Existence of Free Involution on 5-Manifolds

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Introduction. Let M be a closed, oriented, simply-connected manifold of dimension five. If, in addition, $w_2(M) = 0$ then M is classified up to diffeomorphism by the structure of H_2M [4]. In this paper we determine which of these manifolds admit a smooth, free, orientation-preserving involution. Since for these manifolds Tors $(H_2M) \cong B \oplus B$ for some finite abelian group B [1], our answer will be expressed as restrictions on the groups B. In the statement of the theorems, $B_{(2)}$ denotes the 2-primary summand of B and $s(K)$ denotes the direct sum of s copies of a group K.

Theorem 1. *Let M^5 be a closed, oriented, simply-connected manifold of dimension five with $w_2(M) = 0$. If H_2M is finite, let $H_2M \cong B \oplus B$, then M admits a smooth, free, orientation-preserving involution if and only if $B_{(2)}$ is not isomorphic to*

(i) $Z/4$

(ii) $s(Z/4) \oplus t(Z/2)$ *for* t *odd*

or

(iii) $Z/8 \oplus t(Z/2)$ *for* t *odd.*

Theorem 2. *Let M^5 be as in Theorem 1. If rank $(H_2M) \geqq 1$, then M admits a smooth, free, orientation-preserving involution.*

The method given here for the proof of Theorem 1 generalizes to the case of free Z/p actions on 5-manifolds for p an odd prime. The basic algebraic facts needed for this case (as for $p = 2$) are contained in the paper of G. Szekeres [5]. I am indebted to Professor S. Conlon for a letter explaining the results of [5].

Our method also gives necessary conditions for the existence of free involutions on $(n - 1)$-connected $(2n + 1)$-manifolds for $n > 2$ which are n-parallelisable but does not appear to work for 5-manifolds with $w_2 \neq 0$.

Sections 1 and 2 contain the algebraic results needed, including a statement of the relevant parts of [5]. In Section 3 this algebra is related to the geometry of our situation and Sections 4 and 5 contain the proofs of the theorems just stated.

1. Finite $\wedge$-modules. Let $\wedge$ be the integral group ring of $Z/2$ and G be a finite abelian group with an endomorphism T such that T^2 = identity. In this

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section we describe the classification of the isomorphism classes of pairs (G, T) (called *finite* $\wedge$*-modules*) and compute the image of the obvious map:

$$\{\text{finite } \wedge\text{-modules}\} \to \{\text{finite abelian groups}\}.$$

For the application to 5-manifolds, the $\wedge$-modules will satisfy an additional condition:

Definition 1. A finite $\wedge$-module G is *cohomologically trivial* (CT) if $H^i(Z/2; G) = 0$ for $i > 0$.

A necessary and sufficient condition is given for a CT finite $\wedge$-module to admit a non-singular equivariant linking form.

We begin the classification by observing that a finite $\wedge$-module G splits as a $\wedge$-direct sum into its p-primary parts. Furthermore, if p is odd, the indecomposable p-torsion $\wedge$-modules are of the form $(Z/p^k, T)$ where $Tx = \pm x$ and these are all CT.

It remains to consider finite 2-primary $\wedge$-modules G. By the Krull–Schmidt Theorem, every such module splits uniquely (up to isomorphism and rearrangement) into a direct sum of indecomposables. We now describe the classification of the indecomposables due to G. Szekeres [5]. Set $\phi = T - 1$, $\pi = T + 1$ and observe that these are nilpotent endomorphisms of G with the properties:

$$\phi\pi = \pi\phi = 0 \quad \text{and} \quad \pi^k + (-1)^k\phi^k = 2^k.$$

For each $x \neq 0$ in G we define

$$i(x) = \max\{i \in Z \mid \phi^i(x) \neq 0\} \quad \text{and}$$

$$j(x) = \max\{j \in Z \mid \pi^j(x) \neq 0\}.$$

In addition, if 2^l is the minimal exponent of G, then G is a module over

$$R_l = Z[t]/(2^l, t^2 - 1) \cong Z[\phi, \pi]/(\phi^{l+1}, \pi^{l+1}, \pi\phi, 2^l, 2 + \phi - \pi).$$

Using these notations we define the basic $\wedge$-modules.

