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Perturbation of equivariant moduli spaces

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Let X^4 be a smooth, oriented, closed 4-manifold with a smooth action of a finite group π, preserving the orientation. If $P \to X$ is a SU(2)-bundle with $c_2(P) = k$, then the space $\mathscr{A}/\mathscr{G}$ of gauge equivalent classes of connections on P inherits a π-action. We choose a π-equivariant Riemannian metric on X. Then the Yang-Mills functional associated to this metric is invariant with respect to the group action $\mathscr{A}/\mathscr{G}$, and hence the moduli spaces $\mathscr{M}_+$ of self-dual connections or $\mathscr{M}_-$ of anti-self-dual connections, up to gauge equivalence, also have a π-action.

In the work of Donaldson and Uhlenbeck [7, 13], a perturbation theory for these moduli spaces was developed, and from this striking results concerning the diffeomorphism structures of 4-manifolds have been obtained [8, 10]. In the equivariant setting it is known that their perturbation cannot be generalized in a naive manner: an example (see Example 2.15) communicated to us by Fintushel shows that in general there is no generic equivariant metric for which the sub-moduli space $\mathscr{M}^*$ of irreducible connections admits a smooth π-manifold structure of the expected dimension. In constrast Fintushel and Stern prove that the space of π-invariant connections does have good generic properties [15], and Furuta [16] investigated a special case of the relationship between the fixed point set $\mathscr{M}^\pi$ of $\mathscr{M}$ and the set of invariant connections (cf. [19, 14]). This approach was developed further by Braam and Matić [3]. They showed how further information about finite group actions on definite 4-manifolds can be obtained from the $c_2 = -1$ moduli space, particularly if the reducible connections fixed under the group action have neighbourhoods which are cones over linear actions on $P^2(\mathbf{C})$. In the case when π is a cyclic 2-group, this assertion was proved by Cho [4, 5].

The failure of the equivariant space of connections to behave well under perturbations may be seen as a failure of equivariant transversality. In this paper, we construct an equivariant perturbation $(\pi, \mathscr{M})$ of the above moduli space based

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upon the equivariant general position of Bierstone [2] (see Sect. 1). This notation of general position is an appropriate replacement for equivariant transversality: maps in general position form an open and dense subset of the space of equivariant maps, and the preimages of π-submanifolds have natural Whitney stratifications.

Theorem A. *Let* $(\pi, \mathscr{M})$ *denote the equivariant moduli space of (anti-) self-dual connections on P, defined as in* (2.10). *Let* $\mathscr{M}^*$ *denote the subspace in* $\mathscr{M}$ *consisting of irreducible connections. Then* $\mathscr{M}^*$ *has a Whitney stratification with invariant subspaces* $\mathscr{M}^*_{(\pi')}$, $\pi' \subseteq \pi$, *as its strata. For a given conjugacy class* (π') *of subgroups in* π, *the corresponding stratum* $\mathscr{M}^*_{(\pi')}$ *is either empty or a disjoint union of submanifolds whose dimensions are determined by the topological data on* $(\pi'; P \to X)$.

In the statement, $\mathscr{M}^*_{(\pi')}$ denotes the subspace of points with isotropy subgroup conjugate to π', and "Whitney stratification" means that locally each stratum of $\mathscr{M}^*$ is the preimage of certain equivariant minimum Whitney stratification of some affine variety (see [2, p. 456]). We remark that there is a corresponding version of this theorem for equivariant moduli spaces of SO(3) connections.

We give two applications of this moduli space $(\pi, \mathscr{M})$ to smooth finite group actions on positive definite 4-manifolds. In this situation, $(\pi, \mathscr{M})$ can be compactified equivariantly by adding a smooth collar $(\pi, X \times I)$, and we can try to obtain information about (π, X) by studying the stratified moduli space near the isolated reducible connections. The following result was first proved by Edmonds and Ewing [12] using the G-Signature Theorem, number theory and a formidable computer-assisted calculation (see also [17]).

Theorem B. *Let* π *be a cyclic group acting smoothly and semi-freely on the complex projective plane* $P^2(\mathbf{C})$ *with three isolated fixed points. Then the local tangential representations at the fixed points agree with those in some linear action of* π *on* $P^2(\mathbf{C})$, $\pi \subseteq \mathrm{PGL}_3(\mathbf{C})$.

The main step in the proof is to show that $(\pi, \mathscr{M})$ contains a neighbourhood around the reducible connection which is a cone over some linear action on $P^2(\mathbf{C})$.

Theorem C. *Let* π *be a cyclic group of odd order acting smoothly and semi-freely on a simply-connected, positive definite 4-manifold X. If the fixed point set of* (π, X) *contains at most three points, then the action is homeomorphic to a connected sum* $(\pi, \#^n P^2(\mathbf{C}))$ *of linear actions on* $P^2(\mathbf{C})$.

The first part of the proof, determining the local tangential representations at the fixed points, applies more generally if $\pi_1(X)$ has non-trivial representations into SU(2). We then apply the results of [26, 27]. For the special case of actions on $P^2(\mathbf{C})$ the result follows from Theorem B and [17] or [26].

1 Equivariant general position

Let G be a compact Lie group, let M and N denote two smooth G-manifolds and let P be a smooth G-submanifold inside N, $(G, P) \subseteq (G, N)$. A smooth equivariant map $f: (G, M) \to (G, N)$ is said to be *G-transverse to P at a point* $x \in M$ if either

$f(x) \notin P$ or it satisfies the usual transversality condition, i.e.

$$df_x(TM)_x \oplus (TP)_{f(x)} = (TN)_{f(x)} . \tag{1.1}$$

If this equation is satisfied at every point in $f^{-1}(P)$, then f is said to be *G-transverse* to P. Clearly this is a natural generalization of transversality to the equivariant category; however, as shown by the following example, there exist nontrivial obstructions to G-transversality.

Example 1.2. Let us consider the case when M, N, and P are G-vector spaces denoted by $V, U \times W$ and $U \times 0$. Let $f = (f_1, f_2)$ be a smooth equivariant map $f: V \to U \times W$ between these G-vector spaces which sends the origin to the origin, $f(0) = 0$. From (1.1), it is easy to see the condition for f to be transverse to the subspace $U \times 0$ is the same as those for the second factor $f_2: V \to W$ to be transverse with respect to the origin. Furthermore, if we are allowed to deform f by an equivariant homotopy which sends the origin to the origin, then the obstruction to arrive at a position transversal to $U \times 0$ is that W is a subrepresentation of V. Hence an irreducible constituent χ in W appears also in V and with greater multiplicity $\langle \chi, \chi(V) \rangle \geqq \langle \chi, \chi(W) \rangle$. In fact, let $\mathrm{RO}(G)$ denote the real representation ring of G. Let $\mathrm{RO}_+(G)$ denote the positive cone in $\mathrm{RO}(G)$ generated by actual representations of G, i.e.

$$\mathrm{RO}_+(G) = \{ V \in \mathrm{RO}(G) | \langle \chi, \chi(V) \rangle \geqq 0 \text{ for every irreducible } \chi \} .$$

Then condition (1.1) can be formulated as requiring the difference element $[V] - [W]$ to lie in $\mathrm{RO}_+(G)$.

The above example represents, in some sense, the obstruction for G-transversality over the 0-skeleton. Over higher skeletons, there exist a series of obstructions in deforming an equivariant map to a transversal position with respect to an invariant subspace. (see [24] for details). From the existence of these obstructions, one aspect of the problem is clear: in the space $C_G^\infty(M, N)$ of equivariant maps the subspace $C_G^\infty(M, N \cap P)$ of maps transverse to P is neither open nor dense.

A partial remedy for this is the notion of *stratumwise* transversality. A G-manifold M has a natural stratification defined by subspaces of the same orbit type. Given a conjugacy class (H) of subgroups in G, let $M_{(H)}$ denote the submanifold in M consisting of all the points x whose isotropy subgroup G_x belong to (H). Then $M = \coprod_{(H)} M_{(H)}$; and the closure of $M_{(H)}$ is the union $\coprod_{(H') \subseteq (H)} M_{(H')}$ of all those submanifolds $M_{(H')}$ with bigger isotropy subgroups $H' \supseteq H$. A generic equivariant map $f: M \to N$ may send a stratum $M_{(H)}$ into a different stratum $N_{(H')}$ in N. To clarify the situation, we consider two other invariant subspaces associated to a subgroup H in G:

$$\begin{aligned} M^H &= \{x \in M \mid \text{ the isotropy subgroup } G_x \supseteq H\} \\ &= \text{the fixed point submanifold of } H \text{ in } M . \end{aligned} \tag{1.3}$$

$$M_H = M_{(H)} \cap M^H = \{x \in M \mid \text{the isotropy subgroup } G_x = H\} .$$

Then an equivariant map $f: M \to N$ has the property that it sends M_H into N^H, $f(M_H) \subseteq N^H$. It is called *stratumwise transverse* if for every subgroup H, the induced mapping $M_H \to N^H$ is transverse to P^H.

The above notion of stratumwise transversality provides us some control on the topology of the preimage $f^{-1}(P)$ of P. For instance, the dimension of $f^{-1}(P)_H$ can be computed by the formula:

(1.4) $$\dim f^{-1}(P)_H = \dim M_H - \dim \mathrm{N}^H + \dim P^H$$

whenever $f^{-1}(P)$ is nonempty. Another useful fact is that given a smooth equivariant map $f\colon M \to N$, we can perturb this map by an arbitrarily small amount (in the C^∞ topology) to make it stratumwise transverse. In other words, in the function space $C_G^\infty(M, N)$, the subspace $C_G^\infty(M, N \cap \{P^H\})$ of stratumwise transverse maps with respect to $\{P^H\colon H \subseteq G\}$ is dense. The only drawback is that this subspace is not necessarily open [2, Example (2.1)] and therefore a further refinement is needed.

