pin^-. In the introduction we talked about a complex line bundle with first Chern class c_Φ induced by a Pin^c-structure $\Phi = \overline{c}_E\colon X \to BPin^c$ on some vector bundle E over X. We point out that this complex line bundle is constructed via the projection π^+. Since $\pi^{\pm}$ induce isomorphisms on π_2, we see that the map $\pi_2(\Phi)\colon \pi_2 X \to \pi_2(BPin^c) = \mathbb{Z}_+$ is an evaluation on c_Φ.

Lemma 3. *If Φ is a Pin^c-structure on a vector bundle E over X then $c_\Phi \equiv w_2(E) \pmod 2$. Moreover, if $x \in H^2(X; \mathbb{Z})$ is given then $c_{(\Phi+x)} = c_\Phi + 2x$, where the addition on the left-hand side denotes the action of $H^2(M; \mathbb{Z})$ on its affine space of Pin^c-structures.*

Proof. The first claim is easily checked in the universal case and the second follows directly from the following commutative diagram (μ the multiplication maps):

$$\begin{tikzcd} U(1) \times Pin^c \arrow[r, "\mu"] \arrow[d, "\cdot 2 \times \pi_+"'] & Pin^c \arrow[d, "\pi_+"] \\ U(1) \times U(1) \arrow[r, "\mu"] & U(1) \end{tikzcd}$$

□

We now come back to manifolds with fundamental group $\mathbb{Z}/2$.

Lemma 4. *Let $(M, \overline{\nu})$ be a nonorientable Pin^c-manifold with fundamental group $\mathbb{Z}/2$. Then $\pi_1(\overline{\nu})$ is an isomorphism and $c_{\overline{\nu}}$ is a primitive cohomology class if and only if $\pi_2(\overline{\nu})$ is surjective.*

The proof of this lemma is as easy as the former ones, so we skip it. The above discussions apply to smooth manifolds, in the topological case we only have a stable normal Gauss-map $\nu\colon M \to BTop$. But since the natural map $q\colon BO \to BTop$ is a 3-equivalence, the above three fibrations $p^\dagger\colon BPin^\dagger \to BO$ are in fact pullbacks of fibrations $BTopPin^\dagger \to BTop$ via q. In other words, there are topological groups $TopPin^\dagger$, $\dagger \in \{c, +, -\}$, constructed from the group Top by the same central extensions used for the construction of $Pin^\dagger$ from the group O. Also, the obstruction theoretic considerations (Lemmas 1–4) do not change due to the fact that q is a 3-equivalence. For example, we can deduce from Lemma 1 that a 4-dimensional closed nonorientable manifold M with fundamental group $\mathbb{Z}/2$ always admits a $TopPin^c$-structure because by Poincaré-duality

$$H^3(M; \mathbb{Z}_+) \cong H_1(M; \mathbb{Z}_-) \cong H_1(\mathbb{Z}/2; \mathbb{Z}_-) = 0.$$

Next we list the necessary facts on the topological bordism groups we need (for the definition of B-bordism groups compare [14]):

B	Ω_4^B	invariants	generators
$BTopPin^c$	$\mathbb{Z}/2 \times \mathbb{Z}/8 \times \mathbb{Z}/2$	(KS, arf, w_4)	$E_8, \mathbb{RP}^4, \mathbb{CP}^2$
$BTopPin^+$	$\mathbb{Z}/2$	KS	E_8
$BTopPin^-$	$\mathbb{Z}/2 \times \mathbb{Z}/8$	(KS, arf)	$E_8, \mathbb{RP}^4$

The B-structures on the generators are *standard* in the following sense: Firstly, any $TopSpin$-structure (e.g., the unique $TopSpin$-structure on E_8) induces a $TopSpin^\dagger$-structure since there are canonical homomorphisms $TopSpin \to TopPin^\dagger$. Secondly, by Lemma 1, $\mathbb{CP}^2$ has a priori two primitive Pin^c-structures but we will not distinguish these since complex conjugation induces a diffeomorphism between them. Finally, by Lemma 2, $\mathbb{RP}^4$ has exactly two Pin^--structures. But one is the inverse of the other in the group $\Omega_4^{TopPin^-}$. This follows from the following general fact, compare [12, p. 190]: If $(M, \overline{\nu})$ represents an element in the bordism group $\Omega_*^{Pin^\pm}$ then the inverse element is represented by $(M, \overline{\nu} + w_1 M)$. This also implies that $[E_8]$ has order 2 in the above bordism groups $\Omega_4^{TopPin^\dagger}$ and therefore splits the Kirby-Siebenmann invariant in all three cases. Moreover, by construction we have the following relation in $\Omega_4^{TopPin^\pm}$:

$$[*M, \Phi_{*M}] = \pm[M, \Phi_M] + [E_8].$$

Convention. Without mentioning, we always mean the standard $TopPin^\dagger$-structures on the above manifolds.

The computations for $\Omega_4^{TopPin^\pm}$ can be found in [12, §9] but note that the rôles of Pin^+- and Pin^--bordism are reversed in that paper because the authors consider tangential structures whereas we look at the normal Gauss-maps. The extensions

$$1 \to TopPin^\pm \xrightarrow{i^\pm} TopPin^c \xrightarrow{\pi^\pm} U(1) \to 1$$

induce Gysin-sequences (compare [14, pp. 354, 218])

$$\cdots \to \Omega_4^{TopPin^\pm} \xrightarrow{i_*\pm} \Omega_4^{TopPin^c} \xrightarrow{\bigcap c} \Omega_2^{TopPin^\pm}(BU(1)) \to \Omega_3^{TopPin^\pm} \to \cdots.$$