Definition 2. An R_l-module G is an *open chain* if it is generated by $k > 0$ elements $x_1, \cdots, x_k$ satisfying the following conditions:

1) Let $i_1 = i(x_1) + 1$, $i_r = i(x_r)$ for $r > 1$, $j_r = j(x_r)$ for $r < k$, and $j_k = j(x_k) + 1$. Then $i_r > 0$ and $j_r > 0$ for $r = 1, \cdots, k$.
2) Set $y_r = \phi^{i_r}x_r$ and $z_r = \pi^{j_r}x_r$. Then $z_r = y_{r+1}$ for $r = 1, 2, \cdots, k - 1$.
3) Write A for the subgroup of G generated by $\langle z_1, \cdots, z_{k-1}\rangle$ and denote by x_r^* the coset of A in G represented by x_r. Then $A = \langle z_1\rangle \oplus \cdots \oplus \langle z_{k-1}\rangle$ and $G/A = \langle x_1^*\rangle \oplus \cdots \oplus \langle x_k^*\rangle$.

Remark. The order of G is then $2^{\sum(i_r + j_r) - 1}$ and the module G is uniquely determined by the set $[i_1, j_1; \cdots; i_k, j_k]$.

Definition 3. An R_l-module G is a *closed chain* if it is generated by $k > 0$ elements $x_1, \cdots, x_k$ satisfying the following conditions:

1) Set $i_r = i(x_r)$, $j_r = j(x_r)$ for $r = 1, \cdots, k$. Then $i_r > 0$ and $j_r > 0$ for $r = 1, \cdots, k$.
2) Let $\bar{k}$ be the smallest divisor of k, $k = \bar{k}d$ such that $i_r = i_s$ and $j_r = j_s$ whenever $r \equiv s \pmod{\bar{k}}$. Write

$$y_r = \phi^{i_r} x_r \quad \text{and} \quad z_r = \pi^{j_r} x_r\,, \quad \text{then} \quad z_r = y_{r+1} \quad \text{for} \quad r = 1, \cdots, k-1.$$

3) $z_k = \sum_{s=0}^{d-1} \lambda_s y_{s\bar{k}+1}$ where the λ_s equal 0 or 1 and satisfy
   (i) $\lambda_0 \neq 0$
   (ii) the polynomial $f(z) = z^d - \sum_{s=0}^{d-1} \lambda_s z^s$ is either irreducible or a power of an irreducible polynomial over the field F_2.
4) Write $A = \langle z_1, \cdots, z_k \rangle$ and denote by x_r^* the coset of A in G represented by x_r. Then

$$A = \langle z_1 \rangle \oplus \cdots \oplus \langle z_k \rangle \quad \text{and} \quad G/A = \langle x_1^* \rangle \oplus \cdots \oplus \langle x_k^* \rangle.$$

Remark. In this case the order of G is $2^{\sum (i_r + j_r)}$ and the module can be specified by the set $[i_1, j_1; \cdots; i_{\bar{k}}, j_{\bar{k}}, f(z)]$.

Theorem 3 [5. p. 11]. *Suppose G is a finite $\wedge$-module with exponent 2^l. Then G is indecomposable if and only if G is an open or closed chain.*

As mentioned above, in our applications the modules G with $H^i(Z/2; G) = 0$ for $i > 0$ will be needed, Since

$$H^{\text{even}}(Z/2; G) = \ker \phi / \operatorname{im} \pi$$

and

$$H^{\text{odd}}(Z/2; G) = \ker \pi / \operatorname{im} \phi$$

we observe:

Corollary 4. *The closed chains are the only finite indecomposable $\wedge$-modules of 2-primary order which are cohomologically trivial.*

From the explicit definition of the closed chains, we calculate which finite abelian groups admit a CT $\wedge$-module structure. It is enough to consider the 2-primary part again since every p-primary group has such a structure.

