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ROUND L-THEORY

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Introduction

The surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ were introduced by Wall [19]. Geometrically the L-groups are defined to be the bordism groups of normal maps with fundamental group π. Algebraically, they are stable isomorphism groups of quadratic forms over $\mathbb{Z}[\pi]$ and their automorphisms. The L-groups $L_*(A)$ of a ring with antistructure A were expressed in Ranicki [12] as the algebraic cobordism groups of A-module chain complexes with Poincaré duality.

For computational purposes Wall [21,22] introduced variant L-groups using only forms on finitely generated free modules of even rank. We denote these variant L-groups by $L^{\mathrm{r}}_*(A)$, giving them the following chain complex description.

A f.g. free A-module chain complex C is 'round' if it has Euler characteristic

$$\chi(C)=0\in\mathbb{Z}.$$

The 'round L-groups' $L^{\mathrm{r}}_*(A)$ are defined here to be the cobordism groups of round chain complexes with Poincaré duality. We also define round symmetric L-groups $L_{\mathrm{r}}^*(A)$, by analogy with the symmetric L-groups $L^*(A)$ appearing in the surgery product formula. A finite n-dimensional geometric Poincaré complex X with Euler characteristic $\chi(X)=0$ has a round symmetric signature

$$\sigma_{\mathrm{r}}^*(X)\in L_{\mathrm{r}}^n(\mathbb{Z}[\pi_1(X)]),$$

and a normal map $(f,b):M\to X$ of such complexes has a round surgery obstruction

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$$\sigma_*^{\mathrm{r}}(f,b) \in L_n^{\mathrm{r}}(\mathbb{Z}[\pi_1(X)]).$$

The term round was introduced by Asimov [1] in connection with the handle decompositions of compact manifolds with Euler characteristic 0. Previously, Reinhart [17] had considered cobordisms with Euler characteristic 0.

The round L-groups have several advantages over the ordinary L-groups:

(i) $L_*^{\mathrm{r}}(A)$ depends functorially on the additive category with involution of based f.g. free A-modules.

(ii) For any rings with antistructure A, B there are defined products

$$L_{\mathrm{r}}^m(A) \otimes L_n^{\mathrm{p}}(B) \to L_{m+n}^{\mathrm{h}}(A \otimes B) \quad (m, n \geq 0),$$

with L_{r}^* the round symmetric L-groups, L_*^{p} the projective quadratic L-groups and L_*^{h} the free L-groups defined using unbased f.g. free modules. In particular, product of this type with the round symmetric signature $\sigma_{\mathrm{r}}^*(S^1) \in L_{\mathrm{r}}^1(\mathbb{Z}[t, t^{-1}])$ $(\bar{t} = t^{-1})$ of the circle S^1 defines the split injection

$$\sigma_{\mathrm{r}}^*(S^1) \otimes - : L_n^{\mathrm{p}}(A) \to L_{n+1}^{\mathrm{h}}(A[t, t^{-1}])$$

in the direct sum decomposition

$$L_{n+1}^{\mathrm{h}}(A[t, t^{-1}]) = L_{n+1}^{\mathrm{h}}(A) \oplus L_n^{\mathrm{p}}(A).$$

(iii) The round L-groups of a product $A_1 \times A_2$ of rings with antistructure are given by

$$L_n^{\mathrm{r}}(A_1 \times A_2) = L_n^{\mathrm{r}}(A_1) \oplus L_n^{\mathrm{r}}(A_2).$$

(iv) If $A = M_m(B)$ is a matrix ring with the conjugate transpose antistructure, then there is a Morita isomorphism

$$L_n^{\mathrm{r}}(A) = L_n^{\mathrm{r}}(B).$$

(v) If π is a finite group, then by Wedderburn's theorem

$$\mathbb{Q}[\pi] = \prod_j M_{m_j}(D_j)$$

for some matrix rings $M_{m_j}(D_j)$ over skewfields D_j. It follows from (ii) and (iv) that with the involution $\bar{g} = g^{-1}$ $(g \in \pi)$ on $\mathbb{Q}[\pi]$ the round L-groups of $\mathbb{Q}[\pi]$ can be expressed as

$$L_n^{\mathrm{r}}(\mathbb{Q}[\pi]) = \prod_j L_n^{\mathrm{r}}(D_j).$$

(vi) Let $(f,b): N^n \to X^n$ be a degree-1 normal map with X a finitely dominated n-dimensional geometric Poincaré complex. Pedersen and Ranicki [10] define a projective surgery obstruction

$$\sigma_*^{\mathrm{p}}(f,b) \in L_n^{\mathrm{p}}(\mathbb{Z}[\pi_1(X)]).$$

If M^m is a closed m-manifold with $\chi(M) = 0$, then the product $M \times X$ is a homotopy finite $(m+n)$-dimensional geometric Poincaré complex. The L-theory product mentioned in (ii) and the product formula for surgery obstructions show that the product $(m+n)$-dimensional normal map

$$(1\times f, 1\times b): M\times N \to M\times X$$

has finite surgery obstruction

$$\sigma_*^{\mathrm{h}}(1\times f, 1\times b)=\sigma_{\mathrm{r}}^*(M)\otimes\sigma_*^{\mathrm{p}}(f,b)\in L_{m+n}^{\mathrm{h}}(\mathbb{Z}[\pi_1(M)\times\pi_1(X)]),$$

with $\sigma_{\mathrm{r}}^*(M)\in L_{\mathrm{r}}^m(\mathbb{Z}[\pi_1(M)])$ the round symmetric signature of M. In particular, if $\sigma_{\mathrm{r}}^*(M)=0$ or $\sigma_*^{\mathrm{p}}(f,b)=0$, then $\sigma_*^{\mathrm{h}}(1\times f, 1\times b)=0$.

1. Round K-theory

Let A be an associative ring with 1. An A-module chain complex C is *finite* if C is a bounded complex of based f.g. free A-modules

$$C:\quad \cdots\to 0\to C_n\to C_{n-1}\to\cdots\to C_1\to C_0.$$

The *Euler characteristic* of a finite A-module chain complex C is defined as usual by

$$\chi(C)=\sum_{r=0}^{\infty}(-)^r\mathrm{rank}_A(C_r)\in\mathbb{Z}.$$

C is *round* if

$$\chi(C)=0\in\mathbb{Z}.$$

In the classical applications of the algebraic K-groups $K_0(A)$, $K_1(A)$ to topology and chain complexes one considers the reduced K-groups

$$\tilde{K}_0(A)=\mathrm{coker}(K_0(\mathbb{Z})\to K_0(A))=K_0(A)/\{[A]\},$$

$$\tilde{K}_1(A)=\mathrm{coker}(K_1(\mathbb{Z})\to K_1(A))=K_1(A)/\{\tau(-1)\}.$$

A finitely dominated A-module chain complex C has a projective class invariant

$$[C]\in K_0(A)$$

such that the reduced projective class $[C]\in\tilde{K}_0(A)$ is the *finiteness obstruction*: C is chain equivalent to a finite complex if and only if $[C]=0\in\tilde{K}_0(A)$. We shall assume that A is such that the rank of f.g. free A-modules is well-defined, so that there is defined an exact sequence

$$0\to K_0(\mathbb{Z})\to K_0(A)\to\tilde{K}_0(A)\to 0.$$

We do not require this sequence to split. If it does split, e.g., if $A=\mathbb{Z}[\pi]$ is a group ring, then the projective class of C can be expressed as

$$[C]=(\chi(C),[C])\in K_0(A)=K_0(\mathbb{Z})\oplus\tilde{K}_0(A).$$

The *(reduced) torsion* of a chain equivalence $f: C \to D$ of finite A-module chain complexes is defined as usual to be the element

$$\tau(f) = \tau(C(f)) \in \tilde{K}_1(A),$$

with $C(f)$ the algebraic mapping cone. Similarly for CW complexes, with $A = \mathbb{Z}[\pi]$ and the Whitehead group $\mathrm{Wh}(\pi) = \tilde{K}_1(A)/\{\pi^{ab}\}$ replacing $\tilde{K}_1(A)$.

The absolute K-groups $K_i(A)$ $(i = 0, 1)$ have several advantages over the reduced K-groups $\tilde{K}_i(A)$:

(i) $K_i(A)$ is the algebraic K_i-group of the exact category of f.g. projective A-modules, so that any categorical construction translates to the absolute K-groups.

(ii) The K_i-groups of a product ring $A_1 \times A_2$ are given by

$$K_i(A_1 \times A_2) = K_i(A_1) \oplus K_i(A_2).$$

(iii) For any rings A, B there are defined products

$$K_0(A) \otimes K_i(B) \to K_i(A \otimes B).$$

(iv) If $A = M_m(B)$ is a matrix ring, then by Morita theory

$$K_i(M_m(B)) = K_i(B).$$

(v) If π is a finite group, then by Wedderburn's theorem

$$\mathbb{Q}[\pi] = \prod_j M_{m_j}(D_j)$$

for some matrix rings $M_{m_j}(D_j)$ over skewfields D_j, so that by (ii) and (iv)

$$K_i(\mathbb{Q}[\pi]) = \prod_j K_i(D_j).$$

Round K-theory is the development of the algebraic K-groups $K_i(A)$ $(i = 0, 1)$ using round finite chain complexes. The main result is that the projective class $[C] \in K_0(A)$ of a finitely dominated A-module chain complex C is the *round finiteness obstruction*: C is chain equivalent to a round finite complex if and only if $[C] = 0 \in K_0(A)$. The *absolute torsion* of a chain equivalence $f: C \to D$ of round finite A-module chain complexes is an element

$$\tau(f) \in K_1(A)$$

with reduction the usual torsion $\tau(f) \in \tilde{K}_1(A)$ – see Ranicki [15,16] for details.