Here $\bigcap c$ is the homomorphism obtained by taking the transverse preimage (see [7, §9]) of $\mathbb{CP}^{N-1}$ under a map $M \to \mathbb{CP}^N$ coming from a $TopPin^c$-structure on M. Let arf: $\Omega_4^{TopPin^c} \to \mathbb{Z}/8$ be the composition

$$\Omega_4^{TopPin^c} \xrightarrow{\bigcap c} \Omega_2^{TopPin^+}(BU(1)) \xrightarrow{f} \Omega_2^{TopPin^+} \underset{\cong}{\xrightarrow{\alpha}} \mathbb{Z}/8,$$

where f forgets the map into $BU(1)$ and α is the 2-dimensional Brown-Arf invariant. Since $\Omega_3^{TopPin^+} = 0$ and $\Omega_2^{TopPin^+}(BU(1)) \cong \mathbb{Z}/8 \times \mathbb{Z}/2$, our result for $\Omega_4^{TopPin^c}$ listed above follows. Note that $\operatorname{arf}(E_8) = \operatorname{arf}(\mathbb{CP}^2) = 0$ and $\operatorname{arf}(\mathbb{RP}^4) = \pm 1$ and thus we obtain the following relation in $\Omega_4^{TopPin^c}$:

$$\operatorname{arf} \equiv w_1^4 \pmod 2.$$

The homomorphism arf: $\Omega_4^{TopPin^-} \to \mathbb{Z}/8$ in our table abbreviates the composition $\Omega_4^{TopPin^-} \xrightarrow{i_*^-} \Omega_4^{TopPin^c} \xrightarrow{\operatorname{arf}} \mathbb{Z}/8$. It will become important to know that this homomorphism is in fact an ordinary Brown-Arf invariant, i.e., comes from a quadratic refinement of the $\mathbb{Z}/2$-intersection form on the second cohomology of a 4-dimensional topological $TopPin^-$-manifold. Such an invariant exists by [3, Proposition (1.9)] since

$$\chi(\mathrm{Sq}^3)(u) = \mathrm{Sq}^2\,\mathrm{Sq}^1(u) = 0$$

where $u \in H^0(MTopPin^-; \mathbb{Z}/2)$ is the Thom class and χ is the canonical antiautomorphism of the Steenrod algebra. It agrees with the above composition since it agrees on the generators E_8 and $\mathbb{RP}^4$ by [3, Example (1.29) and Theorem (1.20)(iv)]. Arguing again with the generators shows that in $\Omega_4^{TopPin^-}$ one has the following relation:

$$\operatorname{arf} \equiv w_4 \pmod 2.$$

3. The topological classification

Since the proof of Theorem 2 will make essential use of the manifolds listed in Theorem 3, we first give a discussion of this list. We begin by noting that our operation $\#_{S^1}$ is $TopPin^-$-bordant to the disjoint union of two $TopPin^-$-manifolds. Thus for w_2-type III, every $TopPin^-$-bordism class appears in our list. For w_2-type II this is clear anyway and for w_2-type I it follows from the next lemma by noting that the connected sum with $\mathbb{CP}^2$ does not change w_1^4 and arf but it changes w_4.

Lemma 5. *Let $z \in H^2(\mathbb{CP}^2; \mathbb{Z})$ and $s \in H^2(S^2; \mathbb{Z})$ be generators. Then for all positive integers n and n_i one has*

(1) $\operatorname{arf}(\mathbb{CP}^2, \Phi_{\mathbb{CP}^2} + n \cdot z) = 2(n^2 + n)$.
(2) $\operatorname{arf}(S^2 \times S^2, \Phi_{S^2 \times S^2} + (n_1 \cdot p_1^*(s) + n_2 \cdot p_2^*(s))) = 4n_1 n_2$.
(3) $\operatorname{arf}(\mathbb{RP}^2 \times S^2, \Phi^+_{\mathbb{RP}^2 \times S^2} + n \cdot p_2^*(s)) = \pm 2n$.

Here $\Phi^+_{\mathbb{RP}^2 \times S^2}$ denotes any Pin^+-structure on $\mathbb{RP}^2 \times S^2$ and $\Phi_{S^2 \times S^2}$ is the Spin-structure on $S^2 \times S^2$. (Both structures induce Pin^c-structures which are needed for the arf*-invariant.)*

Proof. The first two formulas follow from the following generalization of Rohlin's theorem, compare [12, Theorem 6.3]:

$$2 \cdot \alpha(F) = F \circ F - \sigma(M) \mod 16.$$

Here F is a surface in a smooth oriented 4-manifold M which is dual to w_2M. F obtains a Pin^+-structure from a $Spin$-structure on $M\backslash F$ and thus the Brown-Arf invariant $\alpha(F) \in \mathbb{Z}/8$ is defined. If F is the transverse preimage of $\mathbb{CP}^{N-1}$ under a map $M \to \mathbb{CP}^N$ coming from a $Spin^c$-structure Φ on M then by definition $\operatorname{arf}(M, \Phi) = \alpha(F)$.

To obtain formula (1), just apply the above equation to $(M, \Phi) = (\mathbb{CP}^2, \Phi_{\mathbb{CP}^2} + n \cdot z)$, in other words $(M, F) = (\mathbb{CP}^2, (2n+1) \cdot \mathbb{CP}^1)$. For formula (2) take $M = S^2 \times S^2$ and F any surface representing the Poincaré dual to $2(n_1 \cdot p_1^*(s) + n_2 \cdot p_2^*(s))$.

Formula (3) is certainly true for $n = 0$ since $(\mathbb{RP}^2 \times S^2, \Phi_{\mathbb{RP}^2 \times S^2})$ is zero-bordant. It also follows from the fact that a Pin^+-structure on $\mathbb{RP}^2 \times S^2$ induces a null-homotopic map $\mathbb{RP}^2 \times S^2 \to \mathbb{CP}^N$ and thus the transversal inverse image of $\mathbb{CP}^{N-1}$ is zero-bordant. Changing this Pin^c-structure by the element $n \cdot p_2^*(s)$ changes by Lemma 3 this map to the composition

$$\mathbb{RP}^2 \times S^2 \xrightarrow{p_2} S^2 \xrightarrow{2n} S^2 = \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N .$$

This shows that the transversal preimage of $\mathbb{CP}^{N-1}$ in this case is $2n \cdot \mathbb{RP}^2$ and our formula follows from $\alpha(\mathbb{RP}^2) = \pm 1$. □

The manifold $*\mathbb{RP}^4 \# CH$ with its primitive $TopPin^c$-structure has arf-invariant ± 3. This follows by applying the topological version of the equation already mentioned in the above proof (compare [12, Corollary 9.3]) to $M = CH$:

$$\sigma(M) = F \circ F - 2 \cdot \alpha(F) + 8 \cdot KS(M) \mod 16 .$$

The following result will be used for the proof of Theorem 3.