Theorem 5. *Let B be a finite abelian group. Then B admits a cohomologically trivial $\wedge$-module structure if and only if $B_{(2)}$ is not isomorphic to one of the groups: $Z/4$, $Z_8 \oplus t(Z/2)$ for t odd, or $s(Z/4) \oplus t(Z/2)$ for t odd.*

Proof. We suppose first that B has 2-primary order and is not isomorphic to one of the groups listed in the statement. We can write (additively)

$$B = A \oplus s(Z/4) \oplus t(Z/2)$$

where A is a direct sum of groups $Z/2^k$ for various $k \geq 3$ or $A = 0$.

We consider five cases in showing that B admits the required $\wedge$-module structure.

Case 1. $A = 0$. If B is not one of the listed groups, the modules

$$\wedge/2\wedge, \quad \wedge/4\wedge, \quad [1, 1; 1, 1; f(z) = z + 1] \quad \text{and} \quad [1, 1; \cdots; 1, 1; f(z) = z^k + z + 1]$$

provide the $\wedge$-structures needed. The third module is a structure on $Z/4 \oplus Z/2 \oplus Z/2$ and the fourth is a structure on $k(Z/4)$ where $k > 1$ is the number of generators in the bracket.

The remaining cases all assume $A \neq 0$.

Case 2. $s \equiv t \equiv 0 \pmod 2$. In addition to $\wedge/2\wedge$ and $\wedge/4\wedge$ as above, we need only provide a CT structure on $Z/2^n$ for $n \geqq 3$. This is given by $T(x) = (2^{n-1} + 1)x$.

Case 3. $s \equiv t \equiv 1 \pmod 2$. We need a structure on $Z/2^n \oplus Z/4 \oplus Z/2$ for each $n \geqq 3$. This is provided by $[n - 1, 1; 1, 2; f(z) = z + 1]$.

Case 4. $s \equiv 0, t \equiv 1 \pmod 2$. If $A \supseteq Z/2^n$ for $n > 3$ we can use the structure $[n - 1, 2; f(z) = z + 1]$ on $Z/2^n \oplus Z/2$. If $A = r(Z/8)$ (where $r > 1$) we use $[2, 1; 2, 1; \cdots; 2, 2; f(z) = z + 1]$ on $r(Z/8) \oplus Z/2$ (the bracket has r generators). Otherwise, $A = Z/8$ and $s > 0$, so we use $[1, 1; 2, 2; 1, 1; f(z) = z + 1]$ on $Z/8 \oplus Z/4 \oplus Z/4 \oplus Z/2$.

Case 5. $s \equiv 1, t \equiv 0 \pmod 2$. We need the modules $[1, 1; 1, n - 1; f(z) = z + 1]$ on $Z/2^n \oplus Z/4$ $(n \geqq 3)$.

Conversely, suppose B is a finite 2-primary $\wedge$-module which is cohomologically trivial. First we show that no closed chain has underlying group in the list. It then follows easily that no direct sum of closed chains is in the list. Suppose that B is a closed chain of exponent 4. If $B \neq \wedge/4\wedge$ then $i_r < 2$ and $j_r < 2$ for all r so that $i_r = j_r = 1$ for $r = 1, \cdots, k$. In the notation of Definition 2, the relations are $\pi x_r = \phi x_{r+1}$ $r = 1, \cdots, k - 1$ and $\pi x_k = \sum_{s=0}^{d-1} \lambda_s \phi x_{s\bar{k}+1} (*)$. If the number of non-zero coefficients λ_s is odd we use the relation $\pi - \phi = 2$ and (*) to obtain

$$2(x_k + u) = 0$$

where $u \neq 0$ is a sum of some generators x_r for $1 \leqq r < k$. Then, additively we have

$$B \cong \langle x_1 \rangle \oplus \cdots \oplus \langle x_{k-1} \rangle \oplus \langle x_k + u \rangle \oplus \langle \phi x_k \rangle$$
$$\cong (k - 1)(Z/4) \oplus 2(Z/2).$$

We remark here for use in §2 that part 4 of Definition 3 implies $\phi(x_k + u) \neq 0$.