A modification of stratumwise transversality is provided by Bierstone in [2]. To explain Bierstone's idea, we consider first the situation of two G-vector spaces V and W. Let $C_G^\infty(\mathcal{O}(V), W)$ denote the space of germs of C^∞-mappings from invariant open neighbourhoods of $0 \in V$ to W. In an obvious manner, this last space is a module over the ring $C_G^\infty(\mathcal{O}) = C^\infty(\mathcal{O}(V); \mathbf{R})$ of smooth invariant functions. In [2], it is shown that there exists a finite set of polynomial generators $g_1(x), \ldots, g_k(x)$ of this module $C_G^\infty(\mathcal{O}(V), W)$ over $C_G^\infty(\mathcal{O})$. In other words, every element f in $C_G^\infty(\mathcal{O}(V), W)$ can be expressed as a sum

$$f(x) = \sum_{i=1}^{k} h_i(x) \cdot g_i(x)$$

where $h_i(x)$ are smooth invariant functions in $C^\infty(\mathcal{O})$.

Note from the above expression we can also write f as the composite of two functions

$$f(x) = U \circ \operatorname{graph} h(x)$$

where $U\colon V \times \mathbf{R}^k \to \mathbf{R}$ is given by $U(x, h) = \sum_{i=1}^{k} h_i g_i(x)$ and $\operatorname{graph} h(x) = (x, h_1(x), \ldots, h_k(x))$. As the composite of these two functions, the zero set $f^{-1}(0)$ is clearly the intersection of the variety defined by $U(x, h) = 0$ and the graph of h. Hence, in order for $f^{-1}(0)$ to have 'reasonable' behavior, we have to require that these two subspaces $U(x, h) = 0$ and graph h intersect each other in a 'reasonable' manner. An affine G-variety such as $U(x, h) = 0$ has a natural G-stratification called the equivariant minimum Whitney stratification[1].

A manifold in an affine space is said to intersect an affine subvariety transversely if it is transverse to each stratum. This leads us to the following:

Definition 1.5. Let $f\colon V \to W$ be a smooth equivariant map between two G-vector spaces V, W. Then f is said *to be in general position with respect to* $0 \in W$ *at* $0 \in V$ if the graph $\{(x, h_1(x), \ldots, h_k(x))\}$ in $V \times \mathbf{R}^k$ is transverse to the affine algebraic variety $\sum_{i=1}^{k} h_i \cdot g_i(x) = 0$.

In [2], Bierstone proved that (1.5) is well-defined, independent of the choices of the generators $g_i(x)$ and can be extended to cover the situation of general position maps with respect to a subspace.

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[1] Here, we require not only the strata $(V, \mathcal{S})$ to be minimum in the sense of Mather (cf. [22]) but it has to contain the subvarieties $\{V^H, H \subseteq G\}$, as part of the strata

Definition 1.6. Let $f: V \to U \times W$ be a smooth equivariant map between two G-vector spaces, V, $U \times W$. Then f is said to *be in general position with respect to the subspace* $U \times 0 \subset U \times W$ *at* 0 if the projection $Pr_2 \circ f: V \to W$ to the complementary subspace is in general position with respect to $0 \in W$ at $0 \in V$.

In practice all the local considerations can be reduced to (1.6), because of the Slice Theorem.

Definition 1.7. Let $f: M \to N$ be a smooth equivariant map between two G-manifolds, and P a G-submanifold of N, and $x \in f^{-1}(P)$. Then f is *in general position with respect to P at x* if for any slice S of the orbit $G \cdot x$, the G_x-equivariant map $df_x | S: T_x S \to T_{f(x)} N$ is *in general position with respect to* $T_{f(x)} P$ *at* $0 \in T_x S$. A smooth equivariant map $f: M \to N$ is *in general position with respect to a G-submanifold P of N* if it is in general position with respect to P at every point of $f^{-1}(P)$.

This completes our definition of equivariant general position.

Now, to apply (1.7), we need the following properties of general position maps:

(1.8) Equivariant general position implies stratumwise transversality. [2, Proposition (6.4)]. In particular, we can use (1.4) to compute the dimensions of the preimages $f^{-1}(P)_H$, $H \subseteq G$.

(1.9) The preimage $f^{-1}(P)$ has a natural, equivariant, Whitney stratification [2, Proposition (6.5)]. Hence each stratum is a submanifold in M and it has a mapping cone bundle structure in its neighbourhood [22].

(1.10) With respect to the C^∞ topology, the subspace $C^\infty_G(M, N;$ gen. position) of smooth equivariant maps in general position with respect to P is open and dense in $C^\infty_G(M, N)$. [2, Theorems (1.3) and (1.4)].

(1.11) Suppose that V, W are real G representations and $f: V \to W$ a map in G-general position with respect to $0 \in W$. Then if $T = (df)_0$, we can assume that the representations Ker T and Coker T have no non-zero irreducible subrepresentations in common.

Since the proofs of (1.8)–(1.10) can be found in [2], we omit the details. In the situation of (1.11), if Ker T and Coker T have an irreducible subrepresentation U in common, we could perturb f by adding an isomorphism on this subspace to get a new general position map f', removing U from both Ker T and Coker T.

We conclude this section with an example.

Example 1.12. Let C_n be a cyclic group, and let V and W be two complex representation spaces of C_n. Implicit in the complex representation is the action of the circle group $S^1 = \{z \mid |z| = 1\}$ which commutes with the action of the cyclic group C_n. In other words, if we let G denote $C_n \times S^1$ then V and W become natural G-spaces. Our object here is to describe the induced stratification of the zero set $f^{-1}(0)$ for an equivariant general position map $f: V \to W$.

Let χ_k, $0 \leqq k \leqq n-1$, denote the irreducible character of C_n given by the formula $\chi_k(t) = e^{2\pi k\sqrt{-1}/n}$ where t is a fixed generator of C_n. Associated to such a character χ_k, we have a one-dimensional complex representation $\mathbf{C}(\chi_k)$ defined by $t \cdot v = \chi_k(t)v$. Using these representations $\mathbf{C}(\chi_k)$ defined by $t \cdot v = \chi_k(t)v$. Using these representations $\mathbf{C}(\chi_k)$, we can decompose V into a direct sum $V = \bigoplus_{k=0}^{n-1} V(\chi_k)$ of eigenspaces where each $V(\chi_k)$ is isomorphic to a sum of $\langle \chi_k, \chi(V)\rangle$ copies of $\mathbf{C}(\chi_k)$.

Note that a nonzero element v in $V(\chi_k)$ satisfies the formula $t \cdot v = \chi_k(t)v$, or $\chi_k(t)^{-1} t \cdot v = v$. It terms of the group action (G, V), this means the isotropy subgroup G_v of v, $v \in V(\chi_k) - 0$ is the cyclic group $G(\chi_k)$ generated by $(t, \chi_k(t)^{-1})$. In the same manner, if v is an element in the sum $V(\chi_k) \oplus V(\chi_{k'})$, $k \neq k'$ but away from the two components, i.e. $v \in V(\chi_k) \oplus V(\chi_{k'}) - V(\chi_k) - V(\chi_{k'})$, then the isotropy subgroup of v is the intersection $G(\chi_k, \chi_{k'}) = G(\chi_k) \cap G(\chi_{k'})$.

From these observations, it is easy to work out a complete list of isotropy subgroups of (G, V): the group G itself, together with all the intersections $G(\chi_{k_1}, \ldots, \chi_{k_l}) = G(\chi_{k_1}) \cap \ldots \cap G(\chi_{k_l})$ of the subgroups $G(\chi_{k_i})$. Furthermore the singular subspaces $V^G = (0)$ and $V_G = (0)$. In general when $H = G(\chi_{k_1}, \ldots, \chi_{k_l})$, with the property that $G(\chi) \supset H$ implies $\chi = \chi_{k_j}$ for some k_j, the spaces V^H and V_H are given by

$$V^{G(\chi_{k_1}, \ldots, \chi_{k_l})} = \bigoplus_{i=1}^{l} V(\chi_{k_i}),$$

$$V_{G(\chi_{k_1}, \ldots, \chi_{k_l})} = \bigoplus_{i=1}^{l} V(\chi_{k_i}) - \bigcup \{ V(\chi_{k_j}) \mid G(\chi_{k_j}) \supsetneqq G(\chi_{k_1}, \ldots, \chi_{k_l})\} \cup \{0\}.$$

Suppose we are given a smooth equivariant map $f\colon V \to W$ which is in general position with respect to 0. Then from the above list of isotropy subgroups, the dimension of $f^{-1}(0)_H$ can be computed by the transversality condition on $f\colon V_H \to W^H$.

Proposition 1.13. *Let $f\colon V \to W$ be a smooth equivariant map as above. Then $f^{-1}(0)$ contains the origin as the only fixed point, $f^{-1}(0)^G = 0$. In a neighbourhood near this fixed point, $f^{-1}(0)_{G(\chi_{k_1}, \ldots, \chi_{k_l})}$ is the empty set if $\sum_{i=1}^{l} \langle \chi_{k_i}, \chi(V)\rangle < \sum_{i=1}^{l} \langle \chi_{k_i}, \chi(W)\rangle$, and is a submanifold of dimension $2\left\{ \sum_{i=1}^{l} \langle \chi_{k_i}, \chi(V)\rangle - \langle \chi_{k_i}, \chi(W)\rangle \right\}$, otherwise.*

Suppose the actions (G, V) and (G, W) are effective, i.e. the principal orbit type is the free orbit. Then $f^{-1}(0)_e$ (=the free orbits in $f^{-1}(0)$) is either empty or is of dimension $2(\dim V - \dim W)$. Suppose in addition we have $\sum_{i=1}^{l} \langle \chi_{k_i}, \chi(V)\rangle - \langle \chi_{k_i}, \chi(W)\rangle \geqq \dim V - \dim W$ for a sequence of characters $\chi_{k_1}, \ldots, \chi_{k_l}$. Then the dimension of $f^{-1}(0)_{G(\chi_{k_1}, \ldots, \chi_{k_l})}$ is as big as the dimension of the free stratum $f^{-1}(0)_e$. From the property of Whitney stratification, this means that $f^{-1}(0)_{G(\chi_{k_1}, \ldots, \chi_{k_l})}$ is disjoint from $f^{-1}(0)_e$. In the application to moduli spaces, we will have to deal with this phenomenon, where some of the strata are disjoint from the free stratum.