Proposition 1. *Let M be a closed nonorientable topological 4-manifold with fundamental group $\mathbb{Z}/2$. Then the list of Theorem 3 contains exactly one manifold R of the same w_2-type and Euler characteristic as M which is $TopPin^\dagger$-bordant to M for some primitive $TopPin^\dagger$-structures on M and R. Here $\dagger \in \{c, +, -\} \leftrightarrow \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ encodes the w_2-type of M and R.*

Proof. For the arguments below we can assume that $KS(M) = 0$ and we choose a fixed primitive $TopPin^\dagger$-structure on M. We first recall that by Poincaré-duality we always have

$$(*) \qquad \dim_{\mathbb{Z}/2} H^2(M; \mathbb{Z}/2) = e(M) \geq 1$$

and that $e \equiv w_4 \pmod 2$ is a bordism invariant. For our proof, we have to consider three different cases corresponding to the possible w_2-types of M, the easiest being

w_2-type II. Then our model R is $\mathbb{RP}^2 \times S^2 \# k \cdot (S^2 \times S^2)$ and it is clear that since M is $TopPin^+$-zerobordant (and thus $e(M)$ is even), there exists a unique $k \in \mathbb{N}_0$ such that R satisfies the desired properties.

w_2-type III. It is clear that there exist a model R from our list with w_2-type III which is $TopPin^-$-bordant to M. Let R_0 be the corresponding model with $k = 0$, i.e., $R = R_0 \# k \cdot (S^2 \times S^2)$. Then also R_0 and M are $TopPin^-$-bordant and we have to show that we can choose $k \in \mathbb{N}_0$ such that $e(R) = e(M)$. This will certainly follow once we show that the model R_0 has minimal Euler characteristic in its $TopPin^-$-bordism class.

1. *Case.* $w_1(R_0)^4 \neq 0$, i.e., $R_0 = \#_{S^1} \cdot \mathbb{RP}^4$, $r = 1, 3$. For $r = 1$, the minimality is clear, for $r = 3$ assume $e(R_0) = 3$ was not minimal. Then there existed a $TopPin^-$-manifold R_0' with $e(R_0') = 1$. But this means that the $\mathbb{Z}/2$-cohomology ring of R_0' is isomorphic to the one of $\mathbb{RP}^4$ and thus by [3, Theorem (1.20)(iv)] it follows that $\operatorname{arf}(R_0') = \pm 1 \neq \operatorname{arf}(R_0)$.

2. *Case.* $w_1(R_0)^4 = 0$. By the inequality $(*)$, the only critical case is $R_0 = \#_{S^1} 4 \cdot \mathbb{RP}^4$. If $4 = e(R_0)$ was not minimal, there existed a $TopPin^-$-manifold R_0' with $\operatorname{arf}(R_0') = 4$ and $e(R_0') = 2$. Since by assumption we have

$$0 \neq w_1(R_0')^2 = w_2(\nu R_0') = \nu_2(R_0') \qquad (v_2 \text{ the second Wu-class})$$

the intersection form on $H^2(R_0'; \mathbb{Z}/2)$ is odd and thus has in some basis the intersection matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. As above, one has $\operatorname{arf}((1)) = \pm 1$ and thus by additivity $\operatorname{arf}(R_0') \in \{0, \pm 2\}$ which is again a contradiction.

<u>w_2-type I</u>. Then $e(M) \geq 2$ because $e(M) = 1$ would imply that the degree 1 map $\widetilde{M} \to S^4$, obtained by pinching off the complement of a coordinate neighbourhood, is a homotopy equivalence and thus $w_2(\widetilde{M}) = 0$, a contradiction to w_2-type I.

1. *Case.* $w_1(M)^4 \neq 0$, i.e., $\operatorname{arf}(M) \in \{\pm 1, \pm 3\}$. Let

$$R_0 := \begin{cases} \mathbb{RP}^4 \# \mathbb{CP}^2 & \text{if } \operatorname{arf}(M) = \pm 1, \\ *\mathbb{RP}^4 \# CH & \text{if } \operatorname{arf}(M) = \pm 3. \end{cases}$$

By the computation of the arf invariant before Proposition 1 and the computation of the corresponding bordism group at the end of §2 there is a primitive $TopPin^c$-structure on R_0 such that $\operatorname{arf}(R_0) = \operatorname{arf}(M)$. We can deduce that $e(M)$ determines unique numbers $k \in \mathbb{N}_0$ and $l \in \{0, 1\}$ such that

$$R := R_0 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$$

with the standard $TopPin^c$-structure is $TopPin^c$-bordant to M with $e(R) = e(M)$. If l or k are not zero then our list in Theorem 3 contains only the manifold

$$R' := \mathbb{RP}^4 \# \mathbb{CP}^2 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2).$$

Therefore, we have to show that we can choose a primitive Pin^c-structure Φ' on R' such that $\operatorname{arf}(R', \Phi') = \pm 3$. For this we take the primitive Pin^c-structure Φ_0 on $\mathbb{RP}^4 \# \mathbb{CP}^2$ and note that then $\Phi_0 \# \Phi$ is primitive for any Pin^c-structure Φ on $l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$. By Lemma 5 Φ can be chosen so that

$$\operatorname{arf}(l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2), \Phi) = 4$$

which proves our claim.