If the number of non-zero coefficients λ_s is even, we obtain similarly a relation

$$2x_k + \phi x_k = 2u$$

where $u \neq 0$ as before is a sum of some generators $x_r (1 \leqq r < k)$. In this case,

$$B \cong \langle x_1 \rangle \oplus \cdots \oplus \langle x_k \rangle$$
$$\cong k(Z/4).$$

We observe that each closed chain of exponent 4 has two $Z/2$ summands or none and the result is established in this case. There remain the groups of the form $Z/8 \oplus t(Z/2)$ for t odd. We reduce the question to the case $t = 1$ where the result can be verified by direct calculation.

Let $\{x, y_1, \cdots, y_t\}$ be an additive base for B where x has order 8 and y_r has order 2 for $1 \leqq r \leqq t$. Let K be the sub-module generated by $\{y_1, \cdots, y_t\}$, then $K \cong (t+1)(Z/2)$ or $t(Z/2)$ and $0 \to K \to B \to B/K \to 0$ is exact where $B/K \cong Z/4$ or $Z/8$. If we suppose B is CT then $H^i(Z/2; B/K) \cong H^{i+1}(Z/2; K)$ and so the possibility $K \cong (t+1)(Z/2)$ and $B/K \cong Z/4$ can be eliminated. This is because $H^i(Z/2; Z/4) = Z/2$ (for all $i > 0$) with any $\wedge$-structure on $Z/4$.

If $K \cong t(Z/2)$ and $B/K \cong Z/8$ then there exists a $\wedge$-direct summand $K_0 \subset K$ such that $K_0 \cong (t-1)(Z/2)$ and K_0 is CT. Then $B/K_0 \cong Z/8 \oplus Z/2$ must have a CT structure. We now suppose $B \cong Z/8 \oplus Z/2$ where $Tx = ax + by$ and $Ty = y$ ($o(x) = 8$ and $o(y) = 2$ as above). Then $a^2 \equiv 1 \pmod 8$ and $b = 1$ because $T^2 = id$, and from the exact cohomology sequence arising from

$$0 \to Z/2 \to B \to Z/8 \to 0,$$

we deduce $a \equiv \pm 1 \pmod 8$. If $Tx = x + y$, $Ty = y$ then $y \in \ker \phi$ but $y \notin \operatorname{im} \pi$, while if $Tx = -x + y$, $Ty = y$ then $y \in \ker \pi$ but $y \notin \operatorname{im} \phi$. In either case B is not CT.

2. Linking forms on finite $\wedge$-modules. In this section we give the structure of CT finite $\wedge$-modules which admit non-singular linking forms. Such forms have been studied in the context of surgery theory (see [3], [7] for example).

Definition 4. Let G be a finite $\wedge$-module. A *linking form* on G is a non-singular bilinear form

$$b : G \times G \to Q/Z$$

such that

1) $b(Tx, Ty) = b(x, y)$ for all $x, y \in G$,
2) $b(x, x) = b(x, Tx) = 0$ for all $x \in G$.

Property 2) implies that b is strictly skew-symmetric. For finite groups G admitting a non-singular skew-symmetric bilinear form it is known that $G \cong B \oplus B$ for some group B. [6]. If B is a $\wedge$-module, let $B^* = \operatorname{Ext}_\wedge{}^1(B, \wedge)$.