2 Equivariant moduli spaces

Let X be a smooth, (simply) connected, oriented, closed 4-manifold, and let (π, X) denote the smooth action of a compact Lie (or finite) group π on X. Throughout the following discussion, we will restrict our attention to the case when the group action preserves the orientation. According to a result of Palais (unpublished[2]), the manifold X carries a real analytic structure invariant under the group action. Compatible both with this analytic structure and the group action, we can easily choose a real analytic Riemannian metric on X. In other words, an element of π operates on X by an orientation preserving, real analytic, isometry $g: X \to X$, $g \in \pi$.

For definiteness, let us fix a real analytic, principal SU(2)-bundle P on X, and a real analytic connection A_0 on this bundle. Let $\mathscr{A}^\infty$ denote the affine space of C^∞-connections on P, $\mathscr{A}^\infty = A_0 + C^\infty(X; \Lambda^1 TX \otimes \operatorname{Ad} P)$, where Ad P is the adjoint bundle of P, and let $\mathscr{A}$ denote the completion of $\mathscr{A}^\infty$ with respect to the metric.

$$d(A_0 + a, A_0 + a') = \int_X \Big(\sum_{i \leqq l-1} |\nabla^{(i)}_{A_0}(a - a')|^2 \Big)^{1/2}.$$

A gauge transformation of P is a principal SU(2)-bundle automorphism $h: P \to P$ which covers the identity map over the base space X. Let $\mathscr{G}^\infty$ denote the group of C^∞-gauge transformations of P and let $\mathscr{G}$ denote the completion of $\mathscr{G}^\infty$ with respect to the metric

$$d(h, h') = \int_X \Big(\sum_{i \leqq l} |\nabla^{(i)}_{A_0}(h - h')|^2 \Big)^{1/2}.$$

For $l \geqq 3$, the affine space $\mathscr{A}$ has a natural Hilbert manifold structure, $\mathscr{G}$ a Hilbert Lie group, and $\mathscr{G}$ operates smoothly on $\mathscr{A}$ [13, p. 53]. The quotient space $\mathscr{A}/\mathscr{G}$ under this last group action has the structure of a Banach "V-manifold". Points in $\mathscr{A}/\mathscr{G}$ are in one-to-one correspondence with gauge equivalence classes of (square integrable) connections; the irreducible connections form a smooth Banach manifold $\mathscr{A}^*/\mathscr{G}$ but around reducible connections $\mathscr{A}/\mathscr{G}$ has singularities modeled after the quotient of a Banach space modulo a circle group action [13, Theorem 3.1].

Let $\mathscr{G}(\pi)$ denote the group of SU(2)-bundle automorphisms $b: P \to P$ of P which cover an isometry $g: X \to X$ given by the action of some element $g \in \pi$. It is not difficult to see that $\mathscr{G}(\pi)$ is a group extension with $\mathscr{G}$ as its normal subgroup and π as its factor group. Furthermore this extended gauge group $\mathscr{G}(\pi)$ operates on $\mathscr{A}$ and hence gives rise to an action of π on the quotient $\mathscr{A}/\mathscr{G}$. (cf. [3, Sect. 1]).

Remark 2.1. The action of $\mathscr{G}(\pi)$ on $\mathscr{A}$ has only compact isotropy subgroups. In the case when D is irreducible the isotropy subgroup $\mathscr{G}(\pi)_D$ is an extension of the centre $\{\pm 1\}$ in SU(2) and a compact subgroup $\pi_D \subseteq \pi$. In the case when D is reducible $\mathscr{G}(\pi)_D$ is an extension of π_D by $U(1)$. In both cases π_D is the isotropy subgroup of the induced action $(\pi, \mathscr{A}/\mathscr{G})$ at the point D, and we denote by $\tilde{\pi}_D$ the group extension. The singularities of $(\mathscr{G}(\pi), \mathscr{A})$ can be interpreted in terms of π'-bundle structures

[2] We are indebted to R. Palais for pointing out to us (February 1990), that the assertion we need can easily be derived from the argument given in [23]

on P. Given an irreducible singular point D, the compact group $\tilde{\pi}_D$ makes P a $\tilde{\pi}_D$ principal SU(2)-bundle compatible with the action (π_D, X) on the base. Conversely, given a subgroup $\pi' \subset \pi$ and a π'-bundle on P, then there exists a connection D with a lifting of π' into a subgroup in π_D.

Let $\mathscr{M}_\pm(P)$ denote the moduli space of (anti-) self-dual connections on P, i.e. the subspace in $\mathscr{A}/\mathscr{G}$ consisting of classes of connections A whose curvature F_A satisfies the (anti-) self dual equation $F_A = \pm *F_A$. Since the Riemann metric is real analytic, this last equation is an elliptic PDE with real analytic coefficients. From the regularity theorem of PDE, it follows that a solution of this equation can be realized by a real analytic connection on P, not just a distribution in some Hilbert space. Furthermore the solution space $\mathscr{M}_\pm(P)$ admits the structure of a real analytic set, and therefore by the well-known result of Łojasiewicz [20] it can be triangulated. In general, not much more can be said about $\mathscr{M}_\pm(P)$ except its dimension (the dimension of its top cell) is no bigger than $8c_2(P) - 3(1 - b_1(X) + b_2^\mp(X))$. To achieve this 'formal' dimension and to remove all the singularities in $\mathscr{M}_\pm^*(P) = \mathscr{M}_\pm(P) - \{\text{reducibles}\}$, Donaldson [7] developed a method, in the setting without group action ($\pi = (e)$), to perturb the equation $F_A = \pm * F_A$. In the presence of a nontrivial group action (π, X), $\pi \neq (e)$, the equation $F_A = \pm * F_A$ is π-equivariant and so its solution space $\mathscr{M}_\pm(P)$ has an induced group action. To perturb this equivariant moduli space, we have to perturb the equation $F_A = \pm * F_A$ in a manner compatible with respect to the group action. Our task is to explain how to generalize Donaldson's procedure to cover this situation.

Let $\Omega^2(\mathrm{Ad}\,P)^\infty = C^\infty(X; \Lambda^2 TX \otimes \mathrm{Ad}\,P)$ denote the space of smooth, $\mathfrak{su}(2)$-valued, 2-forms on X. Once again we form its completion $\Omega^2(\mathrm{Ad}\,P)$ with respect to the Sobolev norm

$$\|F - F'\| = \int_X \Big(\sum_{i \leq l-2} |\nabla_{A_0}^{(i)} (F - F')|^2 \Big)^{1/2}.$$

Let $\Omega^2_\pm(\mathrm{Ad}\,P)$ denote the (± 1)-eigenspace in $\Omega^2(\mathrm{Ad}\,P)$ under the Hodge $*$-operation. Note $\Omega^2_\mp(\mathrm{Ad}\,P)$ is precisely the space where the (anti)-self-dual component $\frac{1}{2}(F_A \mp *F_A)$ of a curvature 2-form F_A takes its value. This leads us to consider the Hilbert bundle $\mathrm{Pr}_1 : \mathscr{A} \times \Omega^2_\pm(\mathrm{Ad}\,P) \to \mathscr{A}$ over $\mathscr{A}$. The formula $\sigma_\mp(A) = (A, F_A \mp *F_A)$ gives us a smooth section of this bundle. The extended gauge group $\mathscr{G}(\pi)$ acts on $\Omega^2_\mp(\mathrm{Ad}\,P)$ in a natural manner. By taking the diagonal action on $\mathscr{A} \times \Omega^2_\mp(\mathrm{Ad}\,P)$ the projection

$$\mathrm{Pr}_1 : \mathscr{A} \times \Omega^2_\mp(\mathrm{Ad}\,P) \to \mathscr{A} \tag{2.2}$$

becomes a $\mathscr{G}(\pi)$-equivariant, Hilbert bundle over $\mathscr{A}$, and $\sigma_\mp$ becomes an equivariant section. Finally if we consider the intersection $\sigma_\mp^{-1}(0)$ of $\sigma_\mp(\mathscr{A})$ with the zero section $\mathscr{A} \times 0$ and factor out the action of the gauge group $\mathscr{G}$, we recover our equivariant moduli space $(\pi, \mathscr{M}_\pm(P)) = (\pi, \sigma_\mp^{-1}(0)/\mathscr{G})$.

Now, to genaralize Donaldson's procedure, we have to perturb the $\mathscr{G}(\pi)$-equivariant section σ so that it is in equivariant general position with respect to the zero section. Unfortunately in the previous section, the notion of equivariant general position is defined only for equivariant maps between finite dimensional G-spaces, and only for compact finite dimensional Lie group G. It remains, therefore, to generalize these notions so as to cover the present situation of sections of infinite dimensional bundles with actions of infinite dimensional groups.

Firstly, as indicated in (1.6), the definition for an equivariant section, such as $\sigma_{\mp}: \mathscr{A} \to \mathscr{A} \times \Omega^2_{\mp}(\operatorname{Ad} P)$, in general position with respect to the 0-section is the same as the requirement that its projection (locally) onto the fiber direction is in general position with respect to 0. In the present setting, this means the composite

$$\text{(2.3)} \qquad \operatorname{Pr}_2 \circ \sigma_{\mp}: \mathscr{A} \xrightarrow{\sigma_{\mp}} \mathscr{A} \times \Omega^2_{\mp}(\operatorname{Ad} P) \xrightarrow{\operatorname{Pr}_2} \Omega^2_{\mp}(\operatorname{Ad} P)$$

is in equivariant general position with respect to $0 \in \Omega^2_{\mp}(\operatorname{Ad} P)$.