2. Round L-theory

Round L-theory is the development of the algebraic L-groups $L^*(A)$ (resp. $L_*(A)$) using round finite chain complexes with Poincaré duality.

Let now A be an associative ring with 1, together with an antiautomorphism

$$\bar{\ }: A \to A; \quad a \mapsto \bar{a},$$

such that

$$(\overline{a+b}) = \bar{a} + \bar{b}, \qquad (\overline{ab}) = \bar{b} \cdot \bar{a}$$

and let $\varepsilon \in A$ be a unit such that

$$\bar{\varepsilon} = \varepsilon^{-1}, \qquad \bar{\bar{a}} = \varepsilon^{-1} a \varepsilon \in A.$$

In the case when ε is central in A the antiautomorphism is an involution. In general $(\bar{\ }, \varepsilon^{-1})$ is an antistructure in the sense of Wall [20, 21]. The ε-symmetric (resp. ε-quadratic) L-groups $L^n(A, \varepsilon)$ (resp. $L_n(A, \varepsilon)$) were defined in Ranicki [12] to be the cobordism groups of n-dimensional ε-symmetric (resp. ε-quadratic) Poincaré complexes over A. For simplicity we shall denote such complexes only by the underlying chain complex C.

Given a $*$-invariant subgroup $X \subseteq \tilde{K}_i(A)$ $(i = 0, 1)$ of the reduced K_i-group such that $\tau(\varepsilon) \in X$ if $i = 1$ there are defined the ε-symmetric (resp. ε-quadratic) L-groups $L^n_X(A, \varepsilon)$ (resp. $L^X_n(A, \varepsilon)$) of n-dimensional symmetric (resp. quadratic) Poincaré complexes C such that for $i = 0$ the underlying chain complex C is finitely dominated with $[C] \in X \subseteq \tilde{K}_0(A)$, while for $i = 1$ C is finite with $\tau(C) \in X \subseteq \tilde{K}_1(A)$, the torsion $\tau(C)$ being that of the Poincaré duality chain equivalence $C^{n-*} \to C$.

We shall now define variant L-groups decorated by $*$-invariant subgroups $X \subseteq K_i(A)$ $(i = 0, 1)$ of the absolute K_i-group. For $X \subseteq K_0(A)$ these are the bordism groups $L^n_X(A, \varepsilon)$ (resp. $L^X_n(A, \varepsilon)$) as defined above, but restricting all projective classes to lie in X. If $[A] \in X \subseteq K_0(A)$ these L-groups can be identified with the L-groups associated to the image $*$-invariant subgroup $\tilde{X} \subseteq \tilde{K}_0(A)$ (Hambleton, Taylor and Williams [5]). For $*$-invariant subgroups $X \subseteq K_1(A)$ we need the following definition.

An n-dimensional algebraic Poincaré complex C is *round* if the underlying chain complex is round finite.

Let $X \subseteq K_1(A)$ be a $*$-invariant subgroup. (It is not necessary to assume that $\tau(\varepsilon) \in X$ in the round case.) The *round ε-symmetric* (resp. *ε-quadratic*) *L-group* $L^n_{\mathrm{r}X}(A, \varepsilon)$ (resp. $L^{\mathrm{r}X}_n(A, \varepsilon)$) is the cobordism group of round n-dimensional ε-symmetric (resp. ε-quadratic) Poincaré complexes C over A with torsion

$$\tau(C) \in X \subseteq K_1(A).$$

In the extreme cases $X = \{0\}$, $K_1(A)$ the round ε-symmetric (resp. ε-quadratic) L-groups are denoted by

$$L^n_{\mathrm{r}\{0\}}(A, \varepsilon) = L^n_{\mathrm{rs}}(A, \varepsilon), \qquad L^n_{\mathrm{r}K_1(A)}(A, \varepsilon) = L^n_{\mathrm{rh}}(A, \varepsilon)$$

$$(\text{resp. } L^{\mathrm{r}\{0\}}_n(A, \varepsilon) = L^{\mathrm{rs}}_n(A, \varepsilon), \qquad L^{\mathrm{r}K_1(A)}_n(A, \varepsilon) = L^{\mathrm{rh}}_n(A, \varepsilon)).$$

For $\varepsilon = 1$ (A, ε) is abbreviated to A.

Algebraic surgery was used in Ranicki [12] to identify the ε-symmetric (resp. ε-quadratic) Poincaré cobordism group $L^n_X(A, \varepsilon)$ (resp. $L^X_n(A, \varepsilon)$) $(X \subseteq \tilde{K}_i(A),\ i = 0, 1)$ for $n = 0, 1$ (resp. $n \geq 0$) with a Witt group of $(-)^k\varepsilon$-symmetric (resp. $(-)^k\varepsilon$-

quadratic) forms if $n=2k$ and formations if $n=2k+1$, and also to identify for a Dedekind (resp. any) ring with antistructure A

$$L_X^n(A,\varepsilon)=L_X^{n+2}(A,-\varepsilon)=L_X^{n+4}(A,\varepsilon)$$

$$(\text{resp. } L_n^X(A,\varepsilon)=L_{n+2}^X(A,-\varepsilon)=L_{n+4}^X(A,\varepsilon)).$$

Similarly:

Proposition 2.1. *The round ε-symmetric (resp. ε-quadratic) L-group $L_{\mathrm{r}X}^n(A,\varepsilon)$ (resp. $L_n^{\mathrm{r}X}(A,\varepsilon)$) $(X\subseteq K_1(A))$ for $n=0,1$ (resp. $n\geq 0$) is naturally isomorphic to a Grothendieck-Witt group of $(-)^k\varepsilon$-symmetric (resp. $(-)^k\varepsilon$-quadratic) forms if $n=2k$ and formations if $n=2k+1$. The round symmetric (resp. quadratic) L-groups of a Dedekind (resp. any) ring with antistructure A are 4-periodic, with*

$$L_{\mathrm{r}X}^n(A,\varepsilon)=L_{\mathrm{r}X}^{n+2}(A,-\varepsilon)=L_{\mathrm{r}X}^{n+4}(A,\varepsilon)$$

$$(\textit{resp. } L_n^{\mathrm{r}X}(A,\varepsilon)=L_{n+2}^{\mathrm{r}X}(A,-\varepsilon)=L_{n+4}^{\mathrm{r}X}(A,\varepsilon)).\quad\square$$

In particular, $L_{\mathrm{r}X}^0(A,\varepsilon)$ (resp. $L_0^{\mathrm{r}X}(A,\varepsilon)$) is the abelian group of equivalence classes of formal differences $[M,\phi]-[M',\phi']$ (resp. $[M,\psi]-[M',\psi']$) of non-singular ε-symmetric (resp. ε-quadratic) forms over A with M, M' based f.g. free A-modules of the same rank and

$$\tau(\phi:M\to M^*)-\tau(\phi':M'\to M'^*)\in X\subseteq K_1(A)$$

$$(\text{resp. } \tau(\psi+\varepsilon\psi^*:M\to M^*)-\tau(\psi'+\varepsilon\psi'^*:M'\to M'^*)\in X\subseteq K_1(A))$$

subject to the usual Witt relation with the evident rank and torsion restrictions. Thus the quadratic round L-groups $L_n^{\mathrm{r}X}(A)$ agree with the variant L-groups defined by Wall [21].

We can also define relative and triad L-groups. Let

$$\begin{array}{ccc} A_0 & \xrightarrow{f_{01}} & A_1 \\ {\scriptstyle f_{02}}\downarrow & \Phi & \downarrow{\scriptstyle f_{13}} \\ A_2 & \xrightarrow[f_{23}]{} & A_3 \end{array}$$

be a commutative square of rings with antistructure. Fix $i=0$ or 1, and let $X_j\subseteq K_i(A_j)$ be $*$-invariant subgroups for $j=0,1,2,3$, such that $(f_{jk})_*(X_j)\subseteq X_k$. Just as in [13] we can define variant and round versions of the relative and triad L-groups to obtain the corresponding versions of the sequences of 2.5.1 and 6.1.1 of [13, pp. 167, 484].

The Mayer-Vietoris sequence for an arithmetic square (discussed in [13, p. 374] for the usual L-groups) also holds for L-groups based on subgroups of K_0 or K_1.

One can use the comparison sequences in Section 3 below to prove that relevant triad L-groups vanish.

