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On the Classification of Topological 4-Manifolds with Finite Fundamental Group

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The recent work of Freedman [6] shows that a simply-connected closed topological 4-manifold is determined up to homeomorphism by its intersection form and Kirby-Siebenmann invariant (a $\mathbb{Z}/2$-valued obstruction to triangulation). Moreover *every* unimodular symmetric bilinear form on a finitely generated free abelian group is realized as the intersection form of such a manifold. In this paper we study the homotopy classification and realization of intersection forms for closed oriented topological 4-manifolds with finite fundamental group.

In the non-simply connected case the obvious homotopy invariants are π_1, π_2 as a $\Lambda=\mathbb{Z}[\pi_1]$ module and the intersection form $S:\pi_2\times\pi_2\to\mathbb{Z}$ with respect to which π_1 acts as isometries. To these should be added the first k-invariant

$$k\in H^3(\pi_1,\pi_2)$$

which together with π_1 and π_2 gives the algebraic 2-type introduced by MacLane and Whitehead [13]. For a closed oriented 4-manifold M^4 we define the *quadratic 2-type* of M to be the quadruple

$$[\pi_1(M),\pi_2(M),k(M),S(\tilde{M})]$$

where $S(\tilde{M})$ denotes the intersection form on $\pi_2(M)\cong H_2(\tilde{M},\mathbb{Z})$. An *isometry* of two such quadruples is an isomorphism on π_1, π_2 inducing an isometry of S and respecting the k-invariant.

In Theorem (1.1) we show that the (polarized) homotopy type of a Poincaré 4-complex with finite fundamental group is determined by the quadratic 2-type and two additional invariants. More precisely, there is an invariant in $\mathbb{Z}/|\pi_1|\cdot\mathbb{Z}$ and a secondary obstruction in $\mathrm{Tors}(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$, where Γ is Whitehead's quadratic functor. To obtain more detailed information about these invariants, and the relations among them induced by change of polarization, we specialize the situation further.

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In Sects. 2 and 3 we consider 4-manifolds M such that $\pi_1(M)$ is a group with periodic cohomology of period 4. The special case where π_1 is cyclic of prime order was studied by Wall [20].

Theorem A. *Let M^4 be a closed oriented 4-manifold with $\pi_1(M)$ a finite group having periodic cohomology of period 4. Then the homotopy type of M is determined by the isometry class of its quadratic 2-type. In addition, an isometry between quadratic 2-types can be realized by a homotopy equivalence.*

The groups with periodic cohomology of period 4 (this means that $H^i(\pi_1, \mathbb{Z}) \cong H^{i+4}(\pi_1, \mathbb{Z})$ for all $i > 0$) are described for example in [26], and in particular any finite subgroup of $SU(2)$ has this property. More generally these are the finite groups which act freely on some simplicial complex homotopy equivalent to S^3.

Theorem A relates the homotopy classification to an algebraic problem, but does not identify which quadratic 2-types actually occur for manifolds. Note that without this information we would have to consider forms on arbitrary torsion-free Λ modules. This is far too complicated. In (4.1) we show that a quadratic 2-type can be realized by a manifold if and only if it can be realized stably, and the stable question is answered in Theorem (4.2). Here a *stabilization* of a quadratic 2-type replaces $[\pi_1, \pi_2, k, S]$ by

$$[\pi_1, \pi_2 \oplus \Lambda^{2r}, i_*(k), S \oplus H(\Lambda^r)] ,$$

where $H(\Lambda^r)$ denotes the hyperbolic form and $i: \pi_2 \to \pi_2 \oplus \Lambda^{2r}$ is the inclusion. This operation corresponds to forming the connected sum of a given manifold with r copies of $S^2 \times S^2$. For realization of even forms by spin manifolds with period 4 fundamental groups, it turns out that there are at most two stable isometry classes of quadratic 2-types with given signature 8ℓ. We denote them by $Q(\pi_1, \ell)$ and $Q'(\pi_1, \ell)$. Similarly for realization of odd forms, there is only one stable isometry class $Q''(\pi_1, \ell)$ for each signature ℓ. These are defined in Sect. 4 just before the statement of (4.2).

Our realization result, Theorem 4.2, combined with Theorem A, suggests a way to obtain the homeomorphism classification. First one studies the cancellation problem for quadratic 2-types. This is again an algebraic problem. Then surgery theory [21, 7] should give information about which isometries of the quadratic 2-type can be realized by homeomorphisms. In Sect. 5 we carry out these steps assuming π_1 is cyclic of odd order.

Theorem B. (i) *Let M^4 be a closed oriented 4-manifold with π_1 cyclic of odd order. Then M is determined up to homeomorphism by its intersection form on $H_2(M, \mathbb{Z})/Tors$ and the Kirby-Siebenmann invariant.*

(ii) *An automorphism of $H_2(M, \mathbb{Z})$ can be realized by a self-homeomorphism if and only if it induces an isometry of the intersection form.*

If the intersection form is indefinite, it is determined by signature, Euler characteristic and type (odd or even). For example, this implies that those Dolgachev surfaces [2] with given odd order fundamental group are all homeomorphic.

For another application consider the Godeaux surface X. This is the orbit space of a free $\mathbb{Z}/5$ action on the quintic in $\mathbb{C}P^3$ defined by $\Sigma z_i^5=0$ (a generator $\alpha\in\mathbb{Z}/5$ acts by $[\alpha z_1, \alpha^2 z_2, \alpha^3 z_3, \alpha^4 z_4]$). It has Euler characteristic 11 and signature -7, so X is homeomorphic (by Theorem B) to

$$Y=\Sigma^4 \# \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2} ,$$

where Σ^4 is the smooth rational homology sphere obtained from $L^3(5,1)\times S^1$ by surgery on the circle. However, Donaldson [5] has recently shown that $\tilde{X}$ and $\tilde{Y}$ are not diffeomorphic.

In [4, Corollary 4.2] Donaldson also proves that there is no self-diffeomorphism f of X such that

$$f^*(\mathscr{K}_X)\cong\mathscr{K}_X\otimes\mathscr{L}^{\otimes i} , \quad i=1,2,3 \text{ or } 4 ,$$

where $\mathscr{K}_X$ is the canonical line bundle and $\mathscr{L}$ is the flat complex line bundle associated to the universal covering $\tilde{X}\to X$. He then asks whether such a self-homeomorphism exists. Since X is homeomorphic to Y it follows from Theorem B that the answer is yes. These observations are summarized in

Corollary. (i) *There is a smooth manifold homeomorphic (but not diffeomorphic* [5]*) to the Godeaux surface.*

(ii) *There exists a homeomorphism (but no diffeomorphism* [4]*) f of the Godeaux surface such that $f^*(\mathscr{K}_X)\cong\mathscr{K}_X\otimes\mathscr{L}^{\otimes i}$ for $i=1,2,3$ or* 4.

There are many obstacles to extending the results above to manifolds with more general fundamental groups. It is however sometimes useful (and technically much easier) to classify only up to finite ambiguity.

Corollary to (1.1). *There are only finitely many homeomorphism types of closed oriented 4-manifolds with given finite π_1 and given Euler characteristic.*

1. On the Homotopy Classification of Polarized Poincaré 4-Complexes

MacLane and Whitehead [13] introduced an invariant for a CW-complex X given by the isomorphism class $[\pi_1(X), \pi_2(X), k(X)]$. Here $\pi_2(X)$ has to be considered as a left $\pi_1(X)$-module and

$$k(X)\in \operatorname{Ext}^3(\mathbb{Z}, \pi_2(X))=H^3(\pi_1(X), \pi_2(X))$$

is given by the exact sequence

$$\begin{array}{c} 0\to H_2(\tilde{X})\to H_2(\tilde{X}, \tilde{X}^{(1)})\to C_1\to C_0\to\mathbb{Z}\to 0 \, . \\ \uparrow\cong \\ \pi_2(X) \end{array}$$

In this sequence $X^{(1)}$ is the 1-skeleton of X, $\tilde{X}$ is the universal cover, C_* is the cellular chain complex of $\tilde{X}$ and the homology groups with $\mathbb{Z}$-coefficients are as usual considered as $\pi_1(X)$-modules. Two triples $[\pi_1, \pi_2, k]$ and $[\pi_1', \pi_2', k']$ are isomorphic if there exist isomorphisms between the π_1's and π_2's respecting the π_1-module structure and mapping k into the corresponding k'.

We call $[\pi_1(X), \pi_2(X), k(x)]$ the *algebraic 2-type* of X. It determines the 2-type of X or equivalently the homotopy type of a 2 stage Postnikov tower of X: This is a 3-coconnected CW-complex B ($\pi_i(B)=\{0\}$ for $i\geqq 3$) such that there exists a 3-equivalence $X\to B$. B is the total space of a fibration $B\to K(\pi_1(X),1)$ with fibre $K(\pi_2(X),2)$ and the fibration is determined by the k-invariant $k(X)$. The homotopy type of B depends only on $[\pi_1(X), \pi_2(X), k(X)]$ and is denoted by $B(X)$.