Secondly, as indicated in (1.7), there is the general principle: an equivariant map is in general position if and only if the restriction to every slice in the domain is in general position with regard to the action by the istropy subgroup of the slice. In the present setting, a slice χ_D to the orbit $\mathscr{G}.\ D$ in $\mathscr{A}$ is given by the formula:

$$\chi_D = \{A \in \Omega^1(\operatorname{Ad} P) \mid D^* A = 0\}$$

and the restriction $(\operatorname{Pr}_2 \circ \sigma \mid \chi_D)$ of $\operatorname{Pr}_2 \circ \sigma$ to this slice by

$$\text{(2.4)} \qquad \mu: \chi_D \to \Omega^2_{\mp}(\operatorname{Ad} P), \qquad \mu(A) = F_A \mp * F_A$$

(see [13, (4.6)]). Moreover, by (2.1), isotropy subgroup $\mathscr{G}(\pi)_D$ is always compact. Hence our problem of determining general position for the $\mathscr{G}(\pi)$-map $\operatorname{Pr}_2 \circ \sigma_{\mp}: \mathscr{A} \to \Omega^2_{\mp}(\operatorname{Ad} P)$ is reduced to one of studying an equivariant map $\mu: \chi_D \to \Omega^2_{\mp}(\operatorname{Ad} P)$ with respect to the compact Lie group $\mathscr{G}(\pi)_D$.

Thirdly, both χ_D and $\Omega^2_{\mp}(\operatorname{Ad} P)$ are Hillbert spaces and the map μ in (2.4) is an equivariant Fredholm map. We claim, in such a setting the definition of equivariant general position can be reduced to a related map on some finite dimensional vector spaces.

Let G be a compact Lie group acting on two Hilbert spaces H_1 and H_2 by isometries. Let $\Psi: H_1 \to H_2$ be an equivariant Fredholm map with $\Psi(0) = 0$, and let $T = (d\Psi)_0: H_1 \to H_2$ denote its differential at the origin. Then, because Ψ is Fredholm, the two G-vector spaces $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are finite dimensional and there is an induced equivariant map

$$\hat{\Psi}: \operatorname{Ker} T \subseteq H_1 \xrightarrow{\Psi} H_2 \xrightarrow{\text{proj.}} \operatorname{Coker} T$$

between them.

Definition 2.5. A G-Fredholm map $\Psi: H_1 \to H_2$, $\Psi(0) = 0$ between two G-Hilbert spaces is said *to be in equivariant general position with respect to* $0 \in H_2$ if and only if the associated finite dimensional map $\Psi: \operatorname{Ker} T \to \operatorname{Coker} T$ is in general position with respect to $0 \in \operatorname{Coker} T$.

To justify the above definition, we have to verify the following.

Proposition 2.6. *In the space* $\operatorname{Fred}_G(H_1, H_2)$ *of G-Fredholm maps, those in equivariant general position with respect to* $0 \in H_2$ *form an open and dense subspace.*

To prove (2.6), we need the following Lemma (for the proof, see [13, Lemma 4.7]).

Lemma 2.7. *Let $\Psi: H_1 \to H_2$ be a G-Fredholm map with $\Psi(0) = 0$ and let $T = (d\Psi)_0$ be its differential at the origin. Then there exist orthogonal decomposition of G-Hilbert spaces $H_1 = \mathrm{Ker} T \oplus H_1'$, $H_2 = \mathrm{Im}\, T \oplus H_2'$ and an equivariant map $\Phi: H_1 \to H_2'$, $\Phi(0) = 0$, $(d\Phi)_0 = 0$ such that Ψ is locally equivalent to $T + \Phi$ via an equivariant self-diffeomorphism α of the manifold H_1 i.e. $\Psi = (T + \Phi) \circ \alpha$ where $\alpha: H_1 \cong H_1$, $\alpha(0) = 0$.*

Note that the subspace H_2' in (2.7) is canonically isomorphic to Coker T, and so by (2.7) we have another equivariant map between Ker T and Coker T by forming the composite

$$(2.8) \qquad \hat{\Phi}: \mathrm{Ker}\, T \hookrightarrow H_1 \xrightarrow{\Psi} H_2' \xrightarrow{\text{proj.}} \mathrm{Coker}\, T .$$

Since $\Psi \circ \alpha^{-1} = (T + \Phi)$, we have $\Phi | \ker T = \Psi \circ \alpha^{-1} | \ker T$, and using this last relation we can rewrite $\hat{\Phi}$ as

$$\hat{\Phi}: \mathrm{Ker}\, T \xrightarrow{\alpha^{-1}} \alpha^{-1}(\mathrm{Ker}\, T) \hookrightarrow H_1 \xrightarrow{\Psi} H_2 \xrightarrow{\text{proj.}} \mathrm{Coker}\, T .$$

Comparing this expression with the definition of $\hat{\Psi}$, it is clear that the two maps $\hat{\Phi}$ and $\hat{\Psi}$ are defined in a similar way: in the case of $\hat{\Psi}$ we form the restriction of proj $\circ \Psi$ to Ker T and in the case of $\hat{\Phi}$ the restriction to α^{-1}(Ker T). In other words the difference between $\hat{\Phi}$ and $\hat{\Psi}$ lies in the choice of two different normal slices Ker T and α^{-1} (Ker T) to H_1' in H_1. Thus, from the Equivariant Tubular Neighbourhood Theorem, there exist equivariant diffeomorphisms β: Ker $T \to$ Ker T and γ: Coker $T \to$ Coker T such that $\hat{\Psi} = \gamma \circ \hat{\Phi} \circ \beta$. Since the property of being in equivariant general position is unchanged under equivariant diffeomorphisms [2, Propositions (5.2) and (5.3)], we have the following reformulation of general position for G-Fredholm maps:

Lemma 2.9. *Let Ψ be a G-Fredholm map $\Psi: H_1 \to H_2$, $\Psi(0) = 0$, and let $\hat{\Phi}$: Ker $T \to$ Coker T be defined as in (2.8). Then Ψ is in equivariant general position if and only if $\hat{\Phi}$ is in equivariant general position.*

We are now in a position to prove (2.6). Given a G-Fredholm map $\Psi: H_1 \to H_2$, $\Psi = (T + \Phi) \circ \alpha$, we can deform the finite dimensional map $\hat{\Phi}$: Ker $T \to$ Coker T defined in (2.8) so that $\hat{\Phi}$ becomes a map $\hat{\Phi}_\varepsilon$ in G-general position with respect to $0 \in$ Coker T and is still within a small ε-distance. Using an equivariant bump function of H_1 concentrated at neighbourhood of Ker T we can extend $\hat{\Phi}_\varepsilon$ to a smooth G-map $\Phi_\varepsilon: H_1 \to$ Coker T defined over H_1. Then the formula $\Psi_\varepsilon = (T + \Phi_\varepsilon) \circ \alpha$ gives a G-Fredholm map which is in G-general position with respect to $0 \in H_2$, by (2.9), and is within ε-distance away from Ψ.

Next suppose we are given a G-Fredholm map $\Psi: H_1 \to H_2$ which is already in G-general position with respect to $0 \in H_2$. Suppose we perturb Ψ to another G-Fredholm map Ψ_ε with ε-distance away. Note that their differentials $T = (d\Psi)_0$ and $T_\varepsilon = (d\Psi_\varepsilon)_0$ are close to each other, and Ker $T_\varepsilon \subseteq$ Ker T. For ε sufficiently small, since the G-index is constant Ker $T -$ Coker $T =$ Ker $T_\varepsilon -$ Coker T_ε as virtual G-representations. However by (1.11), Ker T and Coker T have no irreducible G-representations in common. It follows that these two linear maps T and T_ε have the same corank. In fact, we may assume that there exists an equivariant linear isomorphism $L\colon H_2 \to H_2$ which is within ε-distance to the identity and $L \circ T_\varepsilon = T$. This last equation gives rise to a commutative diagram of G-maps:

$$\begin{array}{ccc} \operatorname{Ker} T_\varepsilon & \xrightarrow{\hat{\Psi}_\varepsilon} & \operatorname{Coker} T_\varepsilon \\ (L|\operatorname{Ker} T_\varepsilon)\downarrow & \widehat{L\circ\Psi_\varepsilon} & \downarrow L \text{ modulo } T_\varepsilon(H_1) \\ \operatorname{Ker} T & \longrightarrow & \operatorname{Coker} T \end{array}$$

Here the vertical arrows are isomorphisms induced by L and the bottom horizontal map $(\widehat{L\circ\Psi_\varepsilon})$ is the finite dimensional map associated to $L\circ\Psi_\varepsilon$ as in (2.5). For ε sufficiently small, $(\widehat{L\circ\Psi_\varepsilon})$ is within a small distance of $\hat{\Psi}$, and therefore by the openness property of general position maps, $(\widehat{L\circ\Psi_\varepsilon})$ is in general position. This proves that $\hat{\Psi}_\varepsilon$ and hence Ψ_ε are in general position. The proof of (2.6) is complete.

Now combining (2.3)–(2.5) we can define our equivariant moduli space.

Definition 2.10. Let $\sigma_\pm$ be an equivariant section of $\mathscr{A} \times \Omega^2_\mp(\operatorname{Ad} P) \to \mathscr{A}$ in general position with respect to the zero section. Then its quotient space $\sigma_\mp^{-1}(0)/\mathscr{G}$ together with the induced π-action is called an *equivariant moduli space* $(\pi, \mathscr{M}_\pm)$ *of (anti-) self dual connections on P.*

Proof of Theorem A. To begin we will show that the general position moduli space (in the sense of Definition 2.10) can be constructed. Suppose we are given an equivariant section $\sigma_\mp$ of $\mathscr{A} \times \Omega^2_\mp(\operatorname{Ad} P) \to \mathscr{A}$, not necessarily in general position. Then we can cover a neighbourhood of $\sigma_\mp^{-1}(0)$ in $\mathscr{A}$ by a locally-finite $\mathscr{G}(\pi)$-equivariant open covering $\mathscr{U}$. By (2.6) and a 'partition of unity' argument, we can perturb $(\sigma_\mp)_\varepsilon$ on one open set of $\mathscr{U}$ after another so that within a prescribed distance ε, $\sigma_\mp$ becomes $(\sigma_\mp)_\varepsilon$, which is in general position with respect to the zero section. Thus we obtain the equivariant moduli space $(\pi, (\sigma_\mp)_\varepsilon^{-1}(0)/\mathscr{G})$.

