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Smooth structures on algebraic surfaces with finite fundamental group

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In this paper we want to discuss the following

Conjecture. A compact non-singular algebraic surface with finite fundamental group has at least two smooth structures which are stable under blow ups.

This means that for each compact algebraic surface X there exists a homeomorphic smooth manifold Y such that $X \# s \cdot \overline{\mathbb{C}P}^2$ is not diffeomorphic to $Y \# s \cdot \overline{\mathbb{C}P}^2$ for all $s \geqq 0$. To consider smooth structures which are stable under blow ups is very natural from the point of view of algebraic geometry since one has only to study minimal surfaces. Note that the conjecture is false if one allows in addition (arbitrary) stabilization with $\mathbb{C}P^2$ ([W] in the 1-connected case and in general [G] or [Kr_1]).

The main result of this paper is the following.

Theorem. *Let G be a finite group. Then there is a constant $c(G)$ such that the conjecture holds for all algebraic surfaces X with $\pi_1(X) \cong G$, Euler characteristic $e(X) \geqq c(G)$ and $c_1^2(X) \geqq 0$.*

The condition $c_1^2 \geqq 0$ holds automatically for minimal surfaces with finite fundamental group ([BPV], p. 188). Note that for each finite group G there exist algebraic surfaces X with $\pi_1(X) \cong G$, $c_1^2(X) \geqq 0$ and arbitrarily large Euler characteristic. Let Y be a minimal model of an algebraic surface with fundamental group the symmetric group S_m [Sh, p. 402 ff]. Embed G into S_m for some m and let X be the corresponding covering over Y with $\pi_1(X) \cong G$. As noted above $c_1^2(Y) \geqq 0$ and thus $c_1^2(X) \geqq 0$ and we can make $e(X)$ arbitrarily large by choosing m appropriately.

In ([HK_1], Corollary 1.5), we proved that there are only finitely many homeomorphism types of closed oriented 4-manifolds with given finite fundamental group G and fixed Euler characteristic.

Thus we obtain:

Corollary. *Let G be a finite group. Then the conjecture holds for all but perhaps a finite number of homeomorphism types of minimal algebraic surfaces X with $\pi_1(X) \cong G$.*

Our conjecture and the partial results should be seen in the context of a conjecture of Friedman and Morgan which states that the natural map from algebraic surfaces modulo deformation equivalence to smooth 4-manifolds modulo orientation preserving diffeomorphism is finite-to-one [FM_2]. The proof of this conjecture was recently announced in [FM_3]. This result solves our problem in some important cases like for instance the $K3$-surfaces.

Our conjecture is also related to a conjecture of A. Van de Ven: algebraic surfaces of different Kodaira dimension are never diffeomorphic. If this is true, our conjecture holds for those algebraic surfaces which are homeomorphic to an algebraic surface of different Kodaira dimension. An interesting test case for Van de Ven's conjecture is the Barlow surface X which is homeomorphic to $Y = \mathbb{C}P^2 \# 8 \cdot \overline{\mathbb{C}P}^2$, but is of general type. Recently it was shown that $X \# s \cdot \overline{\mathbb{C}P}^2$ is not diffeomorphic to $Y \# s \cdot \overline{\mathbb{C}P}^2$ for all $s \geqq 0$ [Ko].

In Sect. 1 we summarize some known methods and results concerning the conjecture and in Sects. 2 and 3 we give the proof of our Theorem.

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1. Background

In this section we summarize some facts and results concerning the conjecture for specific fundamental groups.

The topological s-cobordism Theorem [Fr_2] reduces the homeomorphism classification for nice fundamental groups to finding topological s-cobordisms. On the other hand Donaldson introduced new diffeomorphism invariants ([Do_1], [Do_2]) which are closely related to algebraic geometry.

In particular Donaldson [Do_2] shows that a simply connected compact algebraic surface is essentially indecomposable.

Theorem 1.1 [Do_2]. *Let X be a 1-connected compact algebraic surface. Then X is not diffeomorphic to a connected sum $M_1 \# M_2$ unless M_1 or M_2 have negative definite intersection form.*

One can use this result in combination with Freedman's classification of 1-connected manifolds to prove the conjecture for 1-connected algebraic surfaces which are homeomorphic to decomposable smooth manifolds.

This shows that the conjecture holds for all 1-connected compact algebraic surfaces X which are not homeomorphic to K, $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# s \cdot \overline{\mathbb{C}P}^2$. A more detailed analysis of the Donaldson invariants leads to a proof of the conjecture for $\mathbb{C}P^2 \# 9 \cdot \overline{\mathbb{C}P}^2$ [FM_1], the Kummer surface [FM_3] as well as $\mathbb{C}P^2 \# 8 \cdot \overline{\mathbb{C}P}^2$ [Ko].

We summarize the present status of the conjecture for 1-connected surfaces as follows.