2. *Case.* $w_1(M)^4 = 0$, i.e., $\operatorname{arf}(M) \in \{0, \pm 2, 4\}$. We first show that $e(M) = 2$ is impossible: $w_2(\widetilde{M}) \neq 0$ implies that $w_2(\nu M) = v_2(M)$ is not a torsion class. If $e(M) = 2$ then w_1^2, v_2 would thus form a $\mathbb{Z}/2$-base for $H^2(M; \mathbb{Z}/2)$ with $0 = w_1^4 = w_1^2 \cdot v_2$ which contradicts the unimodularity of the $\mathbb{Z}/2$-intersection form on M. Let

$$(R_0, \Phi_{R_0}) := (\mathbb{RP}^2 \times S^2 \# \mathbb{CP}^2, (\Phi^+_{\mathbb{RP}^2 \times S^2} + \operatorname{arf}(M)/2 \cdot s) \# \Phi_{\mathbb{CP}^2}).$$

By Lemma 5 one has $\operatorname{arf}(R_0) = \pm \operatorname{arf}(M)$. Since $e(R_0) = 3$, there exist unique numbers $k \in \mathbb{N}_0$ and $l \in \{0, 1\}$ such that

$$R := R_0 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$$

with the standard $TopPin^c$-structure is $TopPin^c$-bordant to M with $e(R) = e(M)$. □

To obtain our classification theorem from the above proposition, we have to make use of the surgery theory developed in [13]. We recall the basic definitions.

Let $p: B \to BTop$ be a fibration and $\overline{\nu}_i: M_i^4 \to B$, $i = 1, 2$, a normal 1-smoothing in B, i.e., $\overline{\nu}$ is a lift of the normal Gauss map ν and a 2-equivalence. Suppose that $(M_1, \overline{\nu}_1)$ and $(M_2, \overline{\nu}_2)$ are bordant in Ω_4^B. Let $(W, \overline{\nu})$ be a B-bordism between $(M_1, \overline{\nu}_1)$ and $(M_2, \overline{\nu}_2)$ and suppose $e(M_1) = e(M_2)$. Then there is an obstruction $\theta(W, M_1) \in l_5^s(\pi_1(B), w_1)$ which is *zero bordant* if and only if W is bordant rel. boundary to an s-cobordism, implying that M_1 and M_2 are homeomorphic if π_1 is good in the sense of [6].

$l_5^s(\pi_1, w_1)$ consists of stable equivalence classes of pairs $\theta = (H(\Lambda^r), V)$, where $H(\Lambda^r)$ is the standard quadratic hyperbolic form on Λ^{2r}, $\Lambda = \mathbb{Z}[\pi_1]$, and $V \subset \Lambda^{2r}$ is a based free direct summand of rank r. More precisely, we stabilize by orthogonal sum with $(H(\Lambda^s), \Lambda^s \times 0)$ and identify $(H(\Lambda^r), V)$ with $(H(\Lambda^r), A(V))$, where A is any element of $RSU(\Lambda) = \lim RSU(\Lambda^r)$, $RSU(\Lambda^r)$ the subgroup of the isometry group of $H(\Lambda^r)$ generated by the flip (interchanging two standard base elements in a hyperbolic plane) and isometries with determinant 1 in $\mathrm{Wh}(\pi_1)$ which preserve $\Lambda^r \times 0$ and, restricted to $\Lambda^r \times 0$, have also determinant 1.

Recall that the ordinary L-group $L_5^s(\pi_1, w_1)$ is the subgroup of the monoid $l_5^s(\pi_1, w_1)$ of classes $[H(\Lambda^r), V]$ with $V \subset V^\perp$ and the quadratic refinement μ vanishes on V. Here $\theta \in l_5^s(\pi_1, w_1)$ is called *zero bordant* if it has a representative $(H(\Lambda^r), V)$ such that $V \oplus \Lambda^r \times \{0\} = \Lambda^{2r}$ and the basis of V together with the standard basis of $\Lambda^r \times \{0\}$ is equivalent to the standard basis of Λ^{2r}. Since we are only interested in the case $\pi_1 = \mathbb{Z}/2$ where the Whitehead group vanishes, we will forget the bases from now on. Let $K\pi_2(M)$ denote the Λ-valued quadratic form given by intersection and self-intersection numbers on $\ker(\overline{\nu}^*: \pi_2(M) \to \pi_2(B))$ and let $\theta(W, M_1) \in l_5(\pi_1, w_1)$ be represented by $[H(\Lambda^r), V]$.

Proposition 2. *The $l_5(\pi_1, w_1)$-obstructions satisfy the following properties*:

(1) *There is a surjective isometry of quadratic forms $V \to K\pi_2(M_1)$.*
(2) $\theta(W, M_2) = [H(\Lambda^r), V^\perp]$.
(3) *Let $K\pi_2(M_1) = K_1 \perp K_2$ and $V_i \to K_i$ be surjections from free Λ-modules V_i. Then there are half rank embeddings $V_1 \subset H(\Lambda^s)$ and $V_2 \subset H(\Lambda^t)$ and $\vartheta \in L_5(\pi_1, w_1)$ such that $\theta(W, M_1)$ is equivalent to $(H(\Lambda^s), V_1) \perp (H(\Lambda^t), V_2) \perp \vartheta$ in $l_5(\pi_1, w_1)$.*
(4) *If $L_5(\pi_1, w_1) = 0$ and V possesses a hamiltonian complement in $H(\Lambda^r)$, i.e., a based submodule $U \subset H(\Lambda^r)$ such that $U = U^\perp$, the quadratic refinement μ is zero on U and $V \oplus U = \Lambda^{2r}$, then $[H(\Lambda^r), V]$ is zero bordant.*

Proof. The first two properties follow directly from the construction of $\theta(w, M_i)$ [13]. Property (3) is a purely algebraic consequence of the second property. For convenience we indicate the proof. We first consider the special case $V_2 = 0$. Starting with the surjection $p: V \to K\pi_2(M_1)$ and $p_1: V_1 \to K\pi_2(M_1)$ one constructs a commutative diagram

$$\begin{tikzcd} V \oplus V_1 \arrow[rr, "\cong", "\Phi"'] \arrow[dr, "(p+0)"'] & & V_1 \oplus V \arrow[dl, "(p_1+0)"] \\ & K\pi_2(M_1) & \end{tikzcd}$$

Pull the form on $K\pi_2(M_1)$ back by $(p+0)$ and (p_1+0) to obtain forms such that Φ is an isometry. Now

$$[H(\Lambda^r), V] = [H(\Lambda^r), V] \perp [H(V_1), V_1] = [H(\Lambda^r) \perp H(V_1), \Phi^{-1}(V_1 \oplus V)]$$

which, since the quadratic form vanishes on $\Phi^{-1}(V)$, is equivalent to $(H(\Lambda^s), V_1) \perp \vartheta$ for some isometric embedding V_1 into Λ^{2s} and some ϑ in $L_5^s(\pi_1, w_1)$.