It remains to establish the properties of $(\pi, \mathscr{M})$ listed in Theorem A. In the above construction of $(\sigma_\mp)_\varepsilon$ let us concentrate on an open set in $\mathscr{U}$ centered at an (anti)-self-dual connection D. As discussed in (2.5), to form $(\sigma_\mp)_\varepsilon$ we have to perturb the $\mathscr{G}(\pi)_D$-Fredholm map $\mu\colon \chi_D \to \Omega^2_\mp(\operatorname{Ad} P)$ into a general position which in turn means to perturb the map $\hat{\mu}\colon \operatorname{Ker}(d\mu)_0 \to \operatorname{Coker}(d\mu)_0$. According to [13, (4.8)], the kernel and cokernel of $(d\mu)_0$ can be identified respectively with the first and second cohomology of the elliptic complex:

$$(2.11)\quad \Omega^*_\mp(\operatorname{Ad} P)\colon \{0 \to \Omega^0(\operatorname{Ad} P) \xrightarrow{D} \Omega^1(\operatorname{Ad} P) \xrightarrow{P_\mp D} \Omega^2_\mp(\operatorname{Ad} P) \to 0\}\,.$$

Note there is an action of $\mathscr{G}(\pi)_D$ on (2.11) and so an induced action on cohomology. Under the above identification, these actions $(\mathscr{G}(\pi)_D, H^1(D))$ and $(\mathscr{G}(\pi)_D, H^2(D))$ coincide with the corresponding actions $(\mathscr{G}(\pi)_D, \operatorname{Ker}(d\mu)_0)$ and $(\mathscr{G}(\pi)_D, \operatorname{Coker}(d\mu)_0)$.

After the perturbation, the zero set $(\sigma_\mp)_\varepsilon^{-1}(0)$ near D has the same structure as $\mu_\varepsilon^{-1}(0) \times \mathscr{G}$ where μ_ε is a general position map from $\operatorname{Ker}(d\mu)_0$ to $\operatorname{Coker}(d\mu)_0$. For an irreducible connection D the isotropy subgroup $\mathscr{G}_D$ is isomorphic to $\{\pm 1\}$, and so after factoring out the gauge group action the local structure on the moduli space $\mathscr{M}^*$ near $[D]$ is the same as the local structure on $\mu_\varepsilon^{-1}(0)$. Thus $\mathscr{M}^*$ has an equivariant Whitney stratification by $\{\mathscr{M}^*_{(\pi')}\}$ in the same manner as the finite dimensional situation.

From the above discussion, the dimension of $\mathcal{M}^*_{\pi'}$ near $[D]$ is the same as the dimension of the corresponding subspace $\mu_\varepsilon^{-1}(0)_{\pi'}$ in $\mu_\varepsilon^{-1}(0)$ with the same orbit type. From (1.4), this last dimension is given by:

$$\dim \mathcal{M}^*_{\pi'} = \dim H^1(D)_{\pi'} - \dim H^2(D)^{\pi'} .$$

Note that $H^1(D)_{\pi'}$ is the open subspace in $H^1(D)^{\pi'}$ consisting of the free orbits of the induced Weyl group action $(N(\pi')/\pi', H^1(D)^{\pi'})$. Hence $H^1(D)_{\pi'}$ is either empty or of the same dimension as $H^1(D)^{\pi'}$. Thus, without loss of generality, we may replace $\dim H^1(D)_{\pi'}$ by $\dim H^1(D)^{\pi'}$ in the above formula; also because $H^0(D) = 0$ for irreducible connections we may subtract by the term $\dim H^0(D)^{\pi'}$:

$$\begin{aligned}\dim \mathcal{M}^*_{\pi'} &= \dim H^1(D)^{\pi'} - \dim H^2(D)^{\pi'} - \dim H^0(D)^{\pi'} \\ &= \text{index of the fixed point subcomplex } \Omega^*_\mp(\operatorname{Ad} P)^{\pi'}: \\ &\{0 \to \Omega^0(\operatorname{Ad} P)^{\pi'} \to \Omega^1(\operatorname{Ad} P)^{\pi'} \to \Omega^*_\mp(\operatorname{Ad} P)^{\pi'} \to 0\} .\end{aligned}$$

The above index is a topological invariant depending only on the π'-bundle structure on P. Since the stratum $\mathcal{M}^*_{(\pi')}$ is a translate of $\mathcal{M}^*_{\pi'}$ under the action of π, i.e.

$$\mathcal{M}^*_{(\pi')} = \mathcal{M}^*_{\pi'} \underset{N(\pi')/\pi'}{\times} \pi/N(\pi') ,$$

and its dimension can be computed directly from $\dim \mathcal{M}^*_{\pi'}$. This completes the proof of Theorem A.

Remark 2.12. In the case when $\pi = (e)$, the above construction coincides with that obtained by Donaldson in [7]. For a nontrivial group π, this equivariant moduli space has not been investigated before; although its fixed point set $\mathcal{M}^\pi$ coincides with the moduli space of instantons on the orbifold X/π. This last object has been studied quite extensively by Fintushel-Stern, Lawson, and Furuta [15, 16, 19]. In the case when X is a homotopy 4-sphere, Furuta investigated the equivariant moduli space of X and its fixed point set. In view of Theorem A, some of his results can be recovered from our more general framework.

Remark 2.13. The argument in the above proof can be extended to study the structure of $\mathcal{M}$ near a reducible connection $[D]$. Let us fix attention on the *self-dual case.* From the definition of a reducible connection, the associated 2-plane bundle of P can be written as the sum $\mathcal{L} \oplus \mathcal{L}^{-1}$ of two line bundles $\mathcal{L}$ and $\mathcal{L}^{-1}$. Accordingly, $\operatorname{Ad} P$ is the sum $(\mathbf{R} \times X) \oplus \mathcal{L}^{\otimes 2}$ of the trivial one dimensional real bundle $\mathbf{R} \times X$ and the square $\mathcal{L}^{\otimes 2}$ of $\mathcal{L}$. In turn this leads to a decomposition of the elliptic complex (2.11) into the sum of ordinary self-dual deRham complex $\{0 \to \Omega^0 \to \Omega^1 \to \Omega^2_- \to 0\}$ and the complex

$$\{0 \to \Omega^0(\mathcal{L}^{\otimes 2}) \to \Omega^1(\mathcal{L}^{\otimes 2}) \to \Omega^2_-(\mathcal{L}^{\otimes 2}) \to 0\} ,$$

and also a decomposition in cohomology [13, p. 82]:

$$H^0(D) \cong H^0_{\mathrm{DR}}(X), \qquad H^1(D) \cong H^1(D; \mathcal{L}^{\otimes 2}) \oplus H^1_{\mathrm{DR}}(X), \text{ and}$$
$$H^2(D) \cong H^2(D; \mathcal{L}^{\otimes 2}) \oplus H^2_{\mathrm{DR}}(X) .$$

Recall that in the case of a reducible connection D the isotropy subgroup $\mathcal{G}_D$ is isomorphic to the circle group $U(1)$. This circle group operates on both χ_D and

$\Omega^2_-(\operatorname{Ad} P)$, and the map $\mu: \chi_D \to \Omega^2_-(\operatorname{Ad} P)$ is equivariant with respect to this action. The fixed point subspace $(\chi_D)^{\mathscr{G}_D}$ admits an interpretation as the directions tangent to the subspace $\mathscr{A}_{\text{red}}$ of reducible connections. Therefore to investigate the structure along the reducible connections in $\mathscr{M}$, it is necessary to understand the restriction of μ to the fixed point subspaces

$$\mu|(\chi_D)^{\mathscr{G}_D}: (\chi_D)^{\mathscr{G}_D} \to (\Omega^2_-(\operatorname{Ad} P))^{\mathscr{G}_D} .$$

Note $\mu|(\chi_D)^{\mathscr{G}_D}$ is a Fredholm map equivariant with respect to the induced π_D-actions. Following (2.5) we therefore have to study the associated finite dimensional map from $H^1(D) \to H^2(D)$.

The circle group $\mathscr{G}_D$ acts trivially on the deRham cohomology groups $H^1_{\text{DR}}(X)$ and $P_- H^2_{\text{DR}}(X)$, but by complex multiplication on $H^1(D; \mathscr{L}^{\otimes 2})$ and $H^2(D; \mathscr{L}^{\otimes 2})$. Hence

$$H^1(D)^{\mathscr{G}_D} \cong H^1_{\text{DR}}(X), \qquad H^2(D)^{\mathscr{G}_D} \cong P_- H^2_{\text{DR}}(X) ,$$

and the problem can be further reduced to study a π_D-general position map $H^1_{\text{DR}}(X) \to H^2_{\text{DR}}(X)$.

In the case when D lies in a free orbit in $(\pi, \mathscr{A}/\mathscr{G})$, then $\pi_D = (e)$ and our treatment is the same as in the situation without group actions. If $\dim H^1_{\text{DR}}(X) < \dim P_- H^2_{\text{DR}}(X)$, then the reducible connections in $\mathscr{M}_\pm(P)$ can all be perturbed away. If $\dim H^1_{\text{DR}}(X) \geqq \dim H^1_{\text{DR}}(X)$, then, reducible connections form a submanifold $\mathscr{M}^{\text{red.}}$ in $\mathscr{M}$ of dimension $\dim H^1_{\text{DR}}(X) - \dim P_- H^2_{\text{DR}}(X)$, and the group acts freely on this submanifold.

In general, the reducible connections form an equivariant substratified space

$$\mathscr{M}_{\text{red.}} = \mathscr{M} - \mathscr{M}^*$$

in $\mathscr{M}$. For a subgroup $\pi' \subset \pi_D$, the dimension of π'-fixed point set $\mathscr{M}^{\pi'}_{\text{red.}}$ is determined by the formula

$$\dim H^1_{\text{DR}}(X)^{\pi'} - \dim P_- H^2_{\text{DR}}(X)^{\pi'} .$$

The following special case is important for our applications.