3. Comparison sequences

Given $*$-invariant subgroups $Y \subseteq X \subseteq \tilde{K}_i(A)$ ($i=0,1$) such that $\tau(\varepsilon)\in X$ if $i=1$ let $L^n_{X,Y}(A,\varepsilon)$ (resp. $L_n^{X,Y}(A,\varepsilon)$) be the cobordism group of n-dimensional ε-symmetric (resp. ε-quadratic) Poincaré pairs (D,C) over A such that

$$[C]\in Y,\quad [D]\in X\subseteq \tilde{K}_0(A) \qquad \text{if } i=0$$

$$\tau(C)\in Y,\quad \tau(D,C)\in X\subseteq \tilde{K}_1(A) \qquad \text{if } i=1,$$

so that there is defined an exact sequence

$$\cdots\to L^n_Y(A,\varepsilon)\to L^n_X(A,\varepsilon)\to L^n_{X,Y}(A,\varepsilon)\to L^{n-1}_Y(A,\varepsilon)\to\cdots$$

$$(\text{resp. } \cdots\to L_n^Y(A,\varepsilon)\to L_n^X(A,\varepsilon)\to L_n^{X,Y}(A,\varepsilon)\to L_{n-1}^Y(A,\varepsilon)\to\cdots).$$

Let $\mathbb{Z}_2$ act on X/Y by the duality antistructure, so that the Tate $\mathbb{Z}_2$-cohomology groups $\hat{H}^*(\mathbb{Z}_2;X/Y)$ are defined as usual. It was proved in Ranicki [12] that the maps

$$L^n_{X,Y}(A,\varepsilon) \text{ (resp. } L_n^{X,Y}(A,\varepsilon))\to \hat{H}^n(\mathbb{Z}_2;X/Y);$$

$$(D,C)\mapsto [D] \text{ if } i=0,\quad \tau(D,C) \text{ if } i=1$$

are isomorphisms, so that there is defined a comparison exact sequence of the Rothenberg type

$$\cdots\to L^n_Y(A,\varepsilon)\to L^n_X(A,\varepsilon)\to \hat{H}^n(\mathbb{Z}_2;X/Y)\to L^{n-1}_Y(A,\varepsilon)\to\cdots$$

$$(\text{resp. } \cdots\to L_n^Y(A,\varepsilon)\to L_n^X(A,\varepsilon)\to \hat{H}^n(\mathbb{Z}_2;X/Y)\to L_{n-1}^Y(A,\varepsilon)\to\cdots).$$

Similarly:

Proposition 3.1. *Given $*$-invariant subgroups $Y\subseteq X\subseteq K_1(A)$ there are defined relative round L-groups $L^n_{\mathrm{r}X,\mathrm{r}Y}(A,\varepsilon)$ (resp. $L_n^{\mathrm{r}X,\mathrm{r}Y}(A,\varepsilon)$), with isomorphisms*

$$L^n_{\mathrm{r}X,\mathrm{r}Y}(A,\varepsilon) \text{ (resp. } L_n^{\mathrm{r}X,\mathrm{r}Y}(A,\varepsilon))\to \hat{H}^n(\mathbb{Z}_2;X/Y);\quad (D,C)\mapsto \tau(D,C)$$

and comparison exact sequences

$$\cdots L^n_{\mathrm{r}Y}(A,\varepsilon)\to L^n_{\mathrm{r}X}(A,\varepsilon)\to \hat{H}^n(\mathbb{Z}_2;X/Y)\to L^{n-1}_{\mathrm{r}Y}(A,\varepsilon)\to\cdots$$

$$(\text{resp. } \cdots\to L_n^{\mathrm{r}Y}(A,\varepsilon)\to L_n^{\mathrm{r}X}(A,\varepsilon)\to \hat{H}^n(\mathbb{Z}_2;X/Y)\to L_{n-1}^{\mathrm{r}Y}(A,\varepsilon)\to\cdots \qquad \square$$

Let $X\subseteq K_1(A)$ be a $*$-invariant subgroup such that $\tau(\pm\varepsilon)\in X$, and let $\tilde{X}\subseteq \tilde{K}_1(A)$ be the image $*$-invariant subgroup.

Proposition 3.2. *The relative L-groups $L^n_{\tilde{X},\mathrm{r}X}(A,\varepsilon)$ (resp. $L_n^{\tilde{X},\mathrm{r}X}(A,\varepsilon)$) in the exact sequence*

$$\cdots \to L^n_{\mathrm{r}X}(A,\varepsilon) \to L^n_{\tilde{X}}(A,\varepsilon) \to L^n_{\tilde{X},\mathrm{r}X}(A,\varepsilon) \to L^{n-1}_{\mathrm{r}X}(A,\varepsilon) \to \cdots$$

$$(\textit{resp.} \ \cdots \to L_n^{\mathrm{r}X}(A,\varepsilon) \to L_n^{\tilde{X}}(A,\varepsilon) \to L_n^{\tilde{X},\mathrm{r}X}(A,\varepsilon) \to L_{n-1}^{\mathrm{r}X}(A,\varepsilon) \to \cdots)$$

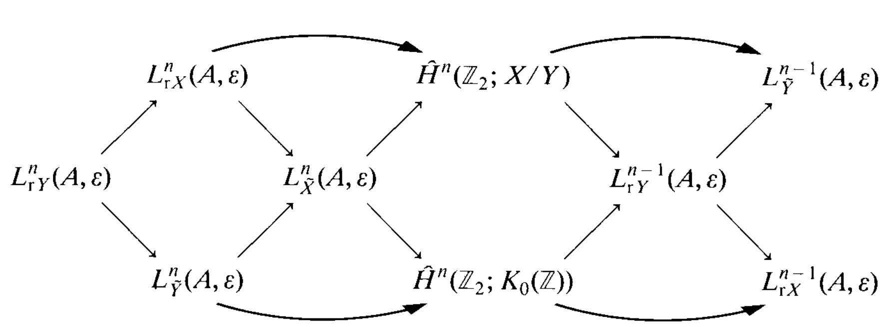
are such that the Euler characteristic defines isomorphisms

$$L^n_{\tilde{X},\mathrm{r}X}(A,\varepsilon) \ (\textit{resp.} \ L_n^{\tilde{X},\mathrm{r}X}(A,\varepsilon)) \to \hat{H}^n(\mathbb{Z}_2; K_0(\mathbb{Z})); \quad (D,C) \mapsto \chi(D). \quad \square$$

The generator $T \in \mathbb{Z}_2$ acts by the identity on $K_0(\mathbb{Z}) = \mathbb{Z}$, so that

$$\hat{H}^n(\mathbb{Z}_2, K_0(\mathbb{Z})) = \begin{cases} \mathbb{Z}_2 \\ 0 \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \end{cases} \pmod 2.$$

Proposition 3.3. *Given $*$-invariant subgroups $Y \subseteq X \subseteq K_1(A)$ such that $\tau(\pm\varepsilon) \in Y$ it is possible to combine the various comparison exact sequences into a commutative braid*



There is a similar braid in the quadratic case. $\square$

The hyperquadratic L-groups $\hat{L}^n(A,\varepsilon)$ were defined in Ranicki [13] to fit into an exact sequence

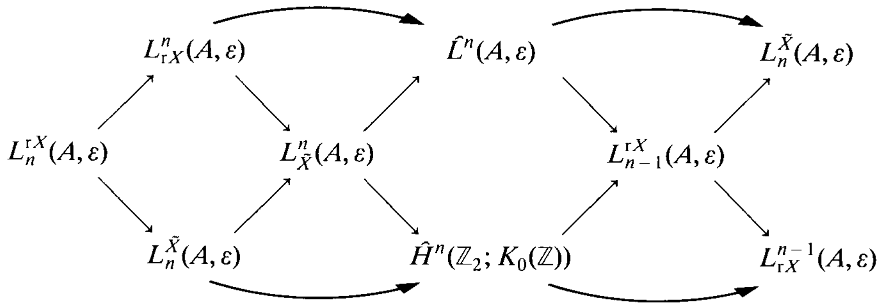
$$\cdots \to L_n^X(A,\varepsilon) \to L^n_X(A,\varepsilon) \to \hat{L}^n(A,\varepsilon) \to L_{n-1}^X(A,\varepsilon) \to \cdots$$

for any $*$-invariant subgroup $X \subseteq \tilde{K}_i(A)$ $(i = 0, 1)$, such that $\tau(\varepsilon) \in X$ if $i = 1$.

Proposition 3.4. *The hyperquadratic L-groups $\hat{L}^*(A,\varepsilon)$ are such that for any $*$-invariant subgroup $X \subseteq K_1(A)$ there is defined an exact sequence*

$$\cdots \to L_n^{\mathrm{r}X}(A,\varepsilon) \to L^n_{\mathrm{r}X}(A,\varepsilon) \to \hat{L}^n(A,\varepsilon) \to L_{n-1}^{\mathrm{r}X}(A,\varepsilon) \to \cdots.$$

If $\tau(\pm\varepsilon) \in X$ there is defined a commutative braid of exact sequences



with $\tilde{X} \subseteq \tilde{K}_1(A)$ *the image of* X. □

4. The round L-theory of $\mathbb{Z}$

We start by recalling the ordinary L-theory of $\mathbb{Z}$:

Proposition 4.1 (Ranicki [12], resp. Kervaire and Milnor [7]). *The symmetric (resp. quadratic) L-groups* $L^*(\mathbb{Z})$ *(resp.* $L_*(\mathbb{Z})$*) of* $\mathbb{Z}$ *are given by*

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ 0 \\ 0 \end{cases}, \qquad L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}_2 \\ 0 \end{cases} \quad \textit{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} (\mathrm{mod}\ 4). \quad \square$$