For the proof of the general case consider the surjection $V_1 \oplus V_2 \to K_1 \perp K_2 = K$, and note that under the pull-back form the summands V_i are orthogonal. Show by elementary base changes that $(H(\Lambda^{s+t}), V_1 \perp V_2)$ is equivalent to $(H(\Lambda^s), V_1) \perp (H(\Lambda^t), V_2)$ for appropriate embeddings of V_1 and V_2. Together with the special case this proves part (3).

To prove part (4) assume that V has a hamiltonian complement U. The nonsingularity of the hyperbolic form implies that there is an isometry which maps U onto the standard lagrangian $\Lambda^r \times \{0\}$. Since $L_5 = 0$, this can be stably done within the subgroup of all isometries which do not change the equivalence class of $[H(\Lambda^r), V] \in l_5$. It follows that for some $s \geq r$ we have

$$[H(\Lambda^r), V] = [H(\Lambda^s), V'] \quad \text{and} \quad \Lambda^{2s} = \Lambda^s \times \{0\} \oplus V'$$

and thus $[H(\Lambda^r), V]$ is zero bordant. $\square$

Before we state the next proposition, we agree that in the following (B, p) *always denotes one of the fibrations* $(BTopPin^\dagger, p^\dagger)$ the superscript $\dagger \in \{c, +, -\}$ determined by the w_2-type of the manifold in question.

Proposition 3. *Let M_1 and M_2 be two closed nonorientable topological 4-manifolds with fundamental group $\mathbb{Z}/2$ and the same w_2-type and Euler characteristic. Then M_1 and M_2 are homeomorphic if and only if there exist normal 1-smoothings $\overline{\nu}_{M_i}\colon M_i \to B$ which are B-cobordant.*

Proof. We first recall from §2 (in particular Lemma 4) that by definition of B a normal 1-smoothing $\overline{\nu}_{M_i}\colon M_i \to B$ is the same as a primitive $TopPin^\dagger$-structure on M_i. By Proposition 1, we can assume that $M_1 =: R$ is one of the manifolds in the list of Theorem 3. Set $M := M_2$. Again, we have to consider three different cases corresponding to the possible w_2-types of M, the easiest being

$\underline{w_2\text{-type II}}$. Then our model R is either $\mathbb{RP}^2 \times S^2 \# k \cdot (S^2 \times S^2)$ or $*(\mathbb{RP}^2 \times S^2) \# k \cdot (S^2 \times S^2)$ depending on $KS(M)$ and $e(M)$. As described above, there is an l-obstruction for making a B-cobordism between R and M into an h-cobordism via surgery. This l-obstruction can be read off from either end of the B-bordism and we clearly prefer to work with the model R. Since $\pi_2 B = 0$, the (self-)intersection form on $K\pi_2(R)$ equals the (self-)intersection form on $\pi_2(R) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_- \oplus \Lambda^{2k}$ which is $\overline{H}(\mathbb{Z}_-) \perp k \cdot H(\Lambda)$. Here the intersection form λ on $\mathbb{Z}_- \oplus \mathbb{Z}_+$ takes the values $\lambda(v_\pm, v_\pm) = 0$, $\lambda(v_-, v_+) = 1 - \tau$

(τ the nontrivial element in $\mathbb{Z}/2$), if $v_\pm$ generate the $\mathbb{Z}_\pm$ summands. The self-intersections μ are given by $\mu(v_-) = \tau$, $\mu(v_+) = 0$. (The hyperbolic form $H(\mathbb{Z}_-)$ would satisfy $\mu(v_\pm) = 0$ and therefore we write $\overline{H}(\mathbb{Z}_-)$.) To compute the l-obstruction we apply Proposition 2(3) which shows that it is additive under orthogonal sum since $L_5(\mathbb{Z}/2, -) = 0$ [15, §13A]. Then parts (a) and (b) of the following proposition (which will be proved in §4) show that the l-obstruction is zero bordant and thus R and M are homeomorphic.

Proposition 4. *$(V, \lambda, \mu) \in l_5(\mathbb{Z}/2, -)$ is zero bordant if one of the following conditions is satisfied:*

(a) *V is freely generated by v_+, v_- with $\lambda(v_\pm, v_\pm) = 0$, $\lambda(v_-, v_+) = 1 - \tau$, and at least one of the two elements $\mu(v_-)$, $\mu(v_+)$ vanishes.*
(b) *(V, λ) is a hyperbolic form on Λ^2.*
(c) *(V, λ) is the zero form on Λ.*

We proceed in the proof of Proposition 3 by assuming that M has $\underline{w_2\text{-type III}}$. Then our model R is one of the models described in Theorem 3(III). As for w_2-type II, we want to show that the l-obstruction for a B-bordism between R and M is zero bordant. Again the (self-)intersection form on $K\pi_2(R)$ is the one on $\pi_2(R)$ since $\pi_2 B = 0$. Let R_0 be defined as in the proof of Proposition 1. Then

$$(\pi_2(R), \lambda, \mu) \cong (\pi_2(R_0), \lambda, \mu) \perp k \cdot H(\Lambda)$$

and by Proposition 2(3) and Proposition 4(b) we can work with R_0. By Lemma 6 below, we see that we are finished using Proposition 4(a), (b).