Proposition 2.14. *Assume that* $H^1_{\text{DR}}(X) = P_- H^2_{\text{DR}}(X) = 0$, *then the perturbed moduli space* $\mathscr{M}_{+\varepsilon}(P)_{\text{red.}}$ *consists of isolated points and the neighbourhood of such a point* D *has a cone structure obtained by factoring the zero set of a* $\mathscr{G}(\pi)_D$*-general position map by the circle group* $\mathscr{G}_D$*-action. In addition, the dimension of the various* π_D*-strata can be determined by the index of various sub-complexes in*

$$\{0 \to \Omega^0(\mathscr{L}^{\otimes 2}) \to \Omega^1(\mathscr{L}^{\otimes 2}) \to \Omega^2_-(\mathscr{L}^{\otimes 2}) \to 0\} .$$

Proof. Under these assumptions, before any perturbation the set of reducible connections in $\mathscr{M}_+(P)$ consists of isolated points. Now in the construction of $\mathscr{M}_{+\varepsilon}(P)$, we chose a transverse slice to the $\mathscr{G}$-orbit through such a point $[D]$ and perturbed the section to a π_D-general position map. By the openness property of general position maps, we continued this perturbation to the nearby slice without further change on a neighbourhood of $[D]$. Since $H^0(D, \mathscr{L}^{\otimes 2} = 0$, we can compute the dimensions of the strata in these neighbourhoods of $\mathscr{M}_{+\varepsilon}(P)$ as in the previous paragraph. □

We conclude this section with an example of Fintushel which explains the failure of equivariant transversality in studying moduli space problems.

Example 2.15. Let X be a K3-surface obtained by taking the branched 2-fold covering of a degree 6 curve in $P^2(\mathbf{C})$. In other words, there is a $\mathbf{Z}/2$-action $(\mathbf{Z}/2, X)$ whose quotient space is $P^2(\mathbf{C})$. Let $\bar{P}$ be the SU(2)-bundle over $P^2(\mathbf{C})$ with Chern number $c_2(\bar{P}) = 1$, and let P be the pull back of $\bar{P}$ over X. It is easy to see that the Chern number $c_2(P) = 2$, $b_2^-(X) = 19$ and $8c_2(P) - 3(1 + b_2^-(X)) = -44$. Since this last number is the formal dimension of the moduli space $\mathscr{M}_+(P)$ of self dual connections on X, the negative sign of $8c_2(P) - 3(1 + b_2^-(X))$ means that this moduli space (in the setting without group action) is generically empty. On the other hand, the moduli space $\mathscr{M}_+(\bar{P})$ of self dual connections on $\bar{P}$ is of dimension $5 = 8c_2(\bar{P}) - 3(1 + b_2^-(P^2(\mathbf{C})))$, and pulling back the self dual connections on $\bar{P}$ to P creates a 5-parameter family of self dual connections on P.

From our viewpoint, this phenomenon can be explained as follows. First of all, the equivariant moduli space $(\mathbf{Z}/2, \mathscr{M})$ is non-empty and is of dimension 5. Let D be a self dual irreducible connection on P obtained from pulling back a irreducible self dual connection on $\bar{P}$. The above calculations indicates that the $\mathbf{Z}/2$-vector spaces $H^1(D)$, $H^2(D)$ satisfy the inequality:

(i) $\dim H^1(D) - \dim H^2(D) = -44 < 0$
(ii) $\dim H^1(D)^{\mathbf{Z}/2} - \dim H^2(D)^{\mathbf{Z}/2} = 5 > 0$.

The criterion for equivariant transversality fails because $(\mathbf{Z}/2; H^2(D))$ cannot be a sub-representation space of $(\mathbf{Z}/2; H^1(D))$. On the other hand, a $\mathbf{Z}/2$-general position map $(\mathbf{Z}/2; H^1(D)) \to (\mathbf{Z}/2; H^2(D))$ exists and its zero set is contained in $H^1(D)^{\mathbf{Z}/2}$. Thus, in this case, the equivariant moduli space $(\mathbf{Z}/2, \mathscr{M})$ is of dimension 5 with trivial $\mathbf{Z}/2$-action.

3 Equivariant collar neighbourhoods

From now on, we will restrict ourselves to the setting: X is a simply connected, positive definite 4-manifold and π is a finite group acting on X. In addition, we will concentrate on the equivariant moduli space $(\pi, \mathscr{M})$ of self-dual connections on an SU(2)-bundle P whose Chern number $c_2(P) = 1$.

In the setting without group action, the compactification of this moduli space was studied by Donaldson [7] and Uhlenbeck [13]. The subspace $\mathscr{M}_+(P) -$ {reducibles} is a 5-dimensional manifold with an end diffeomorphic to $X \times [0, 1)$ and $\mathscr{M}_+(P)$ can be compactified by adding a collar $X \times I$ to this end. The object of this section is to prove the following:

Proposition 3.1. *There exists an equivariant moduli space $(\pi, \mathscr{M})$, as defined in* (2.10), *such that $\mathscr{M}$ has a smooth equivariant end diffeomorphic to the product $(\pi, X) \times [0, 1)$. In particular, it can be compactified by adding an equivariant collar $(\pi, X) \times I$.*

In general, the neighbourhood by neighbourhood perturbation given in the proof of Theorem A may destroy the nice collar structure at the end. Instead, following the procedure in [13] we will start with a single neighbourhood $\mathscr{N}_{r_0}$ of the end and then continue the perturbation to general position in the compact complement $\mathcal{M} - \mathcal{N}_{r_0}$. More precisely, before any perturbation has taken place, the "honest" moduli space $\sigma_{\mp}^{-1}(0)/\mathcal{G}$ has a collar. Recall the proof of the Collar Theorem on p. 157 and p. 162–187 of [13]. There exists a subspace $\mathcal{CC}$ of concentrated connections in $\sigma_{\mp}^{-1}(0)/\mathcal{G}$,

$$\mathcal{CC} = \{D\text{: for some } \mu < \mu_0 \text{ and } y \in X\text{, the inequalities}$$

$$\| e^*_{\mu, y}(\omega(D)) - \hat{I} \|_{L^1} < \varepsilon \text{ and } \int_X \omega(D) \leqq 9\pi^2 \text{ hold}\}$$

where $\omega(D) = -\operatorname{tr} F_D \wedge * F_D$. For the definition of $e^*_{\mu, y}$ see [13, 8.27].

Over this subspace, there exist smooth maps $x\colon \mathcal{CC} \to X$, $\lambda\colon \mathcal{CC} \to (0, \infty)$ defined by sending D to its "center $x(D)$" and to its "radius $\lambda(D)$". Then for connection D with sufficiently small radius $\lambda(D) < r_0$, the homology $H^2(D) = 0$, and so $\mathcal{N}_{r_0} = \lambda^{-1}(0, r_0)$ is smooth. The map $(x, \lambda)\colon \mathcal{N}_{r_0} \to X \times (0, r_0)$ is a diffeomorphism, and via this diffeomorphism $X \times (0, r_0)$ becomes a collar of $\mathcal{M}$. Moreover, the complement $\mathcal{M} - \mathcal{N}_{r_0}$ is compact.

From [13, p. 154], the center $x(D)$ and the radius $\lambda(D)$ are given by the unique solution of a system of nonlinear equations,

$$R(\lambda, x, \omega(D), \text{metric}) = 4\pi^2 ,$$

$$\frac{\partial R}{\partial x}(\lambda, x, \omega(D), \text{metric}) = 0 .$$

It is straightforward to check that this system of equations is invariant under the group action, and therefore from the uniqueness of solutions, $(x(D), \lambda(D))$ is invariant under the group action. In other words, $(x, \lambda)\colon \mathcal{N}_{r_0} \to (0, r_0)$ is equivariant with respect to the product action $(\pi, X \times (0, r_0))$ (c.f. [3, (1.8); 16, Sect. 1]).

Since $H^2(D) = 0$ for $D \in \mathcal{N}_{r_0}$, the section $\sigma_-\colon \mathcal{A} \to \mathcal{A} \times \Omega^2_-(\operatorname{Ad} P)$ is already equivariantly transverse to the zero section. Therefore in the construction of the equivariant moduli space $(\pi, \mathcal{M})$, we can keep this part of the section fixed and so it inherits an equivariant collar neighbourhood. This proves (3.1).

The structure on the equivariant collar neighbourhood also provides us with information about the interior of the moduli space. Let π' be an isotropy subgroup of π, and let $X_{\pi'}$ be the nonempty submanifold of X fixed by π'. Then, by (3.1), $\mathcal{M}_{\pi'}$ is also nonempty and has collar neighbourhood isomorphic to $X_{\pi'} \times (0, r_0)$. An immediate application of Whitney stratification is the following:

Corollary 3.2. *The singular subspace $\mathcal{M}^*_{\pi'}$ in $\mathcal{M}^*$ contains manifold components whose intersection with the collar neighbourhood $\mathcal{N}_{r_0}$ equals $X_{\pi'} \times (0, r_0)$.*

Let x be a point $X_{\pi'}$ and let (π', N_x) denote the normal slice representation of the isotropy subgroup π' at x. Then (π', N_x) is also the slice representation along a connected component of $X_{\pi'} \times (0, r_0)$ in $\mathcal{N}_{r_0}$. In fact, a slightly stronger version of (3.2) states that inside $\mathcal{M}^*$ there exists a 5-dimensional manifold component $\mathcal{H}$ such that a connected component of $\mathcal{H}_{\pi'}$ has normal slice representation (π', N_x) and its intersection with $\mathcal{N}_{r_0}$ equals $X_{\pi'} \times (0, r_0)$.

Corollary 3.3. *Suppose (π, X) has an isolated fixed point x whose isotropy representation (π, N_x) is not isomorphic to (π, N_y) at another fixed point, $y \in X^\pi$, $y \neq x$, via an orientation reversing isomorphism. Then there exists an arc in $\mathscr{M}$ consisting of fixed points which starts out at $x \times (0, r_0)$ in N_{r_0} and ends at a reducible connection of $\mathscr{M}$.*

This implies the result of [16], that a finite group can not operate smoothly on S^4 with one fixed point.