In the following we want to prove a result about the classification of oriented connected 4-dimensional Poincaré complexes X which have the same algebraic 2-type $[\pi_1, \pi_2, k]$ or equivalently with $B(X)\simeq B$. As in similar situations it is easier to determine the corresponding polarized objects. A *B-polarized* oriented 4-dimensional Poincaré complex is a 3-equivalence $f: X\to B$. We denote the set of polarized homotopy types over B by $\mathscr{S}_4^{PD}(B)$.

We need the following notation. For a left π_1-module π_2 let $\Gamma(\pi_2)$ be the left π_1-module of integral symmetric bilinear forms on π_2^*. We denote the intersection form of an oriented Poincaré complex X by $S(X)$. If $f: X\to B$ is a 3-equivalence we consider $S(\tilde{X})$ to be a form on $\pi_2(\mathrm{B}) \underset{f}{\overset{\cong}{\leftarrow}} \pi_2(X)\cong H_2(\tilde{X})$.

Theorem 1.1. *Let $B=B(\pi_1, \pi_2, k)$ be a 3-coconnected CW-complex with fundamental group π_1 and $\mathscr{S}_4^{PD}(B)$ non-empty.*

(i) *If $\pi_1(B)$ is finite there is an exact sequence*

$$0\to \mathrm{Tors}(\Gamma(\pi_2(B))\otimes_\Lambda \mathbb{Z})\to \mathscr{S}_4^{PD}(B)\to \underset{\mathbb{Z}/|\pi_1|\mathbb{Z}}{\underset{\wr\Vert}{H_4(B,\tilde{B})}}\times \underset{\text{on } \pi_2(B)}{\{\text{symmetric } \mathbb{Z}\text{-forms}\}}$$

(ii) *If $\pi_1(B)$ is infinite and $H_2(B;\mathbb{Q})\neq\{0\}$, the map $\mathscr{S}_4^{PD}(B)\to H_4(B;\mathbb{Z})$ sending (X,f) to $f_*[X]$ is injective.*

In the proof we define a free action of the torsion subgroup of $\Gamma(\pi_2(B))\otimes_\Lambda \mathbb{Z}$ on $\mathscr{S}_4^{PD}(B)$, and show that the set of orbits injects into the right side. The map into the right side is given by the image of the fundamental class $f_*[X]\in H_4(B,\tilde{B})$ and the intersection form on $\tilde{X}$.

Remark 1.2. If h is an isometry between the quadratic 2-types of X and Y then by [13] there exist 3-equivalences $f: X\to B$ and $g: Y\to B$ realizing h (i.e. $h=g_*^{-1}\circ f_*$). The Theorem implies that for π_1 finite we have an obstruction in $\mathbb{Z}/|\pi_1|\cdot\mathbb{Z}$ and a secondary obstruction in $\mathrm{Tors}(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$ for realizing h by a homotopy equivalence.

Proof. Let $f: X\to B$ and $g: Y\to B$ be 3-equivalences preserving the intersection form on π_2 in case i) and having same invariant in $H_4(B;\mathbb{Z})$ in case ii) (implying that the intersection form on H_2 is preserved). We first note that (X,f) and (Y,g) are homotopy equivalent over B if there exists a map $h: X\to Y$ such that $g\circ h$ is homotopic to f. We have to show that h is then a homotopy equivalence. As f and g are 3-equivalences, h induces isomorphisms on π_1 and $H_i(\tilde{X})$ for $i<3$. Poincaré duality implies that the same holds for $i\geqq 3$ if h has degree 1. If $H_2(B;\mathbb{Q})\neq\{0\}$ the degree is 1 since h preserves the intersection form. If π_1 is finite and non-trivial we control the degree on the universal covering using the fact that $H_2(\tilde{B};\mathbb{Q})\neq\{0\}$.

These arguments show that we only have to study obstructions for the existence of h.

Lemma 1.3. *The only obstruction for the existence of h as above is the vanishing of* $f_*[X]-g_*[Y]\in H_4(B)$.

We postpone the proof of this Lemma and first finish Theorem 1.1. Since Lemma 1.3 implies statement (ii), it remains to study $H_4(B)$ in more detail to obtain statement (i). The natural homomorphisms give an exact sequence

$$\begin{array}{ccccccc} 0\longrightarrow & H_4(\tilde{B})\otimes_\Lambda \mathbb{Z} & \longrightarrow & H_4(B)\rightarrow & H_4(B,\tilde{B}) & \longrightarrow 0\ . \\ & \uparrow\cong & & & \uparrow\cong & \\ & \Gamma(\pi_2)\otimes_\Lambda\mathbb{Z} & & & \mathbb{Z}/|\pi_1|\cdot\mathbb{Z} & \end{array}$$

The exactness of the sequence and the isomorphism on the right side follows by comparing it with the corresponding exact sequence of a Poincaré complex X admitting a 3-equivalence into B. The isomorphism on the left side follows because $\tilde{B}=K(\pi_2(B),2)$ and thus $H_4(\tilde{B})\cong\Gamma(\pi_2(B))$ [12]. Under this identification the intersection form on $\tilde{X}$ corresponds to $f_*[\tilde{X}]$. For later use we note here that the torsion subgroup of $H_4(\tilde{B})\otimes_\Lambda\mathbb{Z}$ is mapped isomorphically to the torsion subgroup of $H_4(B)$ if $f_*[X]$ is mapped to a primitive element in $H_4(\tilde{B})\otimes_\Lambda\mathbb{Z}$.

Combining this information with the fact that the kernel of the transfer map on $H_4(B)$ is the torsion subgroup, the proof of Theorem (1.1) follows if we can define an action of $\mathrm{Tors}(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$ on $\mathcal{S}_4^{PD}(B)$ which under the map $\mathcal{S}_4^{PD}(B)\rightarrow H_4(B)$ corresponds to addition in $H_4(B)$. For then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_4^{PD}(B)/_{\mathrm{Tors}} & \rightarrow & H_4(B,\tilde{B})\times\{\text{symmetric } \mathbb{Z}\text{-forms on } \pi_2(B)\} \\ \downarrow & & \downarrow\cong \\ H_4(B)/_{\mathrm{Tors}} & \rightarrowtail & H_4(B,\tilde{B})\times H_4(\tilde{B})\ , \end{array}$$

where Tors stands for $\mathrm{Tors}(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$.

Remark 1.4. It is useful to notice that the invariant in $H_4(B,\tilde{B})$ is trivial if $f_*[\tilde{X}]=g_*[\tilde{Y}]$ and $\mathrm{Tors}(H_4(\tilde{B})\otimes_\Lambda\mathbb{Z})\overset{\cong}{\rightarrow}\mathrm{Tors}(H_4(B))$. Hence this invariant is unnecessary whenever these torsion subgroups are isomorphic (which as mentioned above follows if there exist any (X,f) such that $f_*[X]$ maps to a primitive element in $H_4(\tilde{B})\otimes_\Lambda\mathbb{Z}$). We will study this question in Sect. 3.

As $\mathcal{S}_4^{PD}(B)\rightarrow H_4(B)$ is injective, the action with the desired properties exists if for $(X,f)\in\mathcal{S}_4^{PD}(B)$ and $\alpha\in\mathrm{Tors}(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$ we can construct $(X_\alpha,f_\alpha)\in\mathcal{S}_4^{PD}(B)$ with $(f_\alpha)_*[X_\alpha]=f_*[X]+\alpha$.

Given $(X,f)\in\mathcal{S}_4^{PD}(B)$ we write $X=K\cup_g D^4$ with K a 3-complex. We do this in such a way that the orientation of X corresponds to the standard orientation of D^4. For $\alpha\in\mathrm{Tors}(H_4(\tilde{B})\otimes_\Lambda\mathbb{Z})$ let α be the result of mapping a pre-image of α in $H_4(\tilde{B})$ by the map $H_4(\tilde{B})\rightarrow H_4(\tilde{B},\tilde{K})\cong\pi_4(B,K)\rightarrow\pi_3(K)$.

We define $X_\alpha:=K\cup_{g+\alpha'}D^4$. As B is 3-coconnected there is a unique (up to homotopy) extension of $f_{|K}$ to X_α denoted by f_α. Now X_α is a 4-dimensional orientable Poincaré complex (orient it by the standard orientation of D^4). Indeed $H_4(X_\alpha)\cong H_4(X_\alpha,K)\cong\mathbb{Z}$ is generated by $[D^4,S^3]$ denoted $[X_\alpha]$, and X_α is an oriented

Poincaré complex if

$$\cap [X_\alpha] : H^*(\tilde{X}_\alpha) \to H_{4-*}(\tilde{X}_\alpha)$$

is an isomorphism. But this is the case if and only if the intersection form on $\tilde{X}_\alpha$ is unimodular since the homology of $\tilde{X}_\alpha$ is nontrivial only in even dimensions. We will show that X and X_α have the same intersection form.