For the proof of Proposition 3, the last case is that M has $\underline{w_2\text{-type I}}$. The following line of argument does not change if we replace M by $*M$ and thus we assume $KS(M) = 0$.

1. *Case.* $w_1(M)^4 \neq 0$, i.e., $\operatorname{arf}(M) \in \{\pm 1, \pm 3\}$. Then

$$R_0 := \begin{cases} \mathbb{RP}^4 \# \mathbb{CP}^2 & \text{if } \operatorname{arf}(M) = \pm 1, \\ *\mathbb{RP}^4 \# CH & \text{if } \operatorname{arf}(M) = \pm 3, \end{cases}$$

and $R := R_0 \#(l + 2k) \cdot \mathbb{CP}^2 \cong R_0 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$.

The l-obstruction can thus be computed by resolving

$$K\pi_2(R) = \begin{cases} (\mathbb{Z}_-, 0, \mu) \perp k \cdot H(\Lambda) & \text{if } l = 0, \\ (\Lambda \oplus \mathbb{Z}_-, \lambda(\mathbb{CP}^2 \# \mathbb{CP}^2)|_{\Lambda \oplus \mathbb{Z}_-}, \mu) \perp k \cdot H(\Lambda) & \text{if } l = 1. \end{cases}$$

From Proposition 4 we then see that this obstruction is zero bordant: Use parts (b), (c) for $l = 0$ and parts (a), (b) for $l = 1$. It follows in particular that for l or k not zero R is homeomorphic to $\mathbb{RP}^4 \# \mathbb{CP}^2 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$. This means that the choices of different $TopPin^c$-structures destroy the arf-invariant except for Euler characteristic 2 where it still can take two values.

2. *Case.* $w_1(M)^4 = 0$, i.e., $\operatorname{arf}(M) \in \{0, \pm 2, 4\}$. Let

$$R_0 := \begin{cases} \mathbb{RP}^2 \times S^2 \# \mathbb{CP}^2 & \text{if } \operatorname{arf}(M) = 0, \\ \#_{S^1} 2 \cdot \mathbb{RP}^4 \# \mathbb{CP}^2 & \text{if } \operatorname{arf}(M) = \pm 2, \\ *(\mathbb{RP}^2 \times S^2) \# CH & \text{if } \operatorname{arf}(M) = 4, \end{cases}$$

and note that with the standard $TopPin^c$-structures one has $\operatorname{arf}(R_0) = \operatorname{arf}(M)$. Again we have

$$R := R_0 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2).$$

Now we can finish the proof exactly as in the first case by applying Proposition 4 to a free resolution of $K\pi_2(R)$. Moreover, since $\mathbb{RP}^2 \times S^2 \# \mathbb{CP}^2$ has primitive Pin^c-structures with arf-invariant 0, ± 2, and 4, R is homeomorphic to $\mathbb{RP}^2 \times S^2 \# l \cdot \mathbb{CP}^2 \# k \cdot (S^2 \times S^2)$, independently of the arf-invariant. □

Propositions 1 and 4 together imply our Theorems 2 and 3 from the introduction.

Lemma 6. *Let M^4 be a nonorientable manifold with fundamental group $\mathbb{Z}/2$ and w_2-type* III. *Then the hermitian intersection form on $\pi_2 M$ is either hyperbolic or isometric to $H(\mathbb{Z}_-) \perp H(\Lambda^r)$. Moreover, at least one of the two elements $\mu(v_-)$, $\mu(v_+)$ vanishes. (Here $v_\mp$ generate $\mathbb{Z}_- \oplus \mathbb{Z}_+$.)*

Proof. By [16] $\pi_2 M$ is either free or isomorphic to $\mathbb{Z}_+ \oplus \mathbb{Z}_- \oplus F$, where F is a free module. The intersection form is either a unimodular form S on F or it splits as $H(\mathbb{Z}_-) \perp S$, see [10, §6]. Since S admits a quadratic refinement it represents an element in $L_4(\mathbb{Z}/2, -)$. Recall from [15, §13A] that the Arf-invariant induces an isomorphism Arf: $L_4(\mathbb{Z}/2, -) \to \mathbb{Z}/2$. Thus as a hermitian form, S is stably hyperbolic. It is easy to prove that this implies that S is hyperbolic.

To prove the last statement of the lemma, we have to show that the Arf-invariant coming from the self-intersections μ on $\pi_2 M \otimes_{\mathbb{Z}} \mathbb{Z}/2$ vanishes (note that the Arf-invariant on $S \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is zero). This follows since the Arf-invariant is a $TopPin^-$-bordism invariant and it vanishes on the generators. □

4. Proof of Proposition 4

Since $L_5(\mathbb{Z}/2, -) = 0$ [15, §13A], we can use Proposition 2 and have only to show that V possesses a hamiltonian complement in $H(\Lambda^r)$.

ad(c). Let $v \in V \subset H(\Lambda)$ be a free generator such that $\lambda(v, v) = 0$. Then $\mu(v) = 0$ or τ and since V is a direct summand, there exists an element $w \in \Lambda^2$ with $\lambda(v, w) = 1$. By changing w to $w + l \cdot v$, $l \in \Lambda$, we can assume that $\lambda(w, w) = 0$. If $\mu(v) = 0$ then this process also yields $\mu(w) = 0$ and thus $U := \langle w \rangle$ is a hamiltonian complement for V. If $\mu(v) = \tau$ then $\lambda(w, w) = 0$ automatically implies $\mu(w) = 0$, otherwise $H(\Lambda) = \langle v, w \rangle$ would have nontrivial Arf-invariant (in the sense of [15]).