Theorem 6. *If (G, b) is a linking form on a CT finite $\wedge$-module, then $G \cong B \oplus B$ (additively) where B admits a CT $\wedge$-structure. Conversely, if B is a CT finite $\wedge$-module, there exists a linking form on the $\wedge$-direct sum $B \oplus B^*$.*

Proof. If (G, b) is a linking form, we may assume that G is a 2-torsion group. In order to derive a contradiction we also assume that $G \cong B \oplus B$ where $B \cong Z/4$, $s(Z/4) \oplus t(Z/2)$ or $Z/8 \oplus t(Z/2)$ (for t odd).

Case 1. $G \cong 2(\mathbf{Z}/4)$. The only CT structure on this group is $\Lambda/4\Lambda$. However, this implies $b \equiv 0$ on G by property 2 in the definition of linking forms.

Case 2. $G \cong 2s(\mathbf{Z}/4) \oplus 2t(\mathbf{Z}/2)$ for t odd. Let $G \cong G_1 \oplus G_2 \oplus \cdots \oplus G_k$ be the Λ-splitting of G where each G_i is an indecomposable module (a closed chain) of exponent 4. As we remarked in §1 when considering such closed chains, there exists an $x \in G_i$ such that $Tx \neq x$ and x generates an additive direct summand. Let J be the Λ-submodule of G generated by x, so $J \cong \Lambda/2\Lambda$, and let $J^\perp$ be the annihilator of J. Since $J \subset J^\perp$ we may use the following result of W. Pardon [2]:

Lemma 7. *Let $J \subset G$ be a totally isotropic CT submodule of a finite CT module G with linking form. Then $J^\perp/J$ is CT and admits a linking form.*

We assume that if $G \cong 2s(\mathbf{Z}/4) \oplus 2t(\mathbf{Z}/2)$ for t odd admits a linking form then it is the module of this type of smallest order to do so. This, however, leads to a contradiction as follows. Since $0 \to J^\perp \to G \to J^* \to 0$ is exact, the additive structure of $J^\perp$ is:

$$J^\perp \cong \begin{cases} 2s(\mathbf{Z}/4) \oplus (2t-2)(\mathbf{Z}/2), & \text{or} \\ (2s-1)(\mathbf{Z}/4) \oplus 2t(\mathbf{Z}/2). \end{cases}$$

We use here the fact that the subgroup generated by x is a direct summand ($\cong \mathbf{Z}/2$). Since x also generates an additive direct summand of $J^\perp$, we get from

$$0 \to J \to J^\perp \to J^\perp/J \to 0$$

the possibilities:

$$J^\perp/J \cong \begin{cases} 2s(\mathbf{Z}/4) \oplus (2t-4)(\mathbf{Z}/2), & \text{or} \\ (2s-2)(\mathbf{Z}/4) \oplus 2t(\mathbf{Z}/2). \end{cases}$$

(Some additive structures are ruled out by the requirement that $J^\perp/J$ is CT and admits a linking form). Since each of these is a CT module of the required additive type and of smaller order than G it cannot admit a linking form. This contradicts Lemma 7.

Case 3. $G \cong 2(Z/8) \oplus 2t(Z/2)$ for t odd. We first establish:

Lemma 8. *If (G, b) is a linking form where G is as above and $t > 2$, then G contains a submodule $J \cong \Lambda/2\Lambda$ which is an additive direct summand.*

Proof. We choose an additive base for G and let K be the Λ-submodule generated by the basis elements of order 2. Then

$$G/K \cong \begin{cases} 2(\mathbf{Z}/8), \\ \mathbf{Z}/8 \oplus Z/4, & \text{or} \\ 2(\mathbf{Z}/4). \end{cases}$$

If the first possibility occurs, K is an additive summand of G and $H^i(\mathbf{Z}/2; K) = r(\mathbf{Z}/2)$ where $r \leqq 2$ for all $i > 0$. Therefore, K contains a Λ-direct summand

$(t-1)\,\Lambda/2\Lambda$ and the result is verified since $t > 1$. If $G/K \cong \mathbf{Z}/8 \oplus \mathbf{Z}/4$, then K contains a Λ-direct summand $t(\Lambda/2\Lambda)$ and by changing base in K, we can find a Λ-summand $(t-1)\Lambda/2\Lambda$ of K which is an additive summand of G.