The isomorphism

$$L^{4k}(\mathbb{Z}) \to \mathbb{Z}; \quad C \mapsto \sigma(C)$$

$$(\text{resp. } L_{4k}(\mathbb{Z}) \to \mathbb{Z}; \quad C \mapsto \sigma(C)/8)$$

sends a $4k$-dimensional symmetric (resp. quadratic) Poincaré complex C over $\mathbb{Z}$ to the signature (resp. $\frac{1}{8}$(the signature)) $\sigma(C)$ of the non-singular symmetric (resp. even symmetric) form

$$F^{2k}(C) \times F^{2k}(C) \to \mathbb{Z}$$

defined on the f.g. free abelian group $F^{2k}(C) = H^{2k}(C)/\text{torsion}$. The isomorphism

$$L^{4k+1}(\mathbb{Z}) \to \mathbb{Z}_2; \quad C \mapsto d(C)$$

sends a $(4k+1)$-dimensional symmetric Poincaré complex C over $\mathbb{Z}$ to the deRham invariant $d(C)$ of the non-singular skew-symmetric linking form

$$T^{2k+1}(C) \times T^{2k+1}(C) \to \mathbb{Q}/\mathbb{Z}$$

on the finite abelian group $T^{2k+1}(C) = \text{torsion}(H^{2k+1}(C))$, which is the parity of the number of 2-primary components in the decomposition of $T^{2k+1}(C)$ as a direct sum of cyclic groups. (The formula for $d(C)$ of Ranicki [12, p. 243] is wrong, and should have read $d(C)=\mathrm{rank}_{\mathbb{Z}_2}(\mathbb{Z}_2\otimes_{\mathbb{Z}} T^{2k+1}(C))$.) The isomorphism

$$L_{4k+2}(\mathbb{Z})\to\mathbb{Z}_2;\quad C\mapsto a(C)$$

sends a $(4k+2)$-dimensional quadratic Poincaré complex C over $\mathbb{Z}$ to the Arf invariant $a(C)$ of the non-singular quadratic form on the $\mathbb{Z}_2$-vector space $H^{2k+1}(C;\mathbb{Z}_2)$.

In order to compute the round symmetric L-groups (resp. quadratic) L-groups $L_r^*(\mathbb{Z})$ (resp. $L_*^r(\mathbb{Z})$) it is necessary to use the semicharacteristic $\chi_{1/2}(C)$ of Kervaire [6]. This is defined for any $(2n-1)$-dimensional chain complex C over a field F to be

$$\chi_{1/2}(C)=\sum_{i=0}^{n-1}(-)^i\mathrm{rank}_F H_i(C)\in\mathbb{Z},$$

and is such that for a $2n$-dimensional symmetric Poincaré pair (D,C) over F

$$\chi(D)-\chi_{1/2}(C)=\text{rank of the } (-)^n\text{-symmetric form } (H_n(D)\times H_n(D)\to F)$$

$$(=\text{dimension of the image of the adjoint map } H_n(D)\to H_n(D)^*=\mathrm{Hom}_F(H_n(D),F))\ (\mathrm{mod}\ 2)$$

[6, Lemma 4.1]. The mod 2 semicharacteristic played an important role in the work of Kervaire and Milnor [7] on simply-connected surgery.

The deRham invariant $d(C)\in\mathbb{Z}_2$ of a $(4k+1)$-dimensional symmetric Poincaré complex C over $\mathbb{Z}_2$ was expressed by Lusztig, Milnor and Peterson [9] as the difference of the mod 2 and rational semicharacteristics

$$d(C)=\chi_{1/2}(C;\mathbb{Z}_2)-\chi_{1/2}(C;\mathbb{Q})\in\mathbb{Z}_2,$$

where $\chi_{1/2}(C;F)=\chi_{1/2}(F\otimes_{\mathbb{Z}}C)$. Now $d(C)=0\in L^{4k+1}(\mathbb{Z})=\mathbb{Z}_2$ if and only if C is null-cobordant, that is if there exists a $(4k+2)$-dimensional symmetric Poincaré pair (D,C) over $\mathbb{Z}$ with boundary C, in which case

$$\chi(D)\equiv\chi_{1/2}(C;\mathbb{Q})\ (\mathrm{mod}\ 2),$$

since non-singular skew-symmetric forms over $\mathbb{Q}$ have even rank.

Proposition 4.2. *The round symmetric L-groups of $\mathbb{Z}$ are given by*

$$L^n_{\mathrm{rh}}(\mathbb{Z})=\begin{cases}\mathbb{Z}\\ \mathbb{Z}_2\oplus\mathbb{Z}_2\\ 0\\ 0\end{cases},\qquad L^n_{\mathrm{rs}}(\mathbb{Z})=\begin{cases}\mathbb{Z}\\ \mathbb{Z}_2\oplus\mathbb{Z}_2\\ \mathbb{Z}_2\\ 0\end{cases}\quad \text{if } n\equiv\begin{cases}0\\1\\2\\3\end{cases}(\mathrm{mod}\ 4),$$

and the round quadratic L-groups of $\mathbb{Z}$ are given by

$$L_n^{\mathrm{rh}}(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ \mathbb{Z}_2 \\ \mathbb{Z}_2 \end{cases}, \qquad L_n^{\mathrm{rs}}(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathbb{Z}_4 \end{cases} \quad \textit{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} (\mathrm{mod}\ 4),$$

Proof. We define below explicit invariants on all the round L-groups of $\mathbb{Z}$. The quadratic–symmetric–hyperquadratic and the Rothenberg exact sequences show that these invariants give isomorphisms.

Various fractions of the signature $\sigma(C)$ define isomorphisms

$$L_{\mathrm{rh}}^{4k}(\mathbb{Z}) \to \mathbb{Z}; \quad C \mapsto \sigma(C)/2,$$

$$L_{\mathrm{rs}}^{4k}(\mathbb{Z}) \to \mathbb{Z}; \quad C \mapsto \sigma(C)/4,$$

$$L_{4k}^{\mathrm{rh}}(\mathbb{Z}) = L_{4k}^{\mathrm{rs}}(\mathbb{Z}) \to \mathbb{Z}; \quad C \mapsto \sigma(C)/8.$$

The isomorphism

$$L_{\mathrm{rh}}^{4k+1}(\mathbb{Z}) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2; \quad C \mapsto (\chi_{1/2}(C;\mathbb{Z}_2), \chi_{1/2}(C;\mathbb{Q}))$$

sends a round $(4k+1)$-dimensional symmetric Poincaré complex C over $\mathbb{Z}$ to the mod 2 and rational semicharacteristics. (The mod 2 semicharacteristic $\chi_{1/2}(C;\mathbb{Z}_2) \in \mathbb{Z}_2$ is the obstruction to C being round cobordant to a complex C' with torsion homology groups $H_*(C')$.) The isomorphisms

$$L_{2i+1}^{\mathrm{rh}}(\mathbb{Z}) \to \mathbb{Z}_2; \quad C \to \chi_{1/2}(C;\mathbb{Z}_2) = \chi_{1/2}(C;\mathbb{Q})$$

are defined using either the mod 2 or the rational semicharacteristics, which coincide on $(2i+1)$-dimensional quadratic Poincaré complexes C over $\mathbb{Z}$. The isomorphisms

$$L_{4k+1}^{\mathrm{rs}}(\mathbb{Z}) \to \mathbb{Z}_2; \quad C \mapsto \tau(D, C),$$

$$L_{\mathrm{rs}}^{4k+2}(\mathbb{Z}) \to \mathbb{Z}_2; \quad C \mapsto \tau(D, C),$$

$$L_{4k+2}^{\mathrm{rs}}(\mathbb{Z}) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2; \quad C \mapsto (a(C), \tau(D, C))$$

are defined using the torsion $\tau(D, C)$ of a round algebraic Poincaré null-cobordism (D, C), with $a(C)$ the Arf invariant. The isomorphism

$$L_{\mathrm{rs}}^{4k+1}(\mathbb{Z}) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2; \quad C \mapsto (d(C), \tau(D, \mathbb{Z}_3 \otimes C))$$

is defined using the deRham invariant $d(C)$ and the torsion $\tau(D, \mathbb{Z}_3 \otimes C)$ of a round finite null-cobordism $(D, \mathbb{Z}_3 \otimes C)$ over $\mathbb{Z}_3$ of $\mathbb{Z}_3 \otimes C$. Every $(4k+3)$-dimensional round quadratic Poincaré complex C over $\mathbb{Z}$ is the boundary of a $(4k+4)$-dimensional symmetric Poincaré pair (D, C) over $\mathbb{Z}$, and the residue mod 4 of the signature $\sigma(D)$ of the symmetric form on $H^{2k}(D)$/torsion defines the isomorphism

$$L_{4k+3}^{\mathrm{rs}}(\mathbb{Z}) \to \mathbb{Z}_4; \quad C \mapsto \sigma(D). \qquad \square$$

5. Morita theory

Let (A, α, u), (B, β, υ) be two rings with antistructure, expanding the notation to include the antiautomorphisms

$$\alpha : A \to A, \qquad \beta : B \to B,$$

which were previously denoted $x \to \bar{x}$.

Definition 5.1. An (A, α, u)-(B, β, υ) *coform* is a pair $({}_B M_A, \psi)$ with ${}_B M_A$ a B-A-bimodule which is f.g. projective over B and

$$\psi : A \to M^{\mathrm{t}} \otimes_B M$$

an A-A-bimodule map. (Here M^{t} refers to the A-B-bimodule structure obtained from the B-A-bimodule structure of M using α and β.)

Furthermore, we require that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & M^{\mathrm{t}} \otimes_B M \\ \downarrow T & & \downarrow T \\ A & \xrightarrow[\psi]{} & M^{\mathrm{t}} \otimes_B M \end{array}$$

commutes, where

$$T(a) = u\,\alpha(a), \qquad T(m_1 \otimes m_2) = m_2 \otimes \upsilon m_1 .$$

The coform is said to be *non-singular* if the map

$$A \xrightarrow{\psi} M^{\mathrm{t}} \otimes_B M \xrightarrow{j} \mathrm{Hom}_B(\mathrm{Hom}_B(M, B)^{\mathrm{t}}, M)$$

sends $1 \in A$ to a B-module isomorphism

$$j(\psi(1)) : \mathrm{Hom}_B(M, B)^{\mathrm{t}} \to M.$$

Here $j(m_1 \otimes m_2)(f) = \beta^{-1}(f(m_1))m_2$.