ad(b). If $(V, \lambda) \subset H(\Lambda^2)$ is a hyperbolic form on Λ^2, the orthogonal complement $V^\perp$ is stably hyperbolic and thus hyperbolic as in the proof of Lemma 6. Therefore, we can assume that if e_1, e_2, f_1, f_2 is a hyperbolic base for $H(\Lambda^2)$ then $V = \langle e_1, f_1 \rangle$. Note that we ignored the quadratic refinement μ until now but since $\operatorname{Arf}(H(\Lambda^2)) = 0$, we can assume that either $\mu(e_i) = \mu(f_i) = 0$ or $\mu(e_i) = \mu(f_i) = \tau$. It is now easy to check that $U := \langle e_1 + f_2, f_1 - e_2 \rangle$ is a hamiltonian complement for V.

ad(a). Let

$$H := (\Lambda^4, H(\Lambda^2), \lambda, \mu)$$

be the standard hyperbolic form with hyperbolic generators e_1, e_2, f_1, f_2. Furthermore, let v_-, $v_+ \in H$ be two vectors generating a 2-dimensional free direct summand V in Λ^4 such that $\lambda(v_-, v_+) = (1-\tau)$ and $\mu(v_+) = 0$, $\mu(v_-) = \tau$. We want to show that V possesses a hamiltonian complement in H. (The cases $\mu(v_-) = 0$, $\mu(v_+) = \tau$ and $\mu(v_\pm) = 0$ are analogous but do not occur in the geometry of 4-manifolds, therefore we have decided to present only one case.) The aim of the proof is to move V by isometries of H into a position where one can directly read off a hamiltonian complement. For the convenience of the reader, these positions will be enumerated by the labels (V0) to (V3). Given the vectors

(V0) $$v_+ \quad \text{and} \quad v_- \in \Lambda^4,$$

we first note that since H is nonsingular and v_+ is by assumption Λ-unimodular, there exists a vector $w \in \Lambda^4$ with $\lambda(v_+, w) = 1$. Because $\mu(v_+) = 0$, the vectors v_+ and $w - \bar{l} \cdot v_+$ (with $[l] = \mu(w) \in \Lambda/\{l - \bar{l} | l \in \Lambda\}$) generate a hyperbolic summand $H(\Lambda)$ in H. It follows that we can assume $v_+ = e_1$. If

$$v_- = a_1 \cdot e_1 + a_2 \cdot e_2 + b_1 \cdot f_1 + b_2 \cdot f_2, \qquad a_i, b_i \in \Lambda,$$

the equation $1 - \tau = \lambda(v_-, v_+) = \lambda(b_1 \cdot f_1, e_1) = b_1$ shows that we have

(V1) $$v_+ = e_1 \quad \text{and} \quad v_- = a_1 \cdot e_1 + a_2 \cdot e_2 + (1-\tau) \cdot f_1 + b_2 \cdot f_2.$$

Since V is a direct summand of Λ^4, the vector $a_2 \cdot e_2 + (1-\tau) \cdot f_1 + b_2 \cdot f_2$ is Λ-unimodular, in particular there exist $r, s, t \in \Lambda$ such that

$$r \cdot a_2 + s \cdot (1-\tau) + t \cdot b_2 = -a_1.$$

It is easy to check that the matrices

$$R := \begin{pmatrix} \begin{matrix} 1 & r \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ -\bar{r} & 1 \end{matrix} \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} \mathbb{1} & \begin{matrix} 0 & t \\ -\bar{t} & 0 \end{matrix} \\ 0 & \mathbb{1} \end{pmatrix}$$

define isometries of H fixing e_1. The coefficient of e_1 in the vector $(R \circ T)(v_-)$ then equals

$$\begin{aligned} & a_1 + r \cdot a_2 - r \cdot \bar{t} \cdot (1-\tau) + t \cdot b_2 \\ &= a_1 + \overbrace{r \cdot a_2 + s \cdot (1-\tau) + t \cdot b_2}^{-a_1} - (s + r \cdot \bar{t}) \cdot (1-\tau) \\ &= -(s + r \cdot \bar{t}) \cdot (1-\tau) = \varepsilon_-(-(s + r \cdot \bar{t})) \cdot (1-\tau) \end{aligned}$$

where $\varepsilon_- : \Lambda \to \mathbb{Z}$ sends τ to -1. Putting $k := \varepsilon_-(-(s + r \cdot \bar{t}))$, we can therefore assume that

(V2) $$v_+ = e_1 \quad \text{and} \quad v_- = k(1-\tau) \cdot e_1 + a_2 \cdot e_2 + (1-\tau) \cdot f_1 + b_2 \cdot f_2.$$

The ideal $a_2 \cdot \Lambda + b_2 \cdot \Lambda \subseteq \Lambda$ is a free abelian group of rank ≤ 2. By [4, Chapter 74] it is then isomorphic as a Λ-module to $\mathbb{Z}$, $\mathbb{Z}_-$, $\mathbb{Z} \oplus \mathbb{Z}_-$, or Λ. The first three cases are impossible since the vector $a_2 \cdot e_2 + (1-\tau) \cdot f_1 + b_2 \cdot f_2$ is Λ-unimodular. This can be checked case by case just by using that this unimodularity means that

$$a_2 \cdot \Lambda + b_2 \cdot \Lambda + (1-\tau) \cdot \Lambda = \Lambda.$$

Therefore, $a_2 \cdot \Lambda + b_2 \cdot \Lambda \cong \Lambda$ and thus there exists a nonzero divisor $u \in \Lambda$ such that

$$a_2 \cdot \Lambda + b_2 \cdot \Lambda = u \cdot \Lambda \quad \text{and thus} \quad (1-\tau) \cdot \Lambda + u \cdot \Lambda = \Lambda .$$

From the second equation it follows that there exist $z \in \mathbb{Z}$ and $l \in \Lambda$ such that

$$1 = z(1-\tau) + l \cdot u = z(1-\tau) + (l_1 + l_2 \cdot \tau) \cdot (u_1 + u_2 \cdot \tau)$$

which is equivalent to the equations

$$1 = z + l_1 u_1 + l_2 u_2 , \quad 0 = -z + l_1 u_2 + l_2 u_1 .$$

The addition of these equations leads to

$$1 = (l_1 u_1 + l_2 u_2) + (l_1 u_2 + l_2 u_1) = (l_2 + l_1) \cdot (u_2 + u_1) ,$$

which finally implies that

$$(*) \qquad u_1 = \pm 1 - u_2 , \quad \text{in particular } u \cdot \overline{u} = u_1^2 - u_2^2 = 1 \mp 2u_2 \equiv 1 \mod 2 .$$