Remark 3.4. A similar modification of the discussion in [11, 4.4] shows that the moduli spaces $(\pi, \mathscr{M}_k)$ for any $c_2(P) = k$ have an equivariant compactification. Following [11, 4.4.1] we define "ideal connections" $([A], (x_1, \ldots x_l))$ by requiring in addition that $[A]$ lie in our general position moduli space $(\pi, \mathscr{M}_l)$. There is an obvious action of π on $I\mathscr{M}_k$, and we form the corresponding space $\bar{\mathscr{M}}_k$ as the closure of $\mathscr{M}_k$ in the space $(\pi, I\mathscr{M}_k)$ of ideal connections.

Theorem 3.5 [11, 4.4.3]. *The space $\bar{\mathscr{M}}_k$ is compact, π-invariant subspace of $I\mathscr{M}_k$. The action $(\pi, \mathscr{M}_k)$ converges to the action $(\pi, \bar{\mathscr{M}}_k)$.*

A local model for the ends of $\mathscr{M}_k$ is given in [9, Sect. 4(b)]. It again follows that in our construction of $(\pi, \mathscr{M}_k)$, we may assume that our section is in general position on a π-invariant neighbourhood $\mathscr{V}$ of $\bar{\mathscr{M}}_k \cap I\mathscr{M}_k$, and that $\mathscr{M}_k - \mathscr{V}$ is compact.

4 Group actions on $P^2(\mathbf{C})$

We first investigate further the structure near a reducible connection D in $\mathscr{M}$, and then give the proof of Theorem B. Recall in (2.13), the associated 2-plane bundle of P can be written as the sum $\mathscr{L} \oplus \mathscr{L}^{-1}$ of two line bundles. To simplify our discussion, we assume

(4.1) *$\mathscr{L}$ is an equivariant $U(1)$-bundle with respect to the group action (π_D, X).*

This is not always the case; however, we will show in Sect. 6 that the assumption holds when π_D is a cyclic group or when X is a simply-connected, positive definite 4-manifold when π_D is a cyclic group or when X is a simply-connected, positive definite 4-manifold (for example, the connected sum $\overset{n}{\#} P^2(\mathbf{C})$ of complex projective planes, $n > 1$), with π acting trivially on $H^2(X, \mathbf{Z})$. At any rate, under (4.1), the isotropy subgroup $\mathscr{G}(\pi)_D$ becomes the product $\pi_D \times \mathscr{G}_D$, and the actions of $\mathscr{G}(\pi)_D$ on $H^1(D)$ and $H^2(D)$ give rise to complex representations of π_D on $V = H^1(D; \mathscr{L}^{\otimes 2})$ and $W = H^2(D; \mathscr{L}^{\otimes 2})$.

As in (2.13), an equivariant neighbourhood of D can be obtained by first taking a $\mathscr{G}(\pi)_D$-general position map $f\colon V \to W$, forming the zero set $f^{-1}(0)$, and then factoring out the action of circle group $\mathscr{G}_D$. In the situation when π_D is a cyclic group, the geometry of a general position map $f\colon V \to W$ between two representation spaces has been studied in (1.13). This is precisely what we need in the following.

The proof of Theorem B. Suppose first that the fixed point set consists of a 2-sphere and an isolated point. An application of the G-Signature Theorem as in [17] shows that the rotational numbers agree with those in some linear action on $P^2(\mathbf{C})$.

We can therefore assume that we have an odd cyclic group, and a semi-free action $(\pi, P^2(\mathbf{C}))$ on $P^2(\mathbf{C})$ with isolated fixed points $\{p_1, p_2, p_3\}$. Let (a_1, b_1), (a_2, b_2), (a_3, b_3) denote the rotation numbers of the isotropy representations π at the tangent spaces p_1, p_2, p_3. If two of these pairs were the same via an orientation reversing diffeomorphism, i.e., $(a_i, b_i) = (-b_j, a_j)$ or $(-a_j, b_j)$ or $(b_j, -a_j)$ or $(a_j, -b_j)$, then a straightforward calculation, using the Equivariant Signature Theorem, shows that the three rotation numbers (a_1, b_1), (a_2, b_2), (a_3, b_3) coincide with those from a linear action (see [17, Sect. 2; 25]). Otherwise, by (3.3), there exist three arcs $\gamma_1, \gamma_2, \gamma_3$ of fixed points in $\mathscr{M}$, starting from $p_1 \times I$, $p_2 \times I$, $p_3 \times I$ on the equivariant collar neighbourhood $P^2(\mathbf{C}) \times [0, 1]$ and terminating at some reducible connections. Since, up to gauge equivalence, there exists a unique reducible connection $[D]$, all these three arcs γ_i must converge to this point $[D]$ in $\mathscr{M}$.

Associated to these arcs of fixed points we have three families $\{D_i\}$, $i = 1, 2, 3$ of irreducible connections in $\mathscr{A}$ with isotropy subgroups $\mathscr{G}(\pi)_{D_i}$ isomorphic to an extension of π by ± 1. From (2.1), it is difficult to see that the three isotropy subgroups $\mathscr{G}(\pi)_{D_i}$, although isomorphic to each other represent distinct subgroups in $\mathscr{G}(\pi)$. This follows, by a similar argument as before, since we can rule out the possibilities that two of sets of rotation numbers become the same i.e. $(a_i, b_i) = (a_j, b_j)$, or (b_j, a_j), $(-a_j, -b_j)$, or $(-b_j, -a_j)$.

Since the three arcs γ_i converge to D, the three corresponding isotropy subgroups $\mathscr{G}(\pi)_{D_i}$ must appear as subgroups in $\mathscr{G}(\pi)_D$. In view of (1.13), this means there exist three distinct irreducible characters $\chi_{k_1}, \chi_{k_2}, \chi_{k_3}$ such that $\pi_D(\chi_{k_i})$ coincides with $\mathscr{G}(\pi)_{D_i}$ as subgroups of $\mathscr{G}(\pi)_D$. In addition, the difference $\dim V(\chi_{k_i}) - \dim W(\chi_{k_i})$ in the dimensions of the corresponding eigenspaces in V and W equals 1, the same as the dimension of the arcs.

Let V' and W' denote respectively the orthogonal complements of these eigenspaces in V and W, i.e.:

$$V = \bigoplus_{i=1}^{3} V(\chi_{k_i}) \oplus V', \qquad W = \bigoplus_{i=1}^{3} W(\chi_{k_i}) \oplus W',$$

In other words, V' and W' consist of all the irreducible components in V and W different from $\mathbf{C}(\chi_{k_i})$, $i = 1, 2, 3$. To prove Theorem B, it is enough to show that $V' \cong W'$ as π_D-representations. For then

$$[V] - [W] = [\mathbf{C}(\chi_{k_1})] + [\mathbf{C}(\chi_{k_2})] + [\mathbf{C}(\chi_{k_3})] \in \mathrm{RO}_+ \mathscr{G}(\pi)_D)$$

and so there is no obstruction to equivariant transversality (see Example 1.2). Applying the method of Donaldson [13, Theorem 4.11], it follows that a neighbourhood of $[D]$ is equivariantly isomorphic to a cone on $P^2(\mathbf{C})$. The π-action on $P^2(\mathbf{C})$ is linear, and since the rotation numbers (a_i, b_i), $i = 1, 2, 3$, can be translated along the three arcs in $\mathscr{M}$ to the rotation numbers around the fixed point of $(\pi, P^2(\mathbf{C}))$, it follows that they can be identified with the linear model $(\pi, P^2(\mathbf{C}))$.

Suppose if possible that $V' \not\cong W'$, and because $\dim V' = \dim W'$, there would be at least one irreducible representation $C(\chi)$ which appears with greater multiplicity in V' than in W'. By stratumwise transversality, each of these excess eigenspaces in V' would give rise to stratum which contains D in the closure but is disjoint from the arcs γ_i.

Let $\mathcal{N}$ denote one of the lowest strata obtained from the above construction. Let π' denote the associated isotropy subgroup of $\mathcal{N}$ and let $\bar{\mathcal{N}}$ denote the closure. Then the link $\operatorname{lk}(D)$ of D in $\bar{\mathcal{N}}$ forms a closed submanifold. In fact, by further perturbation if necessary, we may assume that $\operatorname{lk}(D)$ is a complex projective space $P^i(\mathbf{C})$, $i \geqq 0$ [15].

Away from the neighbourhood of D, the subspace $\bar{\mathcal{N}}$ may have other singularities. As in (2.1), we can think of $\bar{\mathcal{N}}$ as a submoduli space in $\mathcal{M}$ consisting of connections compatible with a π'-SU(2) bundle structure on P. The singularities in $\bar{\mathcal{N}}$ are due to connections with a bigger isotropy subgroup $\pi'' \supset \pi'$. By forgetting the extra symmetries in π'', we can perturb $\bar{\mathcal{N}}$ − cone(D) using only π'-equivariant maps. This new perturbation has the effect of removing the singularities from $\bar{\mathcal{N}}$ − cone(D) and so the result is a compact manifold with $P^i(\mathbf{C})$ as its boundary.

In (2.11), the dimension of $\mathcal{N}$ is computed by the index of the fixed point subelliptic complex $\Omega^*_-(\operatorname{Ad} P)^{\pi'}$ in the self-dual complex $\Omega^*_-(\operatorname{Ad} P)$. In fact, given a connection $[A]$ in $\mathcal{N}$, we can interpret this index as vector space which in turn can be identified with the tangent space of $\mathcal{N}$ at $[A]$. Moreover, as we vary the connections, the corresponding index vector spaces form an index vector bundle which can be identified with the tangent bundle of $\mathcal{N}$. Its determinant line bundle is just the orientation line bundle of $\mathcal{N}$. From [13, Sect. 5], the determinant line bundle of the self-dual complex $\Omega^*_-(\operatorname{Ad} P)$ is orientable. On the other hand, $\Omega^*_-(\operatorname{Ad} P) = \Omega^*_-(\operatorname{Ad} P)^{\pi'} \oplus [\Omega^*_-(\operatorname{Ad} P)^{\pi'}]^\perp$ and the complementary bundle can be written as a sum of π'-eigenbundles with non-trivial eigenvalues. It follows that this complementary index bundle $[\Omega^*_-(\operatorname{Ad} P)^{\pi'}]^\perp$ has a complex structure and therefore is orientable. As a result, $\mathcal{N}$ is orientable and since the orientation of a determinant line bundle is invariant under perturbation, the above smooth perturbation of $\bar{\mathcal{N}}$ − cone(D) is also orientable.