The intersection form is determined by

$$(\tilde{f}_\alpha)_* [\tilde{X}_\alpha] = \mathrm{tr\,f}((f_\alpha)_* [X_\alpha]) \in H_4(\tilde{B}) \cong \Gamma(\pi_2) \ .$$

We will see from the following diagram that $(f_\alpha)_* [X_\alpha] = f_* [X] + \alpha$. Since $\mathrm{tr\,f}(\alpha) = 0$ this implies $(\tilde{f}_\alpha)_* [\tilde{X}_\alpha] = \tilde{f}_* [\tilde{X}]$. We first look at $X = K \cup_g D^4$ and determine $f_* [X]$ in the diagram

$$\begin{array}{ccccccccc}
 & & & & \Lambda & & & & \\
 & & & & \| & & & & \\
H_4(X) & \xrightarrow{\cong} & H_4(X, K) & \twoheadleftarrow & H_4(\tilde{X}, \tilde{K}) & \xleftarrow{\cong} & \pi_4(X, K) & \longrightarrow & \pi_3(K) \\
\downarrow f_* & & \downarrow & & \downarrow & & \downarrow & & \| \\
H_4(B) & \xrightarrow{\cong} & H_4(B, K) & \twoheadleftarrow & H_4(\tilde{B}, \tilde{K}) & \xleftarrow{\cong} & \pi_4(B, K) & \xrightarrow{\cong} & \pi_3(K) \\
\uparrow & & & & \uparrow & & & & \\
H_4(\tilde{B}) \otimes_\Lambda \mathbb{Z} & & \twoheadleftarrow & & H_4(\tilde{B}) & & & &
\end{array}$$

In the upper line $1 \in \Lambda$ is mapped in $\pi_3(K)$ to g and in $H_4(X)$ to $[X]$. Following the second line g is mapped to $f_*[X]$. We consider now the same diagram for X_α instead of X. The lower two lines are not changed. Thus if we replace g by $g + \alpha'$ the commutativity of the lower part of the diagram implies that $g + \alpha'$ is mapped to $f_*[X] + \alpha$ which by the upper half of the diagram is $(f_\alpha)_* [X_\alpha]$. This finishes the proof of Theorem (1.1).

Proof of (1.3). We choose an orientation preserving embedding $D^4 \subseteq X$ such that $K := X - \mathring{D}^4$ is homotopy equivalent to a 3-complex. Since $g : Y \to B$ is a 3-equivalence we can consider Y as a subcomplex of B with same 3-skeleton. We can homotop $f : X \to B$ such that $f(K) \subseteq Y$.

We want to extend $f_{|K}$ up to homotopy to a map $h : X \to Y$. If we can do this $g \circ h$ and f will automatically be homotopic. For by assumption we have a homotopy on $K \times I$ and since $\pi_4(B) = \{0\}$ there is no obstruction to extending it to $X \times I$.

There is a single obstruction in $H^4(X; \pi_3(Y))$ which decides whether we can extend f to h [1]. As B is 3-coconnected, $\pi_3(Y) \xleftarrow{\cong} \pi_4(B, Y)$ and under this coefficient isomorphism the obstruction is represented by the cocycle

$$\theta : C_4(\tilde{X}) = H_4(\tilde{X}, \tilde{K}) \cong \Lambda \cdot [D^4, S^3] \to \pi_4(B, Y) \ ,$$

defined by sending $[D^4, S^3] \to f_*[D^4, S^3]$.

Thus θ actually sits in $H^4(X; \pi_4(B, Y))$ which by Poincaré duality is isomorphic to $\pi_4(B, Y) \otimes_\Lambda \mathbb{Z}$. This group is isomorphic to $H_4(B, Y)$ as (B, Y) is 3-connected. The image of θ under these isomorphisms is the image of $[X]$ under $H_4(X)\to H_4(B)\to H_4(B, Y)$. Thus θ vanishes if and only if $f_*[X]=g_*[Y]$ in $H_4(B)$. This completes the proof.

For π_1 finite the intersection form is determined up to finite ambiguity by its rank [14, 1.1]. Since the other obstruction groups occuring in Theorem (1.1) as well as $H^3(\pi_1;\pi_2)$ are finite, it implies that there are only finitely many homotopy types with prescribed finite π_1 and Euler characteristic. Furthermore every homotopy equivalence between closed 4-manifolds is weakly simple [23, Proposition 7.2], and since $SK_1(\mathbb{Z}[\pi_1])$ is finite this implies that there are only finitely many simple homotopy types. By surgery theory [21, 7] the difference between the simple homotopy type and the homeomorphism type is measured by normal invariants and surgery obstructions which again can only take finitely many values. Thus we obtain

Corollary 1.5. *There exist only finitely many homeomorphism types of closed oriented 4-manifolds with given Euler characteristic and given finite* π_1.

2. Some Information About $\Gamma(\pi_2)$

We have used Whitehead's functor $\Gamma(\pi_2)$ in the statement of (1.1). This functor is defined for any abelian group L [25] as the group of symmetric bilinear forms on L^*. This is a subgroup of $L\otimes_{\mathbb{Z}}L\cong \mathrm{Hom}_{\mathbb{Z}}(L^*, L)$, the subgroup of symmetric homomorphisms f such that $f=f^*$. Recall that

$$\Gamma(K\oplus L)\cong\Gamma(K)\oplus\Gamma(L)\oplus K\otimes_{\mathbb{Z}}L$$

and that symmetrization defines a homomorphism $s\colon L\otimes_{\mathbb{Z}}L\to\Gamma(L)$.

If a group π acts from the left on L by linear maps (i.e. L is a left $\Lambda=\mathbb{Z}[\pi]$-module) this induces an action of π on $\Gamma(L)$. If we consider $\Gamma(L)$ as a subgroup of $L\otimes_{\mathbb{Z}}L$ this π-action on $\Gamma(L)$ is given by the diagonal action on $L\otimes_{\mathbb{Z}}L$. In terms of homomorphisms $g\in\pi$ maps $f\in\mathrm{Hom}_{\mathbb{Z}}(L^*, L)$ to $g\circ f\circ g^*$. With this convention all maps above become Λ-homomorphisms: $\Gamma(L)$ is a Λ-submodule of $L\otimes_{\mathbb{Z}}L$ and s: $L\otimes_{\mathbb{Z}}L\to\Gamma(L)$ is a Λ-homomorphism. We denote the augmentation ideal in Λ by I.

Theorem 2.1. *Let π be a finite group. If L is either a finitely generated projective Λ-module, I or I^*, then $\Gamma(L)\otimes_\Lambda\mathbb{Z}$ is torsion free.*

Proof. By the additivity formula for the Γ-functor it is enough to show the result for $L=\Lambda$ or I or I^*. To show the result for Λ we actually compute $\Gamma(\Lambda)$ as π_1-module. Let $A\subseteq\pi$ be the subset of all g with $g^2\neq 1$. On A we have a free involution mapping $g\to g^{-1}$. For a given $g\in\pi$ we define a homomorphism $\Lambda\to\Gamma(\Lambda)\subseteq\Lambda\otimes_{\mathbb{Z}}\Lambda$ mapping $1\to 1\otimes g+g\otimes 1$. If we fix for each orbit x of $A/\mathbb{Z}_2$ a representative g_x we obtain a Λ-homomoprhism

$$\Lambda^{|A/\mathbb{Z}_2|}=\prod_{|A/\mathbb{Z}_2|}\Lambda\to\Gamma(\Lambda)\subseteq\Lambda\otimes_{\mathbb{Z}}\Lambda$$

mapping the component corresponding to x by the map $1\to 1\otimes g_x+g_x\otimes 1$.

If $g^2=1$ the Λ-submodule $\Lambda\cdot(1-g)$ is contained in the kernel of the map $\Lambda\to\Gamma(\Lambda)$, given by $1\to 1\otimes g+g\otimes 1$. Thus it induces a Λ-homomorphism $\Lambda/\Lambda\cdot(1-g)\to\Gamma(\Lambda)$. Finally we have a Λ-homomorphism $\Lambda\to\Gamma(\Lambda)$ mapping $1\to 1\otimes 1$.

Lemma 2.2 *The maps above give a Λ-isomorphism*

$$\Lambda\oplus\Lambda^{|A/\mathbb{Z}_2|}\oplus\sum\{\Lambda/\Lambda(1-g): g\neq 1,\ g^2=1\}\xrightarrow{\cong}\Gamma(\Lambda)\ .$$

This implies that $\Gamma(\Lambda)\otimes_\Lambda\mathbb{Z}$ is torsion free if, for $g\in\pi$ with $g^2=1$, $g\neq 1$, the group $\Lambda/\Lambda 1-g)\otimes_\Lambda\mathbb{Z}$ is torsion free. But obviously the augmentation map induces an isomorphism

$$\Lambda/\Lambda(1-g)\otimes_\Lambda\mathbb{Z}\to\mathbb{Z}\ .$$

For $L=I$ or I^* we reduce the proof to the free case by showing

Lemma 2.3. $\Gamma(I)\oplus\Lambda\cong\Gamma(\Lambda)\cong\Gamma(I^*)\oplus\Lambda$.