Since u is not a zero divisor and $a_2 , b_2 \in u \cdot \Lambda$, there exist $\alpha , \beta \in \Lambda$ such that $a_2 = u \cdot \alpha$, $b_2 = u \cdot \beta$, and $\alpha \cdot \Lambda + \beta \cdot \Lambda = \Lambda$. But since we assumed that

$$\begin{aligned} \tau = \mu(v_-) &= \mu(k(1-\tau) \cdot e_1 + a_2 \cdot e_2 + (1-\tau) \cdot f_1 + b_2 \cdot f_2) \\ &= [\lambda(k(1-\tau) \cdot e_1 , (1-\tau) \cdot f_1) + \lambda(a_2 \cdot e_2 , b_2 \cdot f_2)] \\ &= [(u \cdot \overline{u}) \cdot (\alpha \cdot \overline{\beta})] \end{aligned}$$

we can conclude from $(*)$ that

$$\mu(\alpha \cdot e_2 + \beta \cdot f_2) = [\lambda(\alpha \cdot e_1 , \beta \cdot f_2)] = [\alpha \cdot \overline{\beta}] = \tau$$

and

$$\lambda(\alpha \cdot e_2 + \beta \cdot f_2 , \alpha \cdot e_2 + \beta \cdot f_2) = 0 .$$

By the unimodularity of $\alpha \cdot e_2 + \beta \cdot f_2$ there exists a $y \in e_2 \cdot \Lambda + f_2 \cdot \Lambda$ with

$$\lambda(\alpha \cdot e_2 + \beta \cdot f_2 , y) = 1 \quad \text{and} \quad \lambda(y , y) = 0 .$$

But since $\mathrm{Arf}(H|(e_2 \cdot \Lambda + f_2 \cdot \Lambda)) = 0$, it is necessary that $\mu(y) = 0$. Therefore, there is an isometry φ of $H|(e_2 \cdot \Lambda + f_2 \cdot \Lambda)$ satisfying

$$\varphi(y) = f_2 \quad \text{and} \quad \varphi(\alpha \cdot e_2 + \beta \cdot f_2) = e_2 + l \cdot f_2$$

for some l. We can now assume that

$$(\mathrm{V3}) \qquad v_+ = e_1 \quad \text{and} \quad v_- = k(1-\tau) \cdot e_1 + u \cdot e_2 + (1-\tau) \cdot f_1 + l \cdot u \cdot f_2 .$$

We claim that the two vectors

$$w_+ := -u_2 \cdot e_2 + f_1 \quad \text{and} \quad w_- := u_2 \cdot e_1 + f_2$$

generate a hamiltonian complement U for V in H. The fact that U satisfies $U^\perp \subseteq U$ and $\mu|_U \equiv 0$ is obvious and the easiest way to see that U is complementary to V as in (V3), is to compute the determinant (Λ is a commutative ring!) of the matrix for the column vectors v_+, v_-, w_+, w_-. This matrix is

$$\begin{pmatrix} 1 & k(1-\tau) & 0 & u_2 \\ 0 & (\pm 1 - u_2) + u_2 \cdot \tau & -u_2 & 0 \\ 0 & (1-\tau) & 1 & 0 \\ 0 & l \cdot u & 0 & 1 \end{pmatrix}$$

and it has determinant $(\pm 1 - u_2) + u_2 \cdot \tau + u_2(1-\tau) = \pm 1$. □

Remark. The above proof fails if $\mu(v_\pm) = \tau$. But this is not surprising since if Proposition 4 was true also in this case, by taking $B := BTop$ we could prove that the manifolds $\mathbb{RP}^4 \# \mathbb{CP}^2$ and $*\mathbb{RP}^4 \# CH$ are homeomorphic. It should therefore be possible to derive an invariant in the terms of $l_5(\mathbb{Z}/2, -)$ which distinguishes these two manifolds.

5. A topological formula for the η-invariant

Theorem 4. *Let Φ be a Pin^c-structure on a smooth 4-manifold M. Then*

$$8\eta(M, \Phi) = \pm \operatorname{arf}(M, \Phi).$$

Proof. Since both sides are invariants of Pin^c-bordism classes it is enough to check the formula on generators. As in §2 one sees that the bordism group $\Omega_4^{Pin^c}$ is isomorphic to $\mathbb{Z}/8 \times \mathbb{Z}/2$ via $(\operatorname{arf}, w_4)$, generated by $\mathbb{RP}^4$ and $\mathbb{CP}^2$. [8, Theorem 3.3] proves that

$$\eta(\mathbb{RP}^4) = \pm\tfrac{1}{8} \pmod{\mathbb{Z}}.$$

For orientable Pin^c-manifolds one has the formula

$$\eta(M, \Phi) = \tfrac{1}{16}\langle (c_\Phi^2 - \tfrac{1}{3}p_1 M), [M]\rangle \pmod{\mathbb{Z}}.$$

This follows from the general formula in [8, p. 254]

$$\eta(D) = \tfrac{1}{2} \operatorname{Index} D^+$$

and the index formula [1] Index $D^+ = \langle e^{d/2}\widehat{A}(M), [M]\rangle$.

In particular, for $c_\Phi = (2n+1)z$, z a generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$, we have

$$\eta(\mathbb{CP}^2, \Phi) = \tfrac{1}{4}(n^2+n) \pmod{\mathbb{Z}}.$$

Since the value of the η-invariant and the arf-invariant up to sign agree for the generators $\mathbb{RP}^4$ and $(\mathbb{CP}^2, \Phi)$ with $c_\Phi = z$ (for the arf-invariant see §2 and Lemma 5), we are finished. □

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