Theorem 1.2 ([Do_1], [Do_2], [Fr_1], [FM_1], [FM_3], [Ko]). *The conjecture holds for compact 1-connected algebraic surfaces which are not homeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# s \cdot \overline{\mathbb{C}P}^2$ and $s \leqq 7$.*

In the non simply-connected case more work must be done on the homeomorphism classification to apply these ideas.

Theorem 1.3 ([HK_2], [T]). *The conjecture holds for compact algebraic surfaces X in the following cases*:

i) $\pi_1(X) \cong \mathbb{Z}_k$, $k > 1$ *and odd*
ii) $\pi_1(X) \cong \mathbb{Z}_{2k}$ *and* $e(X) \neq 4$
iii) $\pi_1(X) \cong SL_2(F_p)$, $p = 3$ *or* $5 \pmod 8$, $c_1^2(X) \geqq 0$ *and* $e(X) > 12$

Remark. From the known results concerning our conjecture one gets the impression that the simpler the topology of a surface the harder the problem. The hardest case seems to be $\mathbb{C}P^2$. For $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ one has at least a candidate. Note that $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is the 2-fold branched covering of S^4 along the standard embedding of $\mathbb{R}P^2 \# \mathbb{R}P^2$ with normal Euler class 0 (with twisted coefficients). In [FKV] an infinite family of new knottings is constructed. By [Fr_1] each ramified covering is homeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. The same construction can be carried out starting from $\# 2r \cdot \mathbb{R}P^2 \subsetneq S^4$. If $r = 5$ the corresponding ramified covering is diffeomorphic to a Dolgachev surface which is not diffeomorphic to $\mathbb{C}P^2 \# 9 \cdot \overline{\mathbb{C}P}^2$. It would be somewhat surprising if the analogous family of knottings for $r = 1$ would lead to diffeomorphic ramified coverings.

In the situation of Theorems 1.2 and 1.3 the problem simplifies if one blows up an algebraic surface a few times. It is perhaps more realistic to study the following weakening of our conjecture.

Weak conjecture. For each algebraic surface with finite fundamental group there exists an r such that after r blow ups it has at least two smooth structures which are stable under further blow ups.

Theorems 1.2 and 1.3 imply the weak conjecture for all algebraic surfaces with finite cyclic fundamental group.

2. Proof of the theorem

Two 4-manifolds M_0 and M_1 are called stably homeomorphic if for some natural numbers r_0 and r_1, $M_0 \# r_0(S^2 \times S^2)$ is homeomorphic to $M_1 \# r_1(S^2 \times S^2)$. Note that $r_0 = r_1$ if and only if $e(M_0) = e(M_1)$. In [HK_2] we proved the following cancellation result

Theorem 2.1 ([HK_2; 1.3]). *Let M_0 and M_1 be stably homeomorphic closed smooth 4-manifolds with the same finite fundamental group and Euler characteristic. Suppose that M_0 is homeomorphic to $M_0' \# 2(S^2 \times S^2)$. Then M_0 is homeomorphic to M_1.*

We define a $\mathbb{Z}$-action on the set of stable homeomorphism classes of smooth closed oriented 4-manifolds with fixed fundamental group G by connected sum with r copies of the Kummer surface $K:(r, M) \mapsto M \# r \cdot K$. Here for $r < 0$, rK means $|r| \cdot (-K)$. Note that $K \# (-K)$ is homeomorphic to $\# 22(S^2 \times S^2)$. The next step in the proof is the following finiteness result which we will prove in the next section.

Proposition 2.2. *Let G be a finite group. The set of orbits of stable homeomorphism classes of smooth closed oriented 4-manifolds with fundamental group G under the action given by connected sum with K is finite.*

For each orbit choose a representative M_i with $e(M_i)$ minimal such that $-8 \leqq \operatorname{sign} M_i < 8$, and $M_i \cong M_i' \# 2(S^2 \times S^2)$. Then for each closed oriented smooth 4-manifold X with $\pi_1(X) \cong G$ there exist i and r such that X is stably homeomorphic to $M_i \# r \cdot K$. If $e(X) \geqq e(M_i \# r \cdot K)$ then X is homeomorphic to $M_i \# r \cdot K \# s(S^2 \times S^2) = Y$ for some s by Theorem 2.1. On the other hand Donaldson's Theorem (1.1) implies that if X is an algebraic surface, $\tilde{X}$ and $\tilde{Y}$ are not diffeomorphic (note that M_i decomposes as $M_i' \# 2(S^2 \times S^2)$. Thus the proof of our Theorem is finished if we can find a number $c(G)$ such that $e(X) \geqq e(M_i \# r \cdot K)$ for any algebraic surface X with $\pi_1(X) \cong G$, $e(X) \geqq c(G)$ and $c_1^2(X) \geqq 0$.

