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On $G_n(RG)$ for G a Finite Nilpotent Group

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In [1], H. Lenstra determined $G_0(RG)$ for G abelian, by expressing it as a sum of classical objects. In [6], D. L. Webb extended this calculation to $G_n(RG)$ for noetherian rings and for G finite abelian, finite dihedral, or a quaternionic 2-group. In Theorem 2 we do the same calculation for any finite p-group.

We do the calculation in two steps. In Section 1 we reduce the calculation (Theorem 1), and in Section 2 we go from Theorem 1 to the final answer (Theorem 2). In Section 3 we make a conjecture as to the correct answer for any finite group. We prove this conjecture for a large number of groups, including all finite nilpotent groups (Remarks 10 and 11).

We remark that Webb has also proved this result by his methods [7]. We also thank the referee for removing a restriction on the ring R as well as other useful comments.

1. THE FIRST REDUCTION

Let us fix a prime p throughout this section. Let R be a noetherian ring, and let $R[1/p]$ denote the localization of R with respect to p. We prove

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THEOREM 1. *Let G be a finite p-group and let R be noetherian. Then the following sequence is split exact*

$$0 \to G_n(RG) \to G_n(R[1/p]G) \oplus G_n(R) \to G_n(R[1/p]) \to 0.$$

Proof. We begin by considering the localization map

$$G_n(RG) \to G_n(R[1/p]G).$$

The key step in the proof consists in showing that the ring map $RG \to Re$ induces an isomorphism on the relative groups in the long exact sequence associated to the localization map. Here e denotes the trivial group and the ring map is the one induced by the group homomorphism $G \to e$.

Quillen [2] produces a long exact sequence involving the localization map whose third term is the K-theory of the category of finitely generated RG-modules which are p-torsion. This is Theorem 4 of [2], where we use Swan [5] Corollary 5.12 to identify the quotient category. Let $\mathbf{Tor}_p\,\mathbf{G}$ denote the category of finitely generated RG-modules which are p-torsion.

We now apply the usual theory. If M is in $\mathbf{Tor}_p\,\mathbf{G}$, let N denote the set of elements of M fixed by G. Clearly N is an RG-submodule, and it cannot be zero unless M is. An easy induction argument shows that every object in $\mathbf{Tor}_p\,\mathbf{G}$ has a finite filtration whose quotients are RG-modules on which G acts trivially.

Within the abelian category $\mathbf{Tor}_p\,\mathbf{G}$, the full subcategory of G-trivial modules is an abelian subcategory, so we can apply Quillen's Theorem 3 [2]. It says that the map induced by the projection from $\mathbf{Tor}_p\,\mathbf{e}$ to $\mathbf{Tor}_p\,\mathbf{G}$ is an isomorphism on K-theory.

The proof of Theorem 1 is now an easy Mayer-Vietoris argument combined with the splitting of $RG \to Re$ induced by the inclusion $Re \to RG$.

2. THE ANSWER

The answer is given in terms of the rational representation theory of the group G. To each irreducible rational representation ϕ, we can associate a division algebra D_ϕ: if V_ϕ denotes the rational vector space for ϕ, then $D_\phi = \operatorname{End}_{\mathbb{Q}G}(V_\phi)$. It is well known that

$$\mathbb{Q}G = \prod_{\phi} \operatorname{End}_{D_\phi}(V_\phi), \tag{1}$$

where the product runs over the irreducible rational representations of G (see, e.g., [4, p. 92]).

If G is a p-group, $\mathbb{Z}[1/p]G$ is a maximal $\mathbb{Z}[1/p]$-order in $\mathbb{Q}G$ (see, e.g., [3, (41.1) Theorem, p. 379]) and hence is a product of maximal $\mathbb{Z}[1/p]$-orders in the factors. But each of these is Morita equivalent to a maximal $\mathbb{Z}[1/p]$-order in D_ϕ (see, e.g., [3, (21.7) Corollary, p. 189]). If ϕ is not trivial, let Δ_ϕ denote a maximal $\mathbb{Z}[1/p]$-order in D_ϕ (they are all Morita equivalent). If ϕ is trivial, let Δ_ϕ denote $\mathbb{Z}$.

It is now easy to compute $G_n(R[1/p]G)$ since G_n preserves products and Morita equivalences yield isomorphisms. This computation and Theorem 1 above prove

THEOREM 2. *Let R be a noetherian ring and let G be any finite p-group, p any prime. Then*

$$G_n(RG) = \bigoplus G_n(R \otimes_{\mathbb{Z}} \Delta_\phi),$$

where the sum runs over all the irreducible rational representations of G.

Remark 3. If p is a zero divisor in R then Theorem 1 just says that $G_n(RG)$ is isomorphic to $G_n(R)$ via the usual map $RG \to Re$. Theorem 2 says the same thing.

Remark 4. It is a theorem of Schilling that Δ_ϕ is rather restricted (see [3, (41.9) Theorem, p. 383]).

Remark 5. If R is Dedekind, then

$$G_n(R \otimes_{\mathbb{Z}} \Delta_\phi) = K_n(R \otimes_{\mathbb{Z}} \Delta_\phi)$$

via the Cartan map. For example, $K_n(R \otimes_{\mathbb{Z}} \Delta_\phi)$ could be the class group of integers in an algebraic number field if $n = 0$ or it could be $K_n(\mathbb{Z})$ for any n. The literature contains many more such calculations.