A similar definition can be found in Hambleton and Madsen [4].

We can form the set of non-singular (A, α, u)-(B, β, υ) coforms into a Grothendieck group. First of all, (M, ψ) and (M', ψ') are *isomorphic* if there exists a B-A-bimodule isomorphism $f : M \to M'$ with $(f \otimes f) \cdot \psi = \psi'$. The sum operation is defined by $(M, \psi) \oplus (M', \psi')$, with $\psi \oplus \psi'$ given by

$$\psi \oplus \psi' : A \to (M^{\mathrm{t}} \otimes_B M) \oplus ((M')^{\mathrm{t}} \otimes_B M') \subseteq (M \oplus M')^{\mathrm{t}} \otimes_B (M \oplus M').$$

We let $\mathrm{Cf}(A, \alpha, u; B, \beta, \upsilon)$ denote the resulting Grothendieck group, to be denoted $\mathrm{Cf}(A, B)$ for short.

Let (C, γ, w) be yet another ring with antistructure. Given two non-singular coforms $({}_BM_A, \psi)$ and $({}_CN_B, \phi)$ we define $\phi \cdot \psi$ as the composite

$$\phi \cdot \psi : A \xrightarrow{\psi} M^t \otimes_B M = M^t \otimes_B B \otimes_B B \xrightarrow{1 \otimes \phi \otimes 1} M^t \otimes_B N^t \otimes_C N \otimes_B M$$

$$\xrightarrow{\mu \otimes 1} (N \otimes_B M)^t \otimes_C (N \otimes_B M)$$

where μ is defined by

$$\mu : M^t \otimes_B N^t \to (N \otimes_B M)^t; \quad m \otimes n \mapsto n \otimes m.$$

This product defines a pairing of Grothendieck groups

$$\mathrm{Cf}(A, B) \otimes_{\mathbb{Z}} \mathrm{Cf}(B, C) \to \mathrm{Cf}(A, C).$$

There is also defined a pairing

$$\mathrm{Cf}(A, C) \otimes_{\mathbb{Z}} \mathrm{Cf}(B, D) \to \mathrm{Cf}(A \otimes_{\mathbb{Z}} B, C \otimes_{\mathbb{Z}} D);$$

$$({}_CM_A, \psi) \otimes ({}_DN_B, \phi) \mapsto ({}_{(C \otimes_{\mathbb{Z}} D)}(M \otimes_{\mathbb{Z}} N)_{(A \otimes_{\mathbb{Z}} B)}, \psi \otimes_{\mathbb{Z}} \phi),$$

with $\psi \otimes_{\mathbb{Z}} \phi$ the composite

$$\psi \otimes_{\mathbb{Z}} \phi : A \otimes_{\mathbb{Z}} B \xrightarrow{\psi \otimes \phi} (M^t \otimes_C M) \otimes_{\mathbb{Z}} (N^t \otimes_D N)$$

$$\xrightarrow{\varepsilon} (M \otimes_{\mathbb{Z}} N)^t \otimes_{C \otimes_{\mathbb{Z}} D} (M \otimes_{\mathbb{Z}} N).$$

This construction will reappear when we discuss products in Section 6 below.

Next we shall describe our basic transformation. Recall that our object of study is a projective A-module chain complex C and some sort of equivariant homology or cohomology for the involution

$$T : C^t \otimes_A C \to C^t \otimes_A C; \quad c_1 \otimes c_2 \to c_2 \otimes uc_1.$$

Now suppose given a non-singular coform $({}_BM_A, \psi)$. We send the chain complex C to $M \otimes_A C$. We further define a map

$$\lambda : C^t \otimes_A C \to (M \otimes_A C)^t \otimes_B (M \otimes_A C)$$

as the composite

$$C^t \otimes_A C = C^t \otimes_A A \otimes_A C \xrightarrow{1 \otimes \psi \otimes 1} C^t \otimes_A M^t \otimes_B M \otimes_A C$$

$$\xrightarrow{\mu \otimes 1} (M \otimes_A C)^t \otimes_B (M \otimes_B C),$$

where $\mu(c \otimes m) = m \otimes c$. One checks that λ is $\mathbb{Z}_2$-equivariant and then uses λ to transport the quadratic, symmetric or hyperquadratic structure on C to one on $M \otimes_A C$.

So far we have not used the non-singularity of the coform. Only if the coform is non-singular does the above construction send Poincaré complexes (resp. pairs) to Poincaré complexes (resp. pairs). In particular, a non-singular (A, α, u)-(B, β, υ) coform (M, ψ) determines a homomorphism of the projective symmetric L-groups

$$\psi_*: L^n_{\mathrm{p}}(A, \alpha, u) \to L^n_{\mathrm{p}}(B, \beta, \upsilon); \quad (C, \phi) \to (M \otimes_A C, \psi \otimes \phi),$$

with similar maps in the quadratic and hyperquadratic cases. See Proposition 5.6 below for the precise circumstances under which ψ_* is defined for L-groups with decorations other than p.

If the coform (M, ψ) is isomorphic to (M', ψ') then $M \otimes_A C$ is isomorphic to $M' \otimes_A C$ as a symmetric (resp. quadratic, resp. hyperquadratic) complex, for any such complex C. Also, $(M \oplus M') \otimes_A C$ is isomorphic to $(M \otimes_A C) \oplus (M' \otimes_A C)$.

Definition 5.2. The *quadratic Morita category*, Quad-Morita, is the category with objects rings with antistructure, such that the morphisms from A to B are the elements of the Grothendieck group $\mathrm{Cf}(A, B)$.

We have shown that the various types of projective L-group L^*_{p}, L^{p}_*, $\hat{L}^*$ all define functors

$$L: \text{Quad-Morita} \to \text{Abelian groups}.$$

The round L-theory also defines a functor on this category:

Theorem 5.3. *Let (M, ψ) be a non-singular coform. The morphism of K_1-groups*

$$M \otimes_A - : K_1(A) \to K_1(B)$$

sends $*$-invariant subgroups to $*$-invariant subgroups. Let $X \subseteq K_1(A)$ be a $*$-invariant subgroup, and let $Y \subseteq K_1(B)$ be a $*$-invariant subgroup containing the image of X. Then the transformation on chain complexes discussed above defines a map of round symmetric L-groups

$$\psi_*: L^*_{\mathrm{r}X}(A, \alpha, u) \to L^*_{\mathrm{r}Y}(B, \beta, \upsilon); \quad (C, \phi) \mapsto (M \otimes_A C, \psi \otimes \phi),$$

with similar maps in the quadratic and hyperquadratic cases.

Proof. The hardest part is to see that our transformation lands where we claim. We discuss the needed result.

We need to modify slightly the definition of a round chain complex. A *homotopy round complex* is a finite-dimensional f.g. projective A-module chain complex C together with an isomorphism

$$C_{\mathrm{odd}} = \sum_i C_{2i+1} \to C_{\mathrm{even}} = \sum_i C_{2i},$$

so that

$$[C] = 0 \in K_0(A).$$

Then C has a round finite structure in the sense of Ranicki [16], i.e., an equivalence class of round finite complexes D with a chain equivalence $D \to C$, such that $\tau(D \to C \to D') = 0 \in K_1(A)$ for equivalent D, D'.

Given a homotopy round n-dimensional symmetric (resp. quadratic) Poincaré complex C we can define the torsion of the duality map as an element

$$\tau(C) = \tau(C^{n-*} \to C) \in K_1(A),$$

using the round finite structure.

Proposition 5.4. *The round symmetric (resp. quadratic) L-group $L^n_{rX}(A, \alpha, u)$ (resp. $L_n^{rX}(A, \alpha, u)$) $(X \subseteq K_1(A))$ is naturally isomorphic to the cobordism group of homotopy round n-dimensional symmetric (resp. quadratic) Poincaré complexes over A with torsion in $X \subseteq K_1(A)$.* □

The proof of Theorem 5.3 is now immediate from Proposition 5.4. □

Corollary 5.5. *The maps defined in L-theory by (M, ψ) in Theorem 5.3 depend only on the class of (M, ψ) in $\mathrm{Cf}(A, B)$.*

The ordinary (unround) theory is not nearly so nice, since we need based f.g. free A-modules to define torsions, or at least s-based f.g. s-free A-modules. Under certain additional hypotheses, these troubles can be overcome.

Proposition 5.6. *Let (M, ψ) be a non-singular (A, α, u)-(B, β, v) coform. Let $X \subseteq K_1(A)$ be a $*$-invariant subgroup containing $\tau(\pm u)$. Also, let $Y \subseteq K_1(B)$ be a $*$-invariant subgroup containing $\tau(\pm v)$ and the image of X. Finally suppose M is s-free and s-based, and that the torsion of $j\psi(1): \mathrm{Hom}_B(M, B) \to M$ is also in Y. Then the transformation on chain complexes discussed above defines a map of symmetric L-groups*

$$\psi_*: L^*_X(A, \alpha, u) \to L^*_Y(B, \beta, v); \quad (C, \phi) \mapsto (M \otimes_A C, \psi \otimes \phi),$$

and similarly for the quadratic and hyperquadratic cases.