To compare $e(X)$ with $e(M_i \# r \cdot K)$ we express $e(M_i \# r \cdot K)$ in terms of $e(M_i)$, $\operatorname{sign}(M_i)$ and $\operatorname{sign}(X)$:

$$e(M_i \# r \cdot K) = e(M_i) + 22 \cdot |r|$$

and

$$\operatorname{sign}(X) = \operatorname{sign}(M_i \# r \cdot K) = \operatorname{sign} M_i - 16r.$$

This implies

$$e(M_i \# r \cdot K) = e(M_i) + (11/8)|\operatorname{sign}(X) - \operatorname{sign}(M_i)| .$$

Assume now that X is not simply-connected (see (1.2)). We have the following inequality:

$$e(X) - (11/8)|\operatorname{sign}(X)| \geqq (1/12)\, e(X) .$$

If $\operatorname{sign}(X) < 0$ this is an immediate consequence of the signature theorem and the assumption $c_1^2(X) \geqq 0$. If $\operatorname{sign}(X) \geqq 0$ then $c_1^2(X) > 0$ and X is of general type ([BPV], p. 188). We now apply the Miyaoka-Yau inequality [BPV, p. 207].

This implies together with the formula for $e(M_i \# r \cdot K)$ above (note that $\operatorname{sign}(X) \equiv \operatorname{sign}(M_i) \bmod 16$ and $-8 \leqq \operatorname{sign}(M_i) < 8$):

$$\begin{aligned} e(X) - e(M_i \# r \cdot K) &= e(X) - (11/8)|\operatorname{sign}(X)| - e(M_i) \pm (11/8)\operatorname{sign}(M_i) \\ &\geqq (1/12)\, e(X) - e(M_i) - 11 . \end{aligned}$$

Thus, if $e(X) \geqq 12\, e(M_i) + 132$ we have $e(X) \geqq e(M_i \# r \cdot K)$.

As there are only finitely many M_i's we can define

$$c(G) := 12 \cdot \max\{e(M_i) + 11\}$$

finishing the proof of our Theorem.

3. Proof of proposition 2.2

The normal 1-type of a closed smooth oriented 4-manifold M is the fibre homotopy type of a fibration $\xi: B \to BO$ which is completely characterised by two facts: the homotopy groups of the fibre $\pi_r(F)$ vanish for $r \geqq 2$ and there is a lift of the normal Gauss map $\nu: M \to BO$ to B, denoted by $\bar{\nu}: M \to B$, which is a 2-equivalence (isomorphism on π_1 and surjective on π_2, [Kr_2]). In other words, the normal

1-type is the first stage of a Moore–Postnikov decomposition of the normal Gauss map. The pair $(M, \bar{v})$ represents an element in $\Omega_4(\xi)$, the bordism group of oriented 4-dimensional ξ-manifolds. The group of fibre homotopy self-equivalences $\mathrm{Aut}(\xi)$ operates on $\Omega_4(\xi)$ by composition and the class represented by $(M, \bar{v})$ in $\Omega_4(\xi)/\mathrm{Aut}(\xi)$ is independent of the choice of $\bar{v}$[Kr_2; p. 16]. The set of stable homeomorphism classes of smooth oriented closed 4-manifolds M having ξ as its normal 1-type is isomorphic to $\Omega_4(\xi)/\mathrm{Aut}(\xi)$ [Kr_2; 2.4].

The bordism group $\Omega_4(\xi)$ is finitely generated and $\Omega_4(\xi) \otimes \mathbb{Q} \cong H_4(B; \mathbb{Q})$ under $[M, \bar{v}] \mapsto \bar{v}_*[M]$ (compare [MS]). As $\pi_1(B)$ is finite the transfer $H_4(B; \mathbb{Q}) \to H_4(\tilde{B}; \mathbb{Q})$ is injective. Also $\tilde{B}$ is BSO if $w_2(\tilde{M}) \neq 0$ and BSpin if $w_2(\tilde{M}) = 0$. Thus $\Omega_4(\xi) \otimes \mathbb{Q} \cong \mathbb{Q}$ and the isomorphism is given by the signature.

Since the Kummer surface is 1-connected and almost parallelizable, connected sum with K does not change the normal 1-type of M. As $\mathrm{sign}(K) \neq 0$ we see that the orbit space of stable diffeomorphism classes of closed smooth oriented 4-manifolds M with normal 1-type ξ under the action of K is a finite set.

The proof of Proposition 2.2 follows from this if we show that for a fixed finite fundamental group G the number of normal 1-types of manifolds M with $\pi_1(M) \cong G$ is finite.

This is obvious if $w_2(\tilde{M}) \neq 0$ since then the normal 1-type is $k(\pi_1(M), 1) \times \mathrm{BSO} \xrightarrow{p_2} \mathrm{BSO}$. If $w_2(\tilde{M}) = 0$, let $f\colon Y \to M$ be a map inducing an isomorphism on π_1. Then it follows immediately from the characterization of a Moore–Postnikov factorization that the normal 1-type is determined by $(Y, \nu \circ f)$. We can obtain Y as the geometric realization of a fixed presentation of $G \cong \pi_1(M)$. But the set of homotopy classes of maps $Y \to BO$ is finite.

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