Remark 6. Webb [6] remarks that his results actually hold for the defining K-theory spectra. This is easy to see for our results also.

Remark 7. The Lenstra–Webb Theorem for $G_n(RG)$ with G abelian can be derived from Theorem 2. Write $G = H \oplus P$ where P is the p-Sylow subgroup of G. Then $G_n(RG) = \bigoplus G_n(R[\mathrm{H}] \otimes_{\mathbb{Z}} \Delta_\phi)$ where the sum runs over the irreducible rational representations of P. Furthermore, $G_n(R[H] \otimes_{\mathbb{Z}} \Delta_\phi) = G_n((R \otimes_{\mathbb{Z}} \Delta_\phi)[H])$ so we can induct on the order of G.

Remark 8. The above discussion can be generalized. Let P be a p-group which is normal in G. The same argument that proved Theorem 1 will produce a Mayer–Vietoris sequence

$$\cdots \to G_n(R[G/P]) \to G_n(R[1/p][G/P]) \oplus G_n(RG) \to G_n(R[1/p]G) \to \cdots.$$

One can write $\mathbb{Z}[1/p]G = \mathbb{Z}[1/p][G/P] \times \mathbb{A}$ where the ring map back from $\mathbb{Z}[1/p][G/P]$ to $\mathbb{Z}[1/p]G$ sends each $g \in G/P$ to $1/|P| \sum_{h \in \pi^{-1}(g)} h$ (where π: $G \to G/P$ denotes the projection). From this splitting and the Mayer–Vietoris sequence, we get that

$$G_n(RG) = G_n(R[G/P]) \oplus G_n(R \otimes_{\mathbb{Z}} \mathbb{A}).$$

If one can identify $\mathbb{A}$ in a useful way, one may proceed further. See Remark 10 below for an example of this.

3. The Conjecture

The decomposition (1) is valid for any finite group. One might conjecture that, if R is noetherian,

$$G_n(RG) \text{ is isomorphic to } \bigoplus G_n(R \otimes_{\mathbb{Z}} \Delta_\phi),$$

where the product runs over the irreducible rational representations of G and where Δ_ϕ is a maximal $\mathbb{Z}[1/w_\phi]$-order in D_ϕ. The conjectured value for w_ϕ is g/kx, where g is the order of G, k is the order of the kernel of the representation ϕ, and x is the degree of any of the irreducible complex constituents of the complexification of ϕ.

It is not difficult to see that this conjecture is consistent with the results of this note, and with those of Lenstra [1] and Webb [6].

Remark 9. One useful fact about w_ϕ is the following theorem of Jacobinski ([3, (41.3) Theorem, p. 380]). Both $\mathbb{Z}[1/w_\phi]G$ and $\mathbb{Z}[1/w_\phi] \otimes_{\mathbb{Z}} \mathcal{M}$, where $\mathcal{M}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}G$ containing $\mathbb{Z}G$, are subrings of $\mathbb{Q}G$. Their projections into the factor $\mathrm{End}_{D_\phi}(V_\phi)$ of $\mathbb{Q}G$ are equal. More generally, fix r, and suppose that for $\phi_1, ..., \phi_t$ it happens that each w_{ϕ_i} divides r. Then the projections of $\mathbb{Z}[1/r]G$ and $\mathbb{Z}[1/r] \otimes_{\mathbb{Z}} \mathcal{M}$ into $\times_{i=1}^{t} \mathrm{End}_{D_\phi}(V_\phi)$ are equal.

Remark 10. Return to the situation described in Remark 8. The quotient group G/P acts on the irreducible complex representations of P. Suppose that the isotropy group for each non-trivial irreducible complex representation is a p-group. Then Clifford's Theorem (see, e.g., [4, p. 61]) shows that w_ϕ is a power of p for each irreducible rational representation that does not factor through G/P. By Remark 9 we may identify the $\mathbb{A}$ occurring in Remark 8 with a piece of the maximal $\mathbb{Z}[1/p]$-order of $\mathbb{Q}G$. Hence the conjecture holds for $\mathbb{Q}G$ if it holds for $\mathbb{Q}[G/P]$. Concrete examples of this are the alternating and symmetric groups on four letters, and certain meta-cyclic groups. Indeed, let G have a normal cyclic subgroup P

of order p^r and suppose that the composite $G/P \to \mathrm{Aut}(P) \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ has p-torsion kernel. Then the conjecture holds for G if it holds for G/P.

Remark 11. Finally, the idea in Remark 7 is fairly general. If G_0 and G_1 are two groups for which the conjecture holds, and if the orders of these two groups are relatively prime, then the conjecture holds for $G_0 \oplus G_1$. In particular, the conjecture holds for all finite nilpotent groups.

Given any irreducible rational representation ϕ of $G_0 \oplus G_1$ there exist unique irreducible rational representations ϕ_t of G_t such that ϕ is a constituent of $\phi_0 \otimes \phi_1$. The hypothesis that the orders of G_0 and G_1 are relatively prime ensure that there exists an integer r such that

$$r\phi = \phi_0 \otimes \phi_1.$$

This hypothesis further guarantees that Δ_ϕ and $\Delta_{\phi_0} \otimes_{\mathbb{Z}} \Delta_{\phi_1}$ are Morita equivalent since they are each $\mathbb{Z}[1/w_\phi]$-maximal orders in Morita equivalent simple algebras.

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