Proof. The proof consists of tedious verifications. □

Corollary 5.7. *If (M, ψ) and (M', ψ') are isomorphic, then the two maps defined in Proposition 5.6 are equal, provided that the torsion of the isomorphism $M \to M'$ lies in Y.* □

We conclude Section 5 with some examples.

Example 5.8. Let $f: A \to B$ be a map of rings with antistructure. Let ${}_BM_A = {}_BB_A$ with

$$b_1(b)a = b_1bf(a) \in {}_BM_A.$$

Thus M is free of rank 1 as a B-module, with base 1. Define

$$\psi : A \to M^{\mathrm{t}} \otimes_B M; \quad a \mapsto 1 \otimes f(a).$$

The maps in 5.3 and 5.6 are the usual maps covariantly induced by f in L-theory.

Example 5.9. Let ε be a central unit in A such that $\alpha(\varepsilon)=\varepsilon$. Let ${}_AM_A = A$, with $a_1(a)a_2 = a_1aa_2$, and let

$$\psi : A \to A \otimes_A A; \quad a \mapsto \varepsilon \otimes a.$$

Let ε_* denote the induced map in round L-theory, and in ordinary L-theory for subgroups of $\tilde{K}_1(A)$ containing $\tau(\varepsilon)$. Note that if $\varepsilon=-1$, ε_* is just the map taking each element in an L-group to its inverse.

Example 5.10. Let $g: A \to A$ be an inner automorphism, such that

$$g: A \to A; \quad a \mapsto rar^{-1}$$

defines an automorphism of a ring with antistructure. This occurs if and only if $\alpha(r) = r^{-1}\varepsilon$ where ε is a central unit with $\alpha(\varepsilon)=\varepsilon$ and $rur^{-1}=u$. Define ${}_AM_A = A$, with $a_1(a)a_2 = a_1ag(a_2)$, and also ${}_AN_A = A$, with $a_1(a)a_2 = a_1aa_2$. Define $f: N \to M$ by $f(a) = ar^{-1}$. In this case Corollaries 5.5 and 5.7 show that $g_* = \varepsilon_*$ in round L-theory, and also in ordinary L-theory if r is contained in the decoration subgroup of K_1.

Corollary 5.11. *Let ε be a central unit, so that $\alpha(\varepsilon)\varepsilon=-1$. Then all round L-groups are $\mathbb{Z}_2$-vector spaces. The ordinary L-groups with $\tau(\varepsilon)$ in the decoration subgroup of $\tilde{K}_1(A)$ are also $\mathbb{Z}_2$-vector spaces.*

Example 5.12. Let

$$(B, \beta, \upsilon) = (A_1 \times A_2, \alpha_1 \times \alpha_2, u_1 \times u_2).$$

Define ${}_BM_{A_1} = {}_B(A_1)_{A_1}$; $b(a)a_1 = \mathrm{pr}_1(b)aa_1$, where $\mathrm{pr}_1 : B \to A_1$ denotes the projection. Define $\psi : A_1 \to (A_1)^{\mathrm{t}} \otimes_{A_1} A_1$ by $\psi(a) = 1 \otimes a$. The Morita maps induced in round L-theory split the maps covariantly induced by the projection. Moreover, if we include using ψ_* and then project out to A_2, we get the 0 map in round L-theory. This proves:

Corollary 5.13. *Up to natural isomorphism*

$$L^n_{\mathrm{r}(X_1 \times X_2)}(A_1 \times A_2, \alpha_1 \times \alpha_2, u_1 \times u_2) = L^n_{\mathrm{r}X_1}(A_1, \alpha_1, u_1) \oplus L^n_{\mathrm{r}X_2}(A_2, \alpha_2, u_2),$$

and similarly for the quadratic and hyperquadratic L-groups. □

Note that the module M in Example 5.12 is rarely s-free (as a B-module), so we almost never get a decomposition as in 5.13 for ordinary L-theories.

Example 5.14. Hambleton, Taylor and Williams [5] and Hahn [2] define the notion of quadratic Morita equivalence using forms. We explain this as follows. Define an (A,α,u)-(B,β,v) *form* as a pair $({}_BM_A,\lambda)$ with ${}_BM_A$ a B-A-bimodule which is f.g. projective over B and $\lambda: M\otimes_A M^t \to B$ a B-B-bimodule map. Furthermore, it is required that the diagram

$$\begin{array}{ccc} M\otimes_A M^t & \xrightarrow{\lambda} & B \\ \downarrow{\scriptstyle T^0} & & \downarrow{\scriptstyle T^0} \\ M\otimes_A M^t & \xrightarrow[\lambda]{} & B \end{array}$$

commutes, where

$$T^0(b)=v^{-1}\beta^{-1}(b), \qquad T^0(m_1\otimes m_2)=m_2\otimes u^{-1}m_1.$$

The form is *non-singular* if $\mathrm{ad}(\lambda): M\to \mathrm{Hom}_B(M,B)^{t^{-1}}$ is an isomorphism of B-modules, where

$$\mathrm{ad}(\lambda)(m_1)(m_2)=\lambda(m_2,m_1).$$

There is a natural 1-1 correspondence

$$\{\text{non-singular } (A,\alpha,u)\text{-}(B,\beta,v) \text{ coforms}\}$$
$$\rightleftarrows \{\text{non-singular } (A,\alpha,u)\text{-}(B,\beta,v) \text{ forms}\}.$$

The map from forms to coforms is the following: the composite

$$\psi: A\to \mathrm{End}_B(M) \xleftarrow{\cong} \mathrm{Hom}_B(M,B)\otimes_B M \xleftarrow[\cong]{\mathrm{ad}(\lambda)\otimes 1} M^t\otimes_B M$$

defines ψ. Conversely, the composite

$$\lambda: M\otimes_A M^t \xrightarrow{(j\psi(1))^{-1}\otimes 1} (M^*)^t\otimes_A M^t \xrightarrow{\beta^{-1}(\text{evaluation})} B$$

defines λ given ψ.

The Morita equivalence maps defined in Hambleton, Taylor and Williams [5] agree with the maps defined here once one corrects for

(i) the fact that we have switched from right to left modules,

(ii) the switch of units from u to u^{-1},

(iii) the symmetry formula 2.5 in [5] has a typographical error – the last m_1v should be vm_1.

Example 5.15. One way to get a form is to use a trace. Let $i: B\to A$ be a map of rings (not necessarily preserving the antistructures) such that A is a f.g. projective B-module. A *trace* is a linear map

$$X : A \to B$$

such that

(i) X is left B-linear when we regard A as a left B-module.

(ii) For all $a \in A$,

$$v^{-1}\beta^{-1}(X(a)) = X(u^{-1}\alpha^{-1}(a)).$$

Define an (A, α, u)-(B, β, v) form (M, λ) by ${}_BM_A = A$, with $b(a)a_1 = i(b)aa_1$, and

$$\lambda : A \otimes_A A \to B; \quad a_1 \otimes a_2 \mapsto X(a_1\alpha^{-1}(a_2)).$$

If λ is non-singular we can use 5.14 to get a coform and hence maps in L-theory, usually referred to as *transfer maps*. For example

(a) If $i : H \to G$ is an inclusion of a subgroup of finite index define a trace by

$$X : \mathbb{Z}[G] \to \mathbb{Z}[H]; \quad g \to \begin{cases} g & \text{if } g \in H, \\ 0 & \text{otherwise,} \end{cases}$$

with the antistructure $g \to \pm g^{-1}$ $(g \in G)$ on $\mathbb{Z}[G]$, $\varepsilon = \pm 1$. The resulting transfer maps

$$\lambda_* : L_n(\mathbb{Z}[G]) \to L_n(\mathbb{Z}[H])$$

are the usual transfer maps associated to finite covers in topology.

(b) If $i : H \to G$ is an index 2 subgroup and

$$t \in G - H, \qquad \beta(g) = w(g)g^{-1}, \qquad v = -1,$$

$$\alpha(h) = w(h)\phi(h)tht^{-1}, \qquad u = w(t)t^{-2},$$

with $\phi : G \to \mathbb{Z}_2$ the projection such that $\ker \phi = H$, then

$$X : \mathbb{Z}[G] \to \mathbb{Z}[H]; \quad g \to \begin{cases} 0 & \text{if } g \in H, \\ gt & \text{if } g \in G - H \end{cases}$$

is a trace whose resulting transfer map

$$\lambda_* : L_n(\mathbb{Z}[G], \alpha, u) \to L_n(\mathbb{Z}[H], \beta, v)$$

is the 'twisted transfer' of Hambleton [3] and Hambleton, Taylor and Williams [5].

Example 5.16. Lück and Ranicki [8] define a transfer map in quadratic L-theory $L_m(A) \to L_{m+n}(B)$ given an n-dimensional symmetric Poincaré complex (C, ϕ) over B together with a morphism of rings with involution $U : A \to H_0(\mathrm{Hom}_B(C, C))$. (For simplicity we are taking $u = 1$, $v = 1$ here, and ignoring decorations.) For $n = 0$ such a complex is essentially the same as a non-singular A-B coform (M, ψ), with $M = C_0$, $amb = U(a)(m\bar{b})$, and the transfer map agrees with the Morita map $L_m(A) \to L_m(B)$. Moreover, there is defined in [8] a cobordism group $L^n(A\text{-}B)$ of such complexes, such that the transfer is the evaluation of a product pairing

$$L_m(A)\otimes L^n(A\text{-}B)\to L_{m+n}(B).$$

This suggests that the Grothendieck group $\mathrm{Cf}(A\text{-}B)$ of coforms in 5.5 could be replaced by a Grothendieck–Witt group.