Associated to the complex structure on the complementary index bundle, we have a complex determinant line bundle defined on $\mathcal{N}$. Once again we can extend this line bundle to the smooth perturbation of $\bar{\mathcal{N}}$ − cone(D). Restricted to the boundary of this line last manifold, this line bundle coincides with the Hopf bundle H of $P^i(\mathbf{C})$. However, a straightforward calculation shows this is impossible because the pair $(P^i(\mathbf{C}), H)$ represents a non-zero element in the bordism group $\Omega_{2i}(BU(1))$. Thus in conclusion, the supposition on dim $V'(\chi) >$ dim $W'(\chi)$ leads to a contradiction. The proof of Theorem B is complete.

Remark 4.2. The argument in the last few paragraphs is similar to the one in [15, p. 358] where they use SO(3) instead of SU(2) as the structural group.

5 The proof of Theorem C

Next we consider smooth, semi-free, cyclic group actions (π, X) on a positive definite, simply-connected 4-manifold as in Theorem C. For the moment we only assume that the Euler characteristic $\chi(X^\pi) \leqq 3$ and $|\pi|$ is arbitrary. Our object is to compare (π, X) with the following linear models:

(5.1) We begin with the action $(\pi, P^2(\mathbf{C}))$ given by $t \cdot [z_1 : z_2 : z_3] = [\zeta^{r_1} z_1 : \zeta^{r_2} z_2 : \zeta^{r_3} z_3]$ where ζ is a root of unity. Then on the free part of the action we perform equivariant connected sum with m copies of $(\pi, \pi \times P^2(\mathbf{C}))$, where the action is given by cyclic permutation, to get $(\pi, \#^n P^2(\mathbf{C}))$, $n = m|\pi| + 1$.

(5.2) We begin with the linear action (π, S^4) given by $t\cdot(z_1, z_2) = (\zeta^{r_1} z_1, \zeta^{r_2} z_2)$, $|z_1|^2 + |z_2|^2 = 1$. Then we perform the same equivariant connected sum operation as in (5.1) to get $(\pi, \#_n P^2(\mathbf{C}))$, $n = m|\pi|$.

Note that the action $(\pi, \pi \times P^2(\mathbf{C}))$ is free and so, after taking connected sum, the resulting manifolds $(\pi, \#_n P^2(\mathbf{C}))$ in either (5.1) or (5.2) have the same fixed point data as $(\pi, P^2(\mathbf{C}))$ or (π, S^4). In particular, the action is semi-free with the following possible fixed point sets:

(5.3) a disjoint union of a 2-sphere and an isolated point, or

(5.4) two isolated points, or

(5.5) three isolated points.

By P.A. Smith theory, it is not difficult to show that each component of the fixed point set X^π is simply-connected. Then our Euler characteristic assumption $\chi(X^\pi) \leqq 3$ implies that the fixed point set of (π, χ) is described by one of (5.3)–(5.5). Moreover when $|\pi|$ is odd, the representation $(\pi, H^2(X, \mathbf{Z})) = \mathbf{Z}^s \oplus (\mathbf{Z}\pi)^m$ where $s = \chi(X^\pi) - 2$.

In case (5.3), we can apply the G-Signature Theorem to conclude that the action (π, X) has the same fixed-point data as in (5.1) with $r_i = r_j$ for some $i \neq j$. In case (5.4), there are no reducible connections in $\mathcal{M}$ fixed under the action π because in the representation $(\pi, H^2(X, \mathbf{Z}))$ there are no elements $\alpha \in H^2(X, \mathbf{Z})^\pi$ with $\alpha^2 = 1$. Therefore, by (3.2) there are two arcs in $\mathcal{M}^\pi$ coming from the collar neighbourhood $X \times I$. By compactness, these two arcs are connected in the interior of $\mathcal{M}^\pi$ and hence the rotational numbers at the two fixed points in X are of the form (a, b) and $(a, -b)$. This agrees with the linear model (5.2).

It remains to discuss case (5.5). We now assume that π has odd order. Then $\mathcal{M}^\pi$ contains exactly one reducible connection $[D]$, because $(\pi, H^2(X, \mathbf{Z}))$ has (up to sign) only one element $\alpha \in H^2(X, \mathbf{Z})^\pi$ with $\alpha^2 = 1$. In this case, there are three arcs in $\mathcal{M}^\pi$ coming from the collar neighbourhood $X \times I$. If two of the arcs are connected in $\mathcal{M}^\pi$ away from the reducible connection $[D]$, then we conclude that their rotational numbers form a cancelling pair as before. From the G-Signature Theorem it follows again that the local fixed point data for (π, X) is the same as in the suitable linear model (5.1). On the other hand, if all three arcs are connected to $[D]$ we are in a similar situation to that handled in the proof of Theorem B. Any extra fixed-point strata of $\mathcal{M}$ in a neighbourhood of $[D]$ would, by compactness, not intersect either the collar $X \times I$ or any other reducible connections. Similar arguments to those already given in Sect. 4 show that we get equivariant transversality around $[D]$, and hence the local fixed point data for (π, X) is the same as in a linear model (5.1).

For each of the possibilities (5.3)–(5.5) we have now constructed a linear model (π, X') with the same local fixed point data as (π, X). By construction, these linear models also have the same equivariant intersection form as (π, X). Since both are smooth actions, the Kirby-Siebenmann invariants are trivial. According to [26, Sect. 3] the equivariant homotopy type of our action is determined by the local fixed point data, the equivariant intersection form and a k-invariant. In [27, 3.7] the latter invariant was replaced by an element in a double coset $O_h(v)\backslash O_{st}(\lambda)/O(\lambda)$. A direct computation shows that in our case, this double coset consists of a single element. Therefore the actions are equivariantly homeomorphic to each other by [27, Theorem A]. This completes the proof of Theorem C.

6 Existence of an equivariant line bundle

To complete our discussion, we will show that the condition (4.1) is often satisfied for group actions on definite four-manifolds.

Theorem 6.1. *Let X^4 be a four-manifold homeomorphic to $\overset{n}{\#} P^2(\mathbf{C})$. If π is a finite group acting on X inducing the identity on homology and $n > 1$, then for every class α in $H^2(X, \mathbf{Z})$ there exists a π-$U(1)$ equivariant bundle $\mathscr{L}$ on X with $c_1(\mathscr{L}) = \alpha$.*

Proof. We recall the classification of equivariant line bundles over a space X with π-action (the following convenient formulation is given in [21]):

$$[X, \mathrm{BU}(1)]^\pi \cong [X \times_\pi E\pi, \mathrm{BU}(1)] .$$

The right-hand side is just the Borel cohomology group $H^2_\pi(X; \mathbf{Z})$ and the left-hand side is the set of isomorphism classes of $\pi - U(1)$ bundles over X (compare [6]).

In view of this classification, to prove (6.1), it is enough to show that the natural map

$$H^2_\pi(X; \mathbf{Z}) \to H^2(X; \mathbf{Z})$$

is surjective. As is well-known, $H^2_\pi(X; \mathbf{Z})$ can be computed by a spectral sequence whose E_2-terms $E_2^{i,j} = H^i(\pi; H^j(X, \mathbf{Z}))$. In particular, by assumption, $H^2(X; \mathbf{Z}) = H^0(\pi, H^2(X; \mathbf{Z}))$ and so $H^2(X; \mathbf{Z})$ appears as the $E_2^{0,2}$-term. We must prove that this term $E_2^{0,2}$ survives to E_∞.

Since X is simply connected, $H^1(X; \mathbf{Z}) = H^3(X; \mathbf{Z}) = 0$ and so $E_2^{i,1} = E_2^{i,3} = 0$, $E_2^{i,2} = H^i(\pi, \mathbf{Z}) \otimes H^2(X, \mathbf{Z})$. Let α_j denote the canonical generators in $H^2(X; \mathbf{Z})$ with $\langle \alpha_i \cup \alpha_j, [X] \rangle = \delta_{ij}$. Then $E_2^{0,2}$ can be written as $\mathbf{Z}(1 \otimes \alpha_1) \oplus \mathbf{Z}(1 \otimes \alpha_2) \ldots \oplus \mathbf{Z}(1 \otimes \alpha_n)$. Since $d_2(1 \otimes \alpha_i) = 0$, the only obstructions for these terms $(1 \oplus \alpha_i)$ to survive to E_∞ are the differentials $d_3(1 \otimes \alpha_i)$. Denote $\gamma_i = d_3(1 \otimes \alpha_i)$. Then, for $i \neq j$,

$$\begin{aligned} 0 &= d_3[1 \otimes \alpha_i \alpha_j] \\ &= [d_3(1 \oplus \alpha_i) \cdot (1 \otimes \alpha_j) + (1 \otimes \alpha_i) \cdot d_3(1 \otimes \alpha_j)] \\ &= \gamma_i \otimes \alpha_j + \gamma_j \otimes \alpha_i . \end{aligned}$$

Since the two factors $\gamma_i \otimes \alpha_j$, $\gamma_j \otimes \alpha_i$ represent linearly independent elements of $H^3(\pi; \mathbf{Z}) \otimes H^2(\pi; \mathbf{Z})$, we have $\gamma_i = \gamma_j = 0$. This proves (6.1).

For a cyclic group π, we have $H^3(\pi, \mathbf{Z}) = 0$ and so the above argument also gives

Theorem 6.2. *Let (π, X) denote a cyclic group action on a simply connected 4-manifold X. Then every element in $H^0(\pi, H^2(X; \mathbf{Z}))$ can be realized as the first Chern class $c_1(\mathscr{L})$ of a π-$U(1)$ equivariant bundle $\mathscr{L}$ on X.*

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