This completes the proof of (2.1).

Proof of (2.2). As a $\mathbb{Z}$-module Λ has basis $\{g\mid g\in\pi\}$. Thus $\Gamma(\Lambda)$ has $\mathbb{Z}$-basis:

$$\{g\otimes g\mid g\in\pi\}\cup\{g\otimes h+h\otimes g\mid g,h\in\pi, g\neq h\}\ .$$

A simple calculation shows that all these basis elements are contained in the image of the homomorphism above. Furthermore the intersection of the image of two different components in the direct sum is $\{0\}$. Thus it is enough to check that the maps on the components are injective. For the components in $\Lambda\oplus\Lambda^{|A/\mathbb{Z}_2|}$ it is again easy to check that the $\mathbb{Z}$-basis of Λ is mapped to pairwise different basis elements of $\Gamma(\Lambda)$.

Finally we have to check that if $g^2=1$, $g\neq 1$ the map $\Lambda\to\Gamma(\Lambda)$, $1\to 1\otimes g+g\otimes 1$ has kernel $\Lambda\cdot(1-g)$. As mentioned above $\Lambda\cdot(1-g)$ is contained in the kernel. The kernel consists of all $\alpha=\Sigma n_h h$ such that $n_h(h\otimes hg+hg\otimes h)=0$. Now

$$h\otimes hg+hg\otimes h=h'\otimes h'g+h'g\otimes h'$$
$$\Leftrightarrow h'=h\quad\text{or}\quad h'=hg$$

and thus $\Sigma n_h(h\otimes hg+hg\otimes h)=0\Leftrightarrow n_{hg}=-n_h$, for all h.

We are finished if $\alpha=\Sigma n_h h$ with $n_{hg}=-n_h$ for all h implies that α is of the form $\alpha=\beta(1-g)$. For this we consider $B\subset\pi$ such that $x\in B\Rightarrow xg\notin B$ and $B\cup Bg=\pi$. This exists as g has order 2. Let $\beta:=\sum_{x\in B} n_x\cdot x$ then

$$\beta\cdot(1-g)=\sum_{x\in B} n_x\cdot x-\sum_{x\in B} n_x\cdot xg=\sum_{h\in\pi} n_h\cdot h$$

Proof of (2.3). The map $\Gamma(I)\to\Gamma(\Lambda)$ induced by the inclusion $I\to\Lambda$ is injective since $I\to\Lambda$ splits (as a $\mathbb{Z}$-module homomorphism). By using the additivity formula for Γ we note that the cokernel K of $\Gamma(I)\to\Gamma(\Lambda)$ has rank $n=|\pi_1|$ over $\mathbb{Z}$. Let $\{g_1=1,g_2,\dots,g_n\}$ denote the elements of π_1 and consider the $\mathbb{Z}$-basis $\{a_i=g_i-1\mid 2\leqq i\leqq n\}$ for I. As above, a $\mathbb{Z}$-basis for $\Gamma(\Lambda)$ is given by the set

$$1\otimes 1\ ,\quad a_i\otimes a_i\ ,\quad 1\otimes a_i+a_i\otimes 1\ ,\quad a_i\otimes a_j+a_j\otimes a_i\quad(2\leqq i<j\leqq n)\ .$$

Clearly the cokernel K has $\mathbb{Z}$-basis the images of:

$$1\otimes 1\ ,\quad 1\otimes a_i + a_i\otimes 1\quad (2\leqq i\leqq n)$$

so is generated as a Λ-module by the image of $1\otimes 1$. Indeed for any $g\in\pi_1$:

$$g\cdot(1\otimes 1)=g\otimes g\equiv 1\otimes(g-1)+(g-1)\otimes 1+1\otimes 1\quad (\mathrm{mod}\,\Gamma(I))$$

It follows that $K\cong\Lambda$ and $\Gamma(\Lambda)\cong\Lambda\oplus\Gamma(I)$.

For $\Gamma(I^*)$ we apply Γ to the dual homomoprhism to obtain

$$0\to L\to\Gamma(\Lambda^*)\to\Gamma(I^*)\to 0\ .$$

Let $\{\hat{g}_i\}\in\Lambda^*$ be the dual basis to $\{g_i\}$ and set $\hat{\Sigma}=\sum_{i=0}^{n}\hat{g}_i$. Then L has $\mathbb{Z}$-basis

$$\hat{\Sigma}\otimes\hat{\Sigma}\ ,\quad \hat{\Sigma}\otimes\hat{g}_i+\hat{g}_i\otimes\hat{\Sigma}\ ,\quad (2\leqq i\leqq n)\ .$$

Here L is generated as a Λ-module by one element $\hat{\Sigma}\otimes\hat{g}+\hat{g}\otimes\hat{\Sigma}$ for any $g\neq 1$. In fact if $h\in\pi_1$:

$$h\cdot(\hat{\Sigma}\otimes\hat{g}+\hat{g}\otimes\hat{\Sigma})=\hat{\Sigma}\otimes\hat{x}+\hat{x}\otimes\hat{\Sigma}\ ,$$

where $x=hg$. Again we see that $L\cong\Lambda$ and

$$\Gamma(\Lambda)\cong\Gamma(I^*)\oplus\Lambda\ .$$

As a consequence of Theorem 2.1 the term Tors $(\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z})$ occuring in (1.1) is unchanged if we stabilize π_2 by direct sum with a *f.g.* projective module. We conclude this section with a description of the stable isomorphism class of $\pi_2(X)$ as a π_1-module for any finite oriented Poincaré complex X with finite π_1.

Consider an exact sequence

$$0\to\mathscr{Z}_2\to C_2\to C_1\to C_0\to\mathbb{Z}\to 0$$

with C_i finitely generated free Λ-modules. By Schanuel's lemma $\mathscr{Z}_2$ depends up to direct sum with *f.g.* free modules only on π_1 (the notation $\Omega^3\mathbb{Z}$ for $\mathscr{Z}_2$ and $S^3\mathscr{Z}_2$ for $\mathbb{Z}$ will be used below, see [24]).

Proposition 2.4. (i) *Let X be a finite 4-dimensional oriented Poincaré complex. Then there exists an integer r and an exact sequence*

$$0\to\mathscr{Z}_2\to\pi_2(X)\oplus\Lambda^r\to\mathscr{Z}_2^*\to 0\ ,$$

where $\mathscr{Z}_2$ fits into an exact sequence as described above.

(ii) $\pi_2(X)\otimes_{\mathbb{Z}}\mathbb{Q}\cong(I\otimes I^*)\otimes_{\mathbb{Z}}\mathbb{Q}\oplus\mathbb{Q}[\pi_1]^r$ *for some r where I is the augmentation ideal of $\mathbb{Z}[\pi_1(X)]$.*

(iii) *There is a natural isomorphism*

$$\mathrm{Ext}^1_\Lambda(\mathscr{Z}_2^*,\mathscr{Z}_2)\cong H_4(\pi_1(X),\mathbb{Z})$$

under which the class of the extension in (i) corresponds to the image of the fundamental class $c_*[X]$, where $c\colon X\to K(\pi_1(X),1)$ classifies the universal cover.

Proof. Part (ii) follows from (i). In fact over $\mathbb{Q}$ we know $I\mathbb{Q}[\pi_1] \cong I\mathbb{Q}[\pi_1]^*$ and thus K and K^* are both isomorphic to $I\mathbb{Q}[\pi_1]$ as there is an exact sequence

$$0 \to I\mathbb{Q}[\pi_1]^* \to \mathbb{Q}[\pi_1] \to \mathbb{Q}[\pi_1] \to \mathbb{Q}[\pi_1] \to \mathbb{Q} \to 0$$

Furthermore the sequence in (i) splits over $\mathbb{Q}$, thus $\pi_2(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ has stably the desired form. But over $\mathbb{Q}$ cancellation holds so the same is true unstably. For (i) we consider the chain complex C_* of $\tilde{X}$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccc} C_3 \to & C_2 & \xrightarrow{q} & \mathscr{K} & \to 0 \\ \| & \uparrow i & & \uparrow f & \\ C_3 \to & \mathscr{Z}_2 & \xrightarrow{p} & H_2(\tilde{X}) & \to 0 \end{array}$$

where K is the cokernel of $C_3 \to C_2$ anf f the induced map. $\mathscr{Z}_2$ fits into an exact sequence

(2.5) $$0 \to \mathscr{Z}_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

and by Poincaré-duality K^* fits into a similar exact sequence. Thus we are finished if $H_2(\tilde{X})$ is an extension of $\mathscr{K}$ by $\mathscr{Z}_2$. A little diagram chasing shows that the following sequence is exact

$$0 \to \mathscr{Z}_2 \xrightarrow{(p,i)} H_2(\tilde{X}) \oplus C_2 \xrightarrow{f-q} \mathscr{K} \to 0$$

The isomorphism of (iii) is obtained by dimension-shifting in the complete $\hat{\text{Ext}}$ theory [24, Sect. 2] using the indentifications ($k \geqq 0$):

$$\hat{\text{Ext}}_\Lambda^{-k}(K, L) \cong \text{Hom}(S^k K, L)/P\text{Hom}(S^k K, L) \cong \text{Hom}(K, \Omega^k L)/P\text{Hom}(K, \Omega^k L)$$

$$\text{Ext}_\Lambda^k(K, L) \cong \text{Hom}(\Omega^k K, L)/P\text{Hom}(\Omega^k K, L) \; .$$

In particular the extension of (i) is represented by the boundary map $B_3 \to \mathscr{Z}_2$ under the identification

$$\begin{aligned} \text{Ext}_\Lambda^1(\mathscr{K}, \mathscr{Z}_2) &\cong \text{Hom}(\Omega^1 \mathscr{K}, \mathscr{Z}_2)/P\text{Hom}(\Omega^1 \mathscr{K}, \mathscr{Z}_2) \\ &\cong \text{Hom}(B_3, \mathscr{Z}_2)[P\text{Hom}(B_3, \mathscr{Z}_2) \; , \end{aligned}$$

where $\Omega^1 \mathscr{K} \cong B_3 \cong B_2^* \cong S^2 \mathbb{Z}$.