Given a non-singular (A,α,u)-(B,β,v) coform we can define relative L-groups. Specifically, let ${}_BM_A$ be the coform and let $X\subseteq K_1(A)$, $Y\subseteq K_1(B)$ be $*$-invariant subgroups such that Y contains the image of X. The procedures in [13] suffice to define relative quadratic round L-groups $L_n^{\mathrm{r}Y,\mathrm{r}X}({}_BM_A)$ to fit into an exact sequence

$$\cdots\to L_n^{\mathrm{r}X}(A,\alpha,u)\to L_n^{\mathrm{r}Y}(B,\beta,v)\to L_n^{\mathrm{r}Y,\mathrm{r}X}({}_BM_A)\to L_{n-1}^{\mathrm{r}X}(A,\alpha,u)\to\cdots.$$

There are similar sequences for the symmetric and hyperquadratic L-groups. We can also replace K_1 by K_0. If the hypotheses of 5.6 hold we can also define $L_n^{\tilde X,\tilde Y}({}_BM_A)$ with the obvious properties.

There are also triad L-groups defined whenever

$$({}_{A_3}M_{A_1})\otimes_{A_1}({}_{A_1}M_{A_0})\cong({}_{A_3}M_{A_2})\otimes_{A_2}({}_{A_2}M_{A_0}).$$

6. Products

For any rings A,B there is a product pairing of algebraic K-groups

$$K_1(A)\otimes K_0(B)\to K_1(A\otimes B);$$

$$\tau(f:P\to P)\otimes[Q]\mapsto\tau(f\otimes 1:P\otimes Q\to P\otimes Q),$$

where $f\in\mathrm{Hom}_A(P,P)$ is an automorphism of a f.g. projective A-module P and Q is a f.g. projective B-module. There is a similar pairing

$$K_0(A)\otimes K_1(B)\to K_1(A\otimes B).$$

The methods of Ranicki [15,16] show that there are defined similar products in L-groups. Given a homotopy round m-dimensional ε-symmetric Poincaré complex (C,ϕ) over A and a projective n-dimensional η-quadratic Poincaré complex (D,ψ) over B there is defined a homotopy round $(m+n)$-dimensional $(\varepsilon\otimes\eta)$-quadratic Poincaré complex $(C\otimes D,\phi\otimes\psi)$ over $A\otimes_{\mathbb{Z}}B$, with torsion

$$\tau(C\otimes D,\phi\otimes\psi)=\tau(C,\phi)\otimes[D]\in K_1(A\otimes_{\mathbb{Z}}B).$$

In particular, if D is homotopy round, then $(C\otimes D,\phi\otimes\psi)$ is homotopy round simple. There is a similar product for symmetric complexes. Products of Poincaré complexes induce products in L-groups, such as:

Proposition 6.1. *Given rings with antistructure $(A,\varepsilon),(B,\eta)$ and $*$-invariant subgroups $X\subseteq K_1(A)$, $Y\subseteq K_0(B)$, $Z\subseteq K_1(A\otimes_{\mathbb{Z}}B)$ such that $X\otimes Y\subseteq Z$ the product of complexes induces a product of L-groups*

$$L^m_{\mathrm{r}X}(A,\varepsilon)\otimes L^Y_n(B,\eta)\to L^{\mathrm{r}Z}_{m+n}(A\otimes B,\varepsilon\otimes\eta).\quad\square$$

(Recall from Proposition 5.4 that the cobordism of homotopy round Poincaré complexes is isomorphic to the cobordism of round finite Poincaré complexes.)

Example 6.2. Product with the round symmetric signature of the circle

$$\sigma^*_{\mathrm{r}}(S^1)\in L^1_{\mathrm{rh}}(\mathbb{Z}[t,t^{-1}])\quad(\bar{t}=t^{-1})$$

defines split injections of ordinary L-groups

$$\sigma^*_{\mathrm{r}}(S^1)\otimes - : L^{\mathrm{p}}_n(A)\to L^{\mathrm{h}}_{n+1}(A[t,t^{-1}]),$$

$$\sigma^*_{\mathrm{r}}(S^1)\otimes - : L^{\mathrm{h}}_n(A)\to L^{\mathrm{t}}_{n+1}(A[t,t^{-1}]),$$

where the decoration t refers to the $*$-invariant subgroup $\{\tau(t)\}\subseteq\tilde{K}_1(A[t,t^{-1}])$. See Ranicki [14] for details of this application of round L-theory. These remarks and Proposition 3.2 can be used to prove a splitting theorem for the round L-groups.

The Morita maps of Section 5 are compatible with products:

Proposition 6.3. *Let $({}_{A'}M_A,\psi)$ and $({}_{B'}N_B,\phi)$ be non-singular coforms, and let $X\subseteq K_1(A)$, $Y\subseteq K_0(B)$, $Z\subseteq K_1(A\otimes B)$ be $*$-invariant subgroups such that $X\otimes Y\subseteq Z$. Then there is defined a commutative diagram of round L-groups*

$$\begin{array}{ccc} L^m_{\mathrm{r}X}(A,\varepsilon)\otimes L^Y_n(B,\eta) & \longrightarrow & L^{\mathrm{r}Z}_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\\ \Big\downarrow{\scriptstyle \psi_*\otimes\phi_*} & & \Big\downarrow{\scriptstyle (\psi\otimes\phi)_*}\\ L^m_{\mathrm{r}X'}(A',\varepsilon')\otimes L^{Y'}_n(B',\eta') & \longrightarrow & L^{\mathrm{r}Z'}_{m+n}(A'\otimes_{\mathbb{Z}}B',\varepsilon'\otimes\eta') \end{array}$$

with $X'\subseteq K_1(A')$, $Y'\subseteq K_0(B')$, $Z'\subseteq K_1(A'\otimes_{\mathbb{Z}}B')$ $*$-invariant subgroups such that $X'\otimes Y'\subseteq Z'$, and such that X' (resp. Y') contains the image under the Morita map of X (resp. Y).

Proof. We check the commutativity of the diagram on the chain level by using the external product of coforms defined in Section 5. $\square$

Remark 6.4. There are two other useful versions of 6.3:
(i) The roles played by K_0 and K_1 may be reversed.
(ii) The quadratic L-groups may be replaced by symmetric L-groups.

The product of a finite (i.e. based f.g. free) m-dimensional ε-symmetric Poincaré complex (C,ϕ) over A and a finite n-dimensional η-quadratic Poincaré complex (D,ψ) over B is a finite $(m+n)$-dimensional $(\varepsilon\otimes\eta)$-quadratic Poincaré complex $(C\otimes_{\mathbb{Z}}D,\phi\otimes\psi)$ over $A\otimes_{\mathbb{Z}}B$, with torsion

$$\tau(C\otimes_{\mathbb{Z}} D, \phi\otimes\psi)=\tau(C,\phi)\otimes\chi(D)+\chi(C)\otimes\tau(D,\psi)\in\tilde{K}_1(A\otimes_{\mathbb{Z}} B).$$

This product formula for torsions can be used to obtain versions of Propositions 6.1, 6.3 in which the factors are ordinary L-groups with $\tilde{K}_1$-decoration, as follows.

Proposition 6.5. *Given $*$-invariant subgroups $X\subseteq\tilde{K}_1(A)$, $Y\subseteq\tilde{K}_1(B)$, $Z\subseteq\tilde{K}_1(A\otimes_{\mathbb{Z}} B)$ such that*

$$\tau(\varepsilon)\in X,\qquad \tau(\eta)\in Y,\qquad X\otimes[B]+[A]\otimes Y\subseteq Z$$

there is defined a pairing of L-groups

$$L_X^m(A,\varepsilon)\otimes L_n^Y(B,\eta)\to L_{m+n}^Z(A\otimes B,\varepsilon\otimes\eta).$$

Given also non-singular coforms $({}_{A'}M_A,\psi)$ and $({}_{B'}N_B,\phi)$ satisfying the conditions of 5.6 there is defined a commutative diagram of L-groups

$$\begin{array}{ccc} L_X^m(A,\varepsilon)\otimes L_n^Y(B,\eta) & \longrightarrow & L_{m+n}^Z(A\otimes_{\mathbb{Z}} B,\varepsilon\otimes\eta) \\ \Big\downarrow{\scriptstyle \psi_*\otimes\phi_*} & & \Big\downarrow{\scriptstyle (\psi\otimes\phi)_*} \\ L_{X'}^m(A',\varepsilon')\otimes L_n^{Y'}(B',\eta') & \longrightarrow & L_{m+n}^{Z'}(A'\otimes_{\mathbb{Z}} B',\varepsilon'\otimes\eta') \end{array}$$

with $X'\subseteq\tilde{K}_1(A')$, $Y'\subseteq\tilde{K}_1(B')$, $Z'\subseteq\tilde{K}_1(A'\otimes_{\mathbb{Z}} B')$ $*$-invariant subgroups such that $\tau(\varepsilon')\in X'$, $\tau(\eta')\in Y'$, $X'\otimes[B']+[A']\otimes Y'\subseteq Z'$, and such that X' (resp. Y') contains the image under the Morita map of X (resp. Y). □

A statement similar to 6.4 also holds in the case of 6.5.