Finally, if $G/K \cong 2(\mathbf{Z}/4)$ then again K contains a Λ-summand $t(\Lambda/2\Lambda)$. If $p : G \to 2(\mathbf{Z}/8)$ is the projection homomorphism (of abelian groups) and u, v the order 8 generators in an additive base for G, then consider $p(\pi x)$ where the submodule $\langle x, Tx\rangle$ is one of the $\Lambda/2\Lambda$ summands of K. The possible elements $p(\pi x)$ are 0, $4p(u)$, $4p(v)$ and $4p(u+v)$. By a basis change in K we can assume that $p(\pi x_i)$ is non-zero for the generators x_i of at most two of the $\Lambda/2\Lambda$ summands of K. Then K contains a Λ-summand $(t-2)(\Lambda/2\Lambda)$ which is an additive summand of G.

Using this lemma, we can by the method of Case 2, reduce to the groups with $t = 1$ or $t = 2$.

In fact, for $t = 2$ we need only consider the situation described above where $G/K \cong 2(\mathbf{Z}/4)$. An additive base $\{u, v, y_1, y_2, y_3, y_4\}$ can then be chosen with u, v of order 8 and y_i order 2 $(1 \leqq i \leqq 4)$ such that:

$$Ty_1 = y_1 + 4u, \qquad Ty_2 = y_2 + 4v,$$

$$Ty_3 = y_3, \qquad Ty_4 = y_4, \qquad Tu = \epsilon u + z_1 \quad \text{and} \quad Tv = \epsilon v + z_2 .$$

Here z_1, z_2 are of order 2 and ϵ is a unit in $\mathbf{Z}/8$. However, there is no CT structure of this form. This can be verified by calculating:

$$\ker \pi = \begin{cases} \langle z_1, z_2, y_3, y_4, 2u + y_1, 2v + y_2\rangle & \epsilon = 1 \text{ or } 5 \\ \langle z_1, z_2, y_3, y_4, 2u, 2v\rangle & \epsilon = 3 \text{ or } 7 \end{cases}$$

and

$$\operatorname{im} \phi = \begin{cases} \langle z_1, z_2, 4u, 4v\rangle & \epsilon = 1 \text{ or } 5 \\ \langle 2u + z_1, 2v + z_2\rangle & \epsilon = 3 \text{ or } 7 \end{cases}.$$

In either case, $\ker \pi/\operatorname{im} \phi \neq 0$ and so the structure is not CT.

There remains the case $t = 1$, where $G \cong 2(\mathbf{Z}/8) \oplus 2(\mathbf{Z}/2)$. If the module G is decomposable, there is a direct summand $J \cong \mathbf{Z}/8$. Since $J \subset J^{\perp}$ we can proceed as before and get a form on $J^{\perp}/J \cong 2(\mathbf{Z}/2)$. Since this group has only one CT structure, namely $\Lambda/2\Lambda$, which clearly does not support a linking form we have a contradiction. If G is indecomposable, then we calculate that it has the structure $[2, 1; 1, 2; 1, 1; f(z) = z + 1]$ or $[1, 2; 1, 1; 2, 1; f(z) = z + 1]$. Neither of these supports a linking form. The first can be described by generators u, v of order 8 and x, y of order 2 such that: $Tu = -u + x$, $Tv = v + x$, $Tx = x$ and $Ty = y + 4u + 4v$. Now $b(u, v) = b(Tu, Tv)$ implies $b(u, 4v) = 0$ so $4v$ is in the radical of b. The second