Since

$$H_4(\pi_1, \mathbb{Z}) \cong \text{Ext}_\Lambda^{-5}(\mathbb{Z}, \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \Omega^5 \mathbb{Z})/P\text{Hom}(\mathbb{Z}, \Omega^5 \mathbb{Z})$$

on the right-hand side the class $c_*[X]$ is represented by the homomorphism $\mathbb{Z} \to \Omega^5 \mathbb{Z}$ induced by comparing the chain complex C_* with any projective resolution defining $\Omega^5 \mathbb{Z}$. This element goes to $\partial : B_3 \to \mathscr{Z}_2$ under the isomorphism

$$\begin{aligned} \text{Hom}(\mathbb{Z}, \Omega^5 \mathbb{Z})/P\text{Hom}(\mathbb{Z}, \Omega^5 \mathbb{Z}) &\cong \text{Hom}(S^2 \mathbb{Z}, \Omega^3 \mathbb{Z})/P\text{Hom}(S^2 \mathbb{Z}, \Omega^3 \mathbb{Z}) \\ &\cong \text{Hom}(B_3, \mathscr{Z}_2)/P\text{Hom}(B_3, \mathscr{Z}_2) \; . \end{aligned}$$

Remark 2.5. If $\pi_1(X)$ has cohomology of period 4 we have an exact sequence

(2.6) $$0 \to \mathbb{Z} \to \Lambda \to P_2 \to P_1 \to \Lambda \to \mathbb{Z} \to 0$$

with P_i finitely generated projective modules [19]. Proposition (2.4) and $H_4(\pi_1;\mathbb{Z})=0$ imply that $\pi_2(X)$ is isomorphic to $I\otimes I^*$ up to stabilization with f.g. projective modules. Thus we obtain from (2.1): $\mathrm{Tors}(\Gamma(\pi_2(X))\otimes_\Lambda\mathbb{Z})=0$.

3. Poincaré 4-Complexes Whose Fundamental Group has Cohomology of Period 4

Theorem 3.1. *Let π_1 be a finite group of cohomological period 4. Then the homotopy types of finite oriented connected 4-dimensional Poincaré complexes X whose Spivak normal fibrations admit a* TOP*-reduction are in $1-1$ correspondence with the isometry classes of quadratic 2-types*

$$[\pi_1(X),\pi_2(X),k(X),S(\tilde{X})]\ .$$

This is an immediate consequence of Theorem (1.1) and the Remark (2.5) at the end of Sect. 2, once we know that we can we can omit the invariant in $\mathbb{Z}/|\pi_1|\cdot\mathbb{Z}$.

For a Poincaré complex X we denote the classifying map of the universal cover by $c:X\to K(\pi_1(X),1)$.

Proposition 3.2. *Let X be a finite oriented Poincaré 4-complex whose Spivak normal bundle has a* TOP*-reduction, $\pi_1(X)$ finite and $c_*[X]=0\in H_4(\pi_1(X))$. Then* $\mathrm{Tors}(H_4(\tilde{B}(X)\otimes_\Lambda\mathbb{Z})\xrightarrow{\cong}\mathrm{Tors}(H_4(B(X))$ [*implying that we can omit the invariant in* $\mathbb{Z}/|\pi_1|\mathbb{Z}$ *in* (1.1)]

Proof. By Remark (1.4) we have to show that the intersection form considered as element of $H_4(\tilde{B})$ maps to a primitive element in $H_4(\tilde{B})\otimes_\Lambda\mathbb{Z}$.

We will show this now in a very special case. Let A be a finite 2-complex with finite fundamental group. We denote the boundary of a smooth regular neighborhood of an embedding of A into $\mathbb{R}^5$ by $M(A)$. $\pi_1(M(A))\cong\pi_1(A)$, $\pi_2(M(A))\cong\pi_2(A)^*\oplus\pi_2(A)$ as Λ-modules and $S(\tilde{M}(A))=\begin{pmatrix}0 & I\\ I & C\end{pmatrix}$ is a metabolic form. We abbreviate $\pi_2(M(A))$ by π_2.

We want to show that the image of $S(\tilde{M}(A))$ in

$$\Gamma(\pi_2^*\oplus\pi_2)\otimes_\Lambda\mathbb{Z}\cong\Gamma(\pi_2^*)\otimes_\Lambda\mathbb{Z}\oplus\Gamma(\pi_2)\otimes_\Lambda\mathbb{Z}\oplus\pi_2^*\otimes_\Lambda\pi_2$$

is primitive. The component of $S(\tilde{M}(A))$ in $\pi_2^*\otimes_{\mathbb{Z}}\pi_2=\mathrm{Hom}_{\mathbb{Z}}(\pi_2,\pi_2)$ is the identity map. We will show that the image of Id in $\pi_2^*\otimes_\Lambda\pi_2$ is primitive.

We have a commutative diagram

$$\begin{array}{ccccc}\pi_2^*\otimes_{\mathbb{Z}}\pi_2 & = & \mathrm{Hom}_{\mathbb{Z}}(\pi_2,\mathbb{Z})\otimes_{\mathbb{Z}}\pi_2 & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(\pi_2,\pi_2)\\ \downarrow & & \downarrow & & \downarrow\\ \pi_2^*\otimes_\Lambda\pi_2 & = & \mathrm{Hom}(\pi_2,\Lambda)\otimes_\Lambda\pi_2 & \xrightarrow{\beta} & \mathrm{Hom}_\Lambda(\pi_2,\pi_2)\end{array}$$

where the horizontal maps are the obvious maps and the left vertical map is $f\otimes a\to\hat{f}\otimes a$ with $\hat{f}(x):=\sum_{g\in\pi_1} g\cdot f(g^{-1}(x))$ and the right vertical map is

$$\phi\to\hat{\phi}\quad\text{with}\quad\hat{\phi}(x)=\sum_{g\in\pi_1} g\cdot\phi(g^{-1}(x))\ .$$

In particular $\mathrm{Id} \in \mathrm{Hom}_{\mathbf{Z}}(\pi_2, \pi_2)$ is mapped to $|\pi_1| \cdot \mathrm{Id}$ in $\mathrm{Hom}_\Lambda(\pi_2, \pi_2)$ and so the component $S(\tilde{M}(A))$ in $\pi_2^* \otimes_\Lambda \pi_2$ is mapped under β in $\mathrm{Hom}_\Lambda(\pi_2, \pi_2)$ to $|\pi_1| \cdot \mathrm{Id}$. We will show that $\mathrm{cok}\, \beta = \mathbb{Z}/_{|\pi_1| \cdot \mathbf{Z}}$ generated by Id. This implies that the component of $S(\tilde{M}(A))$ in $\pi_2^* \otimes_\Lambda \pi_2$ is primitive and thus $S(\tilde{M}(A))$ is primitive in $\Gamma(\pi_2^* \oplus \pi_2) \otimes_\Lambda \mathbb{Z}$.

To determine cok β we use the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow \pi_2(A) \longrightarrow & C_2(\tilde{A}) & \xrightarrow{d_2} & C_1(\tilde{A}) & \xrightarrow{d_1} & C_0(\tilde{A}) & \longrightarrow \mathbb{Z} \longrightarrow 0 \\ \| & \| & & \| & & \| & \\ \pi_2 & C_2 & & C_1 & & \Lambda & \end{array}$$

We denote the image of d_2 by R and of d_1 by I. The commutative diagram

$$\begin{array}{ccc} C_2^* \otimes_\Lambda \pi_2 & \longrightarrow\!\!\!\!\rightarrow & \pi_2^* \otimes_\Lambda \pi_2 \\ \downarrow \cong & & \downarrow \beta \\ \mathrm{Hom}_\Lambda(C_2, \pi_2) \rightarrow & \mathrm{Hom}_\Lambda(\pi_2, \pi_2) & \rightarrow \mathrm{Ext}^1_\Lambda(R, \pi_2) \rightarrow 0 \end{array}$$

implies $\mathrm{cok}\, \beta \cong \mathrm{Ext}^1_\Lambda(R, \pi_2)$ and the class represented by Id goes to

$$0 \rightarrow \pi_2 \rightarrow C_2 \rightarrow R \rightarrow 0 \ .$$

Dimension shifting using the exact sequence above implies

$$\mathrm{Ext}^1_\Lambda(R, \pi_2) \cong \mathrm{Ext}^2_\Lambda(I, \pi_2) \cong \mathrm{Ext}^3_\Lambda(\mathbb{Z}, \pi_2) \cong \mathrm{Ext}^2_\Lambda(\mathbb{Z}, R) = \mathrm{Ext}^1_\Lambda(\mathbb{Z}, I)$$

and under these isomorphisms $0 \rightarrow \pi_2 \rightarrow C_2 \rightarrow R \rightarrow 0$ is mapped to

$$0 \rightarrow I \rightarrow \Lambda \rightarrow \mathbb{Z} \rightarrow 0$$

which is a generator of $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, I) \cong \mathbb{Z}/|\pi_1| \cdot \mathbb{Z}$.

This finishes the proof of Proposition 3.2 in the special case $B = B(M(A))$. We want to reduce the general case to this special case. This will be done by showing that the result for Poincaré complexes follows from that for manifolds and then deducing the result for manifolds from the special case $M(A)$.

The statement that the intersection from considered as an element in $H_4(\tilde{B}) \cong \Gamma(\pi_2)$ maps to a primitive element in $\Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$ can be combined with the formula $\Gamma(K \oplus L) \cong \Gamma(K) \oplus \Gamma(L) \oplus K \otimes L$ to make several easy observations. First note that forms S_K and S_L on K^* and L^* respectively give the orthogonal sum $S_K \oplus S_L$ on $K^* \oplus L^*$ whose image in this expression for $\Gamma(K \oplus L)$ is $(S_K, S_L, 0)$. Hence if the image of S_K is primitive so is that of $S_K \oplus S_L$. Furthermore, if S_L is a π_1-equivariant form on a free Λ-module L^* then its image in $\Gamma(L) \otimes_\Lambda \mathbb{Z}$ is divisible by $|\pi_1|$. In this situation if the image of $S_K \oplus S_L$ is primitive then the image of S_K is divisible at most by integers relatively prime to $|\pi_1|$.

Now suppose that X is a finite Poincaré 4-complex with a TOP reduction of its Spivak normal fibre space. Since $H_4(\tilde{B}(X)) \cong \Gamma(\pi_2(X))$ is $\mathbb{Z}$-torsion free and $S(\tilde{X})$ is a primitive element, its image in $\Gamma(\pi_2(X)) \otimes_\Lambda \mathbb{Z}$ can be divisible only by primes dividing $|\pi_1|$. Let $M \rightarrow X$ be a degree 1 normal map from a manifold M which is a 2-equivalence and note that $S(\tilde{M}) \cong S(\tilde{X}) \oplus S_L$ where S_L is a non-singular π_1-equivariant form on a stably free Λ-module L^*. From the remarks above we see that Proposition 3.1 for Poincaré complexes follows from the result for manifolds.

Next let M be a closed oriented 4-manifold with finite π_1 and N a closed 1-connected 4-manifold. A similar argument proves that Proposition 3.2 holds for M if and only if it holds for $M \# N$.

After replacing M by an appropriate connected sum with $\pm \mathbb{C}P^2$'s and perhaps the Chern manifold CH we can assume that M has trivial signature and Kirby-Siebenmann obstruction. In this situation the stable homeomorphism (i.e. up to connected sum with $S^2 \times S^2$'s) type of M is determined by $c_*[M] \in H_4(\pi_1; \mathbb{Z})$ [11, Sect. 3] (compare also the discussion in Sect. 4). This completes the proof.

4. On the Realization of Quadratic 2-Types

We want to know which quadruples $[\pi_1, \pi_2, k, S]$ are the quadratic 2-types of a closed oriented topological 4-manifold. If this is the case we say that $[\pi_1, \pi_2, k, S]$ is *realizable* by a topological manifold.

Our first observation is that this is again a stable problem in the following sense. We say $Q = [\pi_1, \pi_2, k, S]$ and $Q' = [\pi_1', \pi_2', k', S']$ are *stably isomorphic* if and only if for some r, r'

$$Q \oplus H(\Lambda^r) = Q' \oplus H(\Lambda^{r'}) \ .$$

For any Λ-module $H(V)$ is the hyperbolic form on $V \oplus V^*$, and we use the following notation. Given a quadruple $[\pi_1, \pi_2, k, S]$ and a quadratic form (V, λ) we abbreviate $[\pi_1, \pi_2 \oplus V, i_* k, S \oplus \lambda]$ by $[\pi_1, \pi_2, k, S] \oplus (V, \lambda)$. Here $i_* k$ is the image of k under the inclusion of coefficients $\pi_2 \to \pi_2 \oplus V$.

Lemma 4.1. *Let π_1 be a finite group. Then $[\pi_1, \pi_2, k, S]$ is realizable by a topological manifold if and only if there is a representative of the stable isomorphism class which is realizable.*

Proof. ("$\Rightarrow$") If $[\pi_1, \pi_2, k, S]$ is realizable by X, $[\pi_1, \pi_2 \oplus \Lambda^2, i_* k, S \oplus H(\Lambda^2)]$ is realizable by $X \# S^2 \times S^2$.

("$\Leftarrow$") Let us first suppose $[\pi_1, \pi_2 \oplus \Lambda^2, i_* k, S \oplus H(\Lambda^2)]$ is realizable by a topological manifold M. According to Freedman [7] topological surgery is possible to kill the hyperbolic summand. More precisely M can be decomposed as $M' \# S^2 \times S^2$ such that $\pi_2(M) = \pi_2 \oplus \Lambda^2$, $S(\tilde{M}) = S \oplus H$ and $k(M) = i_*(k)$.

This Lemma suggests that the realization problem could be solved by studying the stable homeomorphism classification. This in turn can be reduced to a bordism problem.

Let $E \to B\mathrm{TOP}$ be a fibration and denote the bordism group of closed topological 4-manifolds together with a lift of the normal bundle over E by $\Omega_4(E)$. If this lift is a 2-equivalence (which always can be achieved within the bordism class), the bordism class determines the stable homeomorphism type. For then one can do surgery and handle-subtraction on the bordism to obtain a stable s-cobordism [11, Sect. 2]. Assigning to such a 2-equivalence the quadratic 2-type of M one obtains a map

$$\Omega_4(E) \to (\text{stable isomorphism classes of } [\pi_1, \pi_2, k, S]\}$$

and the realization problem is solved by determining the image of this map.

Before we carry this out in the case where π_1 has cohomology of period 4 we introduce some notation. The exact sequence (2.6) represents a generator of $H^4(\pi_1;\mathbb{Z})\cong H^3(\pi_1;I^*)$ denoted by γ. Let $[P]=[P_2]-[P_1]\in\tilde{K}_0(\mathbb{Z}[\pi_1])$ be the Euler characteristic of this exact sequence. The image of γ under the map induced by the first factor inclusion $I^*\subseteq I^*\oplus I\oplus P^*\oplus P\oplus\Lambda^{2r}$ for any r will be denoted $i_*(\gamma)$. Using these we define the quadratic 2-type

$$Q(\pi_1)=[\pi_1, I^*\oplus I\oplus P^*\oplus P,\ i_*(\gamma), H(I\oplus P)]$$

where $H(I\oplus P)$ is the hyperbolic form on the Λ-module $I\oplus P$.

Similarly we define a quadratic 2-type $Q'(\pi_1)$ with the same π_1, π_2, and k and a certain metabolic form S on $I\oplus P$ which differs from $H(I\oplus P)$ only in its restriction to I. A $\mathbb{Z}$-base for I is given by $\{g-1 : g\in G\}$ and S on I is defined by:

$$S(g-1,h-1)=\begin{cases}2 & \text{if } g=h\\ 1 & \text{if } g\neq h\end{cases}.$$

We will prove later that $Q(\pi_1)$ and $Q'(\pi_1)$ are isometric if and only if all Sylow subgroups of π_1 are cyclic (in [15, 4.8] the latter condition was shown to be sufficient). In addition we note that the stable isometry classes of these quadratic 2-types do not depend on the choice of the exact sequence (2.6).

We denote the orthogonal direct sum of our special quadratic 2-type with ℓ copies of ($\pm$) the positive definite E_8 form tensored over Λ by

$$Q(\pi_1,\ell)=Q(\pi_1)\oplus[\Lambda^{8\ell},\Lambda^{\ell}\otimes(\pm E_8)]\ .$$

This has signature $8|\pi_1|\ell$ for any integer ℓ. A similar definition gives $Q'(\pi_1,\ell)$. Finally for realization of odd intersection forms we need

$$Q''(\pi_1,\ell)=Q(\pi_1)\oplus(\Lambda,(1))\oplus(\Lambda,(-1))\oplus\pm(\Lambda,(1))^{\ell}\ .$$

Theorem 4.2. *Let π_1 be a finite group having periodic cohomology of period 4. Then*

(i) *If S is odd, a quadratic 2-type $[\pi_1,\pi_2,k,S]$ can be realized by a closed topological 4-manifold if and only if it is stably isometric to $Q''(\pi_1,\ell)$ where $\ell=(\operatorname{sign} S)/|\pi_1|$.*

(ii) *If S is even, $[\pi_1,\pi_2,k,S]$ is realizable by a closed topological spin 4-manifold if and only if it is stably isometric to $Q(\pi_1,\ell)$ or $Q'(\pi_1,\ell)$ where $\ell=(\operatorname{sign} S)/8\cdot|\pi_1|$.*

Proof. According to the discussion above we have to study the quadratic 2-types of representative elements in $\Omega_4(E)$ for the various fibrations E. In our situation we have to consider only two cases. In case (i) the corresponding bordism group is $\Omega_4^{\mathrm{TOP}}(K(\pi_1,1))$, the bordism group of singular oriented topological manifolds. In case (ii) it is $\Omega_4^{\mathrm{TOPSPIN}}(K(\pi_1,1))$, the bordism group of singular topological Spin manifolds. As we can arbitrarily vary the signature and Kirby-Siebenmann obstruction by connected sum with an appropriate 1-connected manifold it is enough to determine the quadratic 2-types of elements in these bordism groups for which these invariants vanish. In the first case this leaves us with the 0-element. Let $M(A)$ be the manifold introduced in Sect. 3. To obtain a manifold with odd intersection form on π_2 we add to it $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, and we are finished in this case if the quadratic 2-type of $M(A)$ is $Q(\pi_1)$. This will be shown below.

In the second case the bordism group is more complicated. Consider the E_2 term of the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{\mathrm{TOPSPIN}}(K(\pi_1, 1))$.

$$E_2^{0,4} = \mathbb{Z} \ , \qquad E_2^{2,2} = H_2(\pi_1, \mathbb{Z}_2) \ , \qquad E_2^{3,1} = H_3(\pi_1, \mathbb{Z}) \ ,$$

and the other groups on the line corresponding to Ω_4 are trivial. The bordism group splits into the summand $\mathbb{Z}$ generated by the E_8 manifold and the rest. The image of the normal cobordism classes $[N, G/\mathrm{TOP}]$ in Ω_4 factors through the image of $H_2(\pi_1, \mathbb{Z}_2)$ by comparison of E_2 terms. Since for this fundamental group the surgery obstructions of closed 4-manifold problems are detected by the signature [9], it follows that all elements in the image of $H_2(\pi_1, \mathbb{Z}_2)$ (if not killed by differentials) are represented by 2-equivalences homotopy equivalent to N. Thus the corresponding bordism classes give no additional quadratic 2-types.

Next we consider the classes corresponding to $H_3(\pi_1; \mathbb{Z}) = \mathbb{Z}_2$ (if this group is not killed by a differential). Consider a closed singular Spin 3-manifold $V \to K(\pi_1, 1)$ representing the non-trivial element in $H_3(\pi_1; \mathbb{Z}_2)$. The corresponding element in Ω_4 is represented by $V \times S^1$, where S^1 is equipped with the non-trivial Spin structure. We will show that after replacing this in its bordism class by a 2-equivalence its quadratic 2-type is $Q'(\pi_1)$. Again for the 0-element in the bordism group we can take $M(A)$ as representative.

In the remainder of the proof, we determine (by an indirect argument) the quadratic 2-types of $M(A)$ and the manifold resulting from $V \times S^1$. Consider the following construction. Let Y be a (finitely-dominated) 3-dimensional Poincaré complex with TOP normal bundle and $\pi_1(Y) = \pi_1$. Let X be the 4-dimensional Poincaré complex

$$X = (Y - D^3) \times I \cup (Y - D^3) \times I \ ,$$

the double of $(Y - D^3) \times I$. From the construction it follows that $S(\tilde{X}) = H(I)$. Furthermore X has a top-reduction of its Spivak normal bundle.

Notice that X could equally be described as the result of surgery on an S^1 in $Y \times S^1$ to make π_1 correct. As $Y \times S^1$ is finite this implies that X is finite. In this description we should consider $Y \times S^1$ with its zero-bordant spin structure and do the surgery preserving it.

The k-invariant of X is the image of the k-invariant $\gamma \in H^3(\pi_1, I^*)$ of $Y - D^3$ under the map induced by the inclusion

$$i : \pi_2(Y - D^3) \cong I^* \to \pi_2(X) \cong I \oplus I^*$$

mapping I^* onto $\{0\} \times I^*$. This follows from the commutative diagram using $Y^{(1)} = X^{(1)}$:

$$\begin{array}{ccccccccccc} 0 \to \pi_2(Y - D^3) & \to & H_2(\tilde{Y} - \tilde{D}^3, \tilde{Y}^1) & \to & C_1 & \to & C_0 & \to & \mathbb{Z} & \to & 0 \\ \downarrow & & \downarrow & & \| & & \| & & \| & & \\ 0 \to \pi_2(X) & \to & H_2(\tilde{X}, \tilde{X}^1) & \to & C_1 & \to & C_0 & \to & \mathbb{Z} & \to & 0 \ . \end{array}$$

Thus the quadratic 2-type of X is $[\pi_1, I \oplus I^*, i_* k, H(I)]$ where k is the k-invariant of $Y - D^3$. Since Y has a Top normal bundle there exists a degree 1 normal map $f: V \to Y$, V a 3-manifold. By the same construction as above we get from this map a 4-dimensional degree 1 normal map $g: W \to X$. The surgery obstruction of this normal map is $H(P)$ where $[P]=\sigma(\gamma)$ is the finiteness obstruction. By construction W is zero bordant in $\Omega_4^{\mathrm{TOPSPIN}}(K(\pi_1,1))$ and thus stably homeomorphic to $M(A)$. By surgery we can replace g by a 2-equivalence $h: N\to X$. The quadratic 2-type of N is $Q(\pi_1)$ since by Poincaré duality the quadratic form on N splits into the quadratic form on X and the surgery obstruction $H(P)$.

Notice that if in the construction of X from $Y\times S^1$ and similarly W from $V\times S^1$ we preserve the spin structure which is non-trivial on S^1, we are in the situation described above. Its quadratic 2-type is $Q'(\pi_1)$ by the calculation in [15, Sect. 4]. This finishes the proof of our realization result.

Remark 4.3. If π_1 has a quaternion 2-Sylow subgroup then $Q(\pi_1)$ is not even stably isometric to $Q'(\pi_1)$. In fact the differential

$$d_2: E_2^{5,0}\to E_2^{3,1}\cong H_3(\pi_1,\mathbb{Z}/2)\cong\mathbb{Z}/2$$

is just the dual of Sq^2 on $H_5(\pi_1,\mathbb{Z})$, so this $\mathbb{Z}/2$ survives to give a non-trivial element in

$$\Omega_4^{\mathrm{TOPSPIN}}(K(\pi_1,1))=\pi_4(K(\pi_1,1)\wedge M\mathrm{TOPSPIN}) .$$

Since $B\mathrm{TOP}$ is equal to BG through dimension 2, this element maps non-trivially into $\pi_4(K(\pi_1,1)\wedge MG\langle 4\rangle)$, where $MG\langle 4\rangle$ is the Thom spectrum over the 3-connected cover of BG. Therefore N' is not homotopy equivalent to N and hence has a distinct quadratic 2-type. If π_1 has only cyclic 2-Sylow subgroups this differential is non-trivial killing the corresponding element.

Notice that Theorem (4.2) very much restricts both the possible π_2 and the possible intersection forms on π_2 for these manifolds. In the extreme case when the manifold is a rational homology sphere, it gives some information about the possible fundamental groups. Not every finite group arises: for example, an abelian fundamental group for a rational homology 4-sphere must have rank $\lesssim 3$.

Corollary 4.4. *Let π_1 be a finite group with periodic cohomology of period 4. Then there exists a closed topological 4-manifold M with fundamental group π_1 which is a rational homology sphere.*

Proof. Consider a closed topological 4-manifold M realizing the quadratic 2-type

$$Q(\pi_1)=[\pi_1, I\oplus I^*\oplus P\oplus P^*,\ i_*(k),\ H(I\oplus P)] .$$

By the Roiter Replacement Lemma [17], $I\oplus P\cong J\oplus \Lambda^r$ for some ideal J of Λ locally isomorphic to I. Therefore

$$Q(\pi_1)\cong[\pi_1, J\oplus J^*\oplus\Lambda^{2r},\ i_*(k'),\ H(J)\oplus H(\Lambda^r)]$$

and the summand $H(\Lambda^r)$ may be surgered away to obtain Σ^4 with $\pi_2(\Sigma)\cong J\oplus J^*$. Since J is locally isomorphic to I, the closed topological manifold Σ is a rational homology sphere.

If π_1 has a presentation with an equal number of generators and relations, for example the finite subgroups of $SU(2)$, such examples can be constructed smoothly.

The above result may be of interest in comparing smooth and topological realization of the ordinary intersection forms on $H_2(M,\mathbb{Z})/\mathrm{Tors}$ for 4-manifolds. Donaldson [3] has shown that for arbitrary π_1, a definite form which can be realized smoothly is a standard form. For a given π_1 it is not clear that definite forms are even realized topologically. We note that when a group π_1 is the fundamental group of a topological rational homology 4-sphere then (exactly as in the simply connected case) every unimodular symmetric bilinear form over $\mathbb{Z}$ is the intersection form on $H_2(M, \mathbb{Z})/\mathrm{Tors}$ for some topological 4-manifold M with the given π_1.

Remark 4.5. It should be noted that for even forms. Theorem (4.2) does not consider realization by arbitrary non-spin manifolds. To give a complete result further computation of the bordism groups is required. The following example shows the relation between the k-invariant and the second Stiefel Whitney class in a special case.

Let N and M be the sphere bundles of $\eta \oplus \varepsilon^2$ and 3η over RP^2 respectively, where η is the canonical line bundle and ε is the trivial bundle. The universal cover is in both cases $S^2 \times S^2$ with hyperbolic intersection form on $\pi_2 = \mathbb{Z}_- \oplus \mathbb{Z}_-$. If we fix the basis of π_2 given by $S^2 \times (*)$ and $(*) \times S^2$ where $(*) = (0,0,1) \in S^2$ then from the formula

$$(x, y) \to (-x, \varrho(y)) \quad \varrho = \eta \oplus \varepsilon^2 \quad \text{or} \quad \varrho = 3\eta$$

for the covering transformations of $\tilde{N}$ and $\tilde{M}$, we see that RP^2 is embedded by the quotient of the equivariant map

$$x \to (x, (*)) \quad \text{or} \quad (x, x) \ .$$

It follows that the k-invariants of N and M in

$$H^3(\pi_1, \pi_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

are given by $k(N) = (1,0)$ and $k(M) = (1,1)$ using the basis above for π_2. Therefore the quadratic 2-types of N and M are

$$[\mathbb{Z}_2, \mathbb{Z}_- \oplus \mathbb{Z}_-, (1,0), H(\mathbb{Z}_-)] \quad \text{and} \quad [\mathbb{Z}_2, \mathbb{Z}_- \oplus \mathbb{Z}_-, (1,1), H(\mathbb{Z}_-)]$$

respectively, These quadruples are not (stably) isomorphic. In fact let $w \in \mathbb{Z} \oplus \mathbb{Z}$ reduce mod 2 to the k-invariant, then $S(w, w)$ mod 4 is independent of the choice of w but has value 0 for N and 2 for M.

Note that since $w_2(N) = 0$ and $w_2(M) \neq 0$ they are not homotopy equivalent, so that the k-invariant must be included in the quadratic 2-type to obtain a homotopy classification. In particular, the homotopy type is not determined by $[\pi_1, \pi_2, S]$ for $\pi_1 = \mathbb{Z}_2$.

5. Proof of Theorem B

In the situation of Theorem A we can get a homeomorphism classification from a homotopy classification by surgery techniques [7, 21]. It follows easily that a closed oriented 4-manifold with π_1 finite cyclic of odd order is determined up to homeomorphism by its quadratic 2-type and the Kirby-Siebenmann invariant. One needs to know that in this case $L_5^s(\mathbb{Z}\pi_1) = 0$, $SK_1(\mathbb{Z}\pi_1) = 0$, and that the self equivalences (inducing the identity on homology) act transitively on the normal invariants [21, p. 237].

To prove Theorem B, we will show that the quadratic 2-type of M is determined by the intersection form B_0 on $H_2(M, \mathbb{Z})/\text{Tors}$, and then this surgery argument finishes the proof.

For a cyclic group π of odd order n let Γ denote the usual maximal order in $\mathbb{Q}\pi$ containing $\Lambda = \mathbb{Z}\pi$. Consider the fibre square

$$(5.1) \qquad \begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \prod \hat{\Lambda}_p & \to & \prod \hat{\Gamma}_p \end{array}$$

where the products are taken over the primes dividing n. Now $\Gamma \cong \Gamma_0 \oplus \Gamma_1$ with $\Gamma_0 \cong \mathbb{Z}$ and Γ_1 a direct sum of rings $\mathbb{Z}[\zeta_d]$ for all $d|n$, $d \neq 1$. Let (L, h) be a non-singular Λ-lattice and $(\Gamma L, h)$ its extension to a non-degenerate Γ-lattice. We denote the induced Γ_i-lattices by (L_i, h_i) so that

$$(\Gamma L, h) \cong (L_0, h_0) \oplus (L_1, h_1) \ .$$

Two Λ-lattices are stably isometric if they become isometric after orthogonal direct sum with a hyperbolic form on a free module.

Theorem 5.2. *Let π be an odd order cyclic group and (L, h), (L', h') be non-singular $\Lambda = \mathbb{Z}\pi$ lattices. If*

(i) *(L, h) and (L', h') are stably isometric,*

(ii) *$(L_0, h_0) \cong (L'_0, h'_0)$,*

(iii) *(L, h) is the orthogonal sum of $H(I)$ and a projective Λ-lattice, then $(L, h) \cong (L', h')$.*

Proof. First one can show that cancellation holds over Γ. The results of Jacobowitz [10] show that cancellation of hyperbolic planes is possible locally, so that our forms may be assumed to lie in the same unitary genus. Then Shimura [18; 5.24] classifies the forms within a unitary genus. To obtain cancellation over Γ from this information is an easy generalization of the argument in [16, pp. 162–163], and [8, Sect. 3].

Over $\hat{\Lambda}_p$ for $p|n$ we may assume that our forms have quadratic refinements and apply the method of [22, Theorem 2] to reduce to $\mathbb{F}_p\pi$. For this we use (iii) to conclude that $\text{End}(L \otimes \hat{\mathbb{Z}}_p) \to \text{End}(L \otimes \mathbb{F}_p)$ is surjective (in Wall's result the modules are projective so this is automatic, but here the point is that $I^* \cong I$ is a principal ideal in Λ). It follows tha the image of

$$(5.3) \qquad \prod_{p|n} \text{Aut}((L, h) \otimes \hat{\mathbb{Z}}_p) \to \prod_{p|n} \text{Aut}((\Gamma L, h) \otimes \hat{\mathbb{Z}}_p)$$

misses only the factors corresponding to characters $\chi: \pi \to \mathbb{Q}(\zeta_m)$ with m having the same prime divisors as n.

From (5.1) the set of Λ-lattices which are isometric over Γ and $\hat{\Lambda}$ is bijective to

$$\text{Aut}((L, h) \otimes \hat{\mathbb{Z}}) \backslash \text{Aut}((\Gamma L, h) \otimes \hat{\mathbb{Z}}) / \text{Aut}(\Gamma L, h)$$

where as above the completion is over primes dividing n. By the Strong Approximation Theorem [18, 5.12], applied to $SU(L_1, h_1)$ and the remark just made on the map (5.3), this set has only one element [condition (iii) implies that $U(L_1, h_1)$

contains enough reflections; at each representation the Γ-extension of $H(I)$ is scale equivalent to a hyperbolic plane]. Therefore $(L,h)\cong(L',h')$.

To apply (5.2) to our situation we take

$$(L,h)=H(I)\oplus((H_2(M,\mathbb{Z})/\mathrm{Tors},B_0)\otimes\Lambda)$$

and $(L',h')=(\pi_2(M),S)$. These are stably isometric by Theorem B and conditions (ii) and (iii) follow by construction. Therefore an automorphism of $H_2(M;\mathbb{Z})$ which induces an isometry of B_0 gives an isometry of (L,h).

It remains to consider the k-invariant in $H^3(\pi_1,I\oplus I^*)\cong\mathbb{Z}/n\oplus\mathbb{Z}/n$. If this group is equipped with the hyperbolic form Theorem (4.2) shows that the possible k-invariants are primitive elements $k=(k_1,k_2)$ with $k_1k_2\equiv 0 \pmod n$. The automorphisms of $H(I)$ act transitively on this set: use the fact that

$$I^*\cong\mathbb{Z}[g]/(1+g+g^2+\ldots+g^{n-1})\ , \qquad g \text{ generates } \pi\ ,$$

to (i) lift units from $I^*\to I^*/(g-1)\cong\mathbb{Z}/n$ and (ii) find elements $x_i\in I^*$ with reduction k_i such that $x_1x_2=0$, and $x_1\bar{y}_2+x_2\bar{y}_1=1$, $y_1\bar{y}_2=0$ for some $y_i\in I^*$ (such elements can easily be found with cyclotomic polynomials). Finally, an automorphism of $H_2(M;\mathbb{Z})$ is induced by an isometry of (L,h). After composing with a suitable automorphism of $H(I)$ we may assume that the k-invariant is preserved. This proves part (ii) of Theorem B.

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