7. Spectra

Let (A,ε) be a ring with antistructure, and let X be a $*$-invariant subgroup of $\tilde{K}_i(A)$ $(i=0,1)$ such that $\tau(\varepsilon)\in X$ if $i=1$, so that the ε-symmetric (resp. ε-quadratic) L-groups $L_X^*(A,\varepsilon)$ (resp. $L_*^X(A,\varepsilon)$) are defined. In Ranicki [11] ε-symmetric (resp. ε-quadratic) Poincaré n-ads over A were used to define a simplicial spectrum $\mathbb{L}_X^0(A,\varepsilon)$ (resp. $\mathbb{L}_0^X(A,\varepsilon)$) with homotopy groups

$$\pi_n(\mathbb{L}_X^0(A,\varepsilon))=L_X^n(A,\varepsilon)\qquad(\text{resp. } \pi_n(\mathbb{L}_0^X(A,\varepsilon))=L_n^X(A,\varepsilon)).$$

Similarly, given a $*$-invariant subgroup $X\subseteq K_i(A)$ $(i=0,1)$ there are defined simplicial spectra with homotopy groups the variant L-groups decorated by X, using projective algebraic Poincaré n-ads with classes in X if $i=0$, and round algebraic Poincaré n-ads with torsions in X if $i=1$. For $i=0$ the spectrum is denoted by $\mathbb{L}_X^0(A,\varepsilon)$ (resp. $\mathbb{L}_0^X(A,\varepsilon)$), and for $[A]\in X$ it is naturally isomorphic to $\mathbb{L}_{\tilde{X}}^0(A,\varepsilon)$ (resp. $\mathbb{L}_0^{\tilde{X}}(A,\varepsilon)$) with $\tilde{X}\subseteq\tilde{K}_0(A)$ the image $*$-invariant subgroup. For $i=1$ the spectrum is denoted by $\mathbb{L}_{rX}^0(A,\varepsilon)$ (resp. $\mathbb{L}_0^{rX}(A,\varepsilon)$), with homotopy groups the round ε-

symmetric (resp. ε-quadratic) L-groups

$$\pi_n(\mathbb{L}^0_{\mathrm{r}X}(A,\varepsilon)) = L^n_{\mathrm{r}X}(A,\varepsilon) \qquad (\text{resp. } \pi_n(\mathbb{L}_0^{\mathrm{r}X}(A,\varepsilon)) = L_n^{\mathrm{r}X}(A,\varepsilon)).$$

In this section we list a few of the spectrum level maps which induce maps previously considered on the level of homotopy groups.

Example 7.1. A non-singular coform $({}_BM_A, \psi)$ induces maps of symmetric L-spectra

$$\psi_*: \mathbb{L}^0_X(A,\varepsilon) \to \mathbb{L}^0_Y(B,\eta) \quad \text{if } i=0,$$

$$\psi_*: \mathbb{L}^0_{\mathrm{r}X}(A,\varepsilon) \to \mathbb{L}^0_{\mathrm{r}Y}(B,\eta) \quad \text{if } i=1,$$

for any $*$-invariant subgroups $X \subseteq K_i(A)$, $Y \subseteq K_i(B)$ $(i=0,1)$ such that $M \otimes_A X \subseteq Y$. The induced maps in the homotopy groups are the Morita maps ψ_* in the symmetric L-groups. Furthermore, if $({}_BM_A, \psi)$ satisfies the conditions of 5.6, then there are also such maps for the L-spectra decorated by $X \subseteq \tilde{K}_i(A)$ $(i=0,1)$. Similarly in the quadratic and hyperquadratic cases. Isomorphic coforms give rise to homotopic maps.

Example 7.2. The product pairings of L-groups obtained in Section 6 are all induced by product pairings of the corresponding L-spectra. In particular, the spectrum version of 6.1 is a map

$$\mathbb{L}^0_{\mathrm{r}X}(A,\varepsilon) \wedge \mathbb{L}_0^Y(B,\eta) \to \mathbb{L}_0^{\mathrm{r}Z}(A \otimes_{\mathbb{Z}} B),$$

with $X \subseteq K_1(A)$, $Y \subseteq K_0(B)$, $Z \subseteq K_1(A \otimes_{\mathbb{Z}} B)$ $*$-invariant subgroups such that $X \otimes Y \subseteq Z$.

Remark 7.3. The spectrum maps of 7.1 and 7.2 are compatible. The resulting commutative diagrams of spectra give rise to the commutative diagrams of 6.2–6.4 on the level of homotopy groups.

Example 7.4. The usual symmetric L-spectrum of $\mathbb{Z}$, $\mathbb{L}^0_{\mathrm{h}}(\mathbb{Z})$, is a ring spectrum. The round L-spectra $\mathbb{L}^0_{\mathrm{r}X}(A,\varepsilon)$, $\mathbb{L}_0^{\mathrm{r}X}(A,\varepsilon)$ defined for any $*$-invariant subgroup $X \subseteq K_1(A)$ are module spectra over $\mathbb{L}^0_{\mathrm{h}}(\mathbb{Z})$. Therefore by Taylor and Williams [18] they are generalized Eilenberg–MacLane spectra when localized at 2.

Given a $\mathbb{Z}[\mathbb{Z}_2]$-module G let $\hat{\mathbb{H}}^\bullet(\mathbb{Z}_2; G)$ denote the simplicial spectrum obtained by the Kan–Dold construction from the $\mathbb{Z}$-module chain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, G)$, with $\hat{W}$ the complete free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$, so that

$$\pi_n(\hat{\mathbb{H}}^\bullet(\mathbb{Z}_2; G)) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, G)) = \hat{H}^n(\mathbb{Z}_2; G).$$

Example 7.5. The comparison sequence of 3.2 is induced by a fibration of spectra

$$\mathbb{L}_0^{\mathrm{r}X}(A,\varepsilon) \to \mathbb{L}_0^{\tilde{X}}(A,\varepsilon) \to \hat{\mathbb{H}}^\bullet(\mathbb{Z}_2; K_0(\mathbb{Z})).$$

Similarly for all the other comparison sequences in Section 3.

Note that 7.4 applied to 7.5 shows that

$$\hat{\mathbb{H}}^{\bullet}(\mathbb{Z}_2; K_0(\mathbb{Z})) = \bigvee_{i=0}^{\infty} \Sigma^{2i} K(\mathbb{Z}_2, 0),$$

a product of generalized Eilenberg–MacLane spectra.

Example 7.6. In some cases the 'rank map'

$$\mathrm{rk} : L_{2k}^{\tilde{X}}(A, \varepsilon) \to \hat{H}^{2k}(\mathbb{Z}_2; K_0(\mathbb{Z})) \quad (\tilde{X} \subseteq \tilde{K}_1(A))$$

can be determined. Wall [22] observed that if $A = \mathbb{Z}[\pi]$ with involution $g \to w(g)g^{-1}$, then rk is trivial for all k, and the resulting short exact sequence

$$0 \to \mathbb{Z}/2 \to L_{2k-1}^{X}(A, \varepsilon) \to L_{2k-1}^{\tilde{X}}(A, \varepsilon) \to 0$$

is split. This splitting is induced by a splitting of spectra.

The rank map is a split surjection on the spectrum level in certain other cases.

Proposition 7.7. *Let A be a ring with involution containing an element $e \in A$ such that*

$$e + \bar{e} = 1, \qquad e^2 = e \in A.$$

For a $*$-invariant subgroup $\tilde{X} \subseteq \tilde{K}_1(A)$ such that $\tau(-1 : Ae \to Ae) \in \tilde{X}$ there is a map of spectra

$$\tau : \hat{\mathbb{H}}^{\bullet}(\mathbb{Z}_2; K_0(\mathbb{Z})) \to \mathbb{L}_0^{\tilde{X}}(A)$$

such that $\mathrm{rk} \cdot \tau \simeq 1$. *We are taking $\varepsilon = 1$ here, abbreviating (A, ε) to A.*

Proof. The rank map of spectra splits if (and only if) it induces a split surjection of homotopy groups

$$\mathrm{rk}_* : L_{2i}^{\tilde{X}}(A) \to \hat{H}^0(\mathbb{Z}_2; K_0(\mathbb{Z})) = \mathbb{Z}_2.$$

The non-singular $(-)^i$-quadratic form (A, e) over A has rank 1 and torsion

$$\begin{aligned} \tau(A, e) &= \tau(e + (-)^i e : A \to A) \\ &= \begin{cases} 0 \\ \tau(-1 : Ae \to Ae) \end{cases} \in \tilde{X} \subseteq \tilde{K}_1(A) \quad \text{if } i \equiv \begin{cases} 0 \\ 1 \end{cases} (\mathrm{mod}\ 2). \end{aligned}$$

The element $(A, e) \in L_{2i}^{\tilde{X}}(A)$ is of order 2, since the isomorphism of $(-)^i$-quadratic forms over A

$$f = \begin{pmatrix} 1-e & e \\ (-)^i e & 1-e \end{pmatrix} : \left(A \oplus A, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \to \left(A \oplus A, \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\right)$$

has torsion

$$\tau(f)=\begin{cases}\tau(-1:Ae\to Ae)\\ 0\end{cases}\in\tilde{X}\subseteq\tilde{K}_1(A)\quad\text{if } i\equiv\begin{cases}0\\ 1\end{cases}\pmod 2.\qquad\square$$

In particular, the condition of Proposition 7.7 is satisfied for $\tilde{X}=\tilde{K}_1(A)$, so that there is a splitting map

$$\tau:\hat{\mathbb{H}}^{\bullet}(\mathbb{Z}_2;K_0(\mathbb{Z}))\to\mathbb{L}_0^h(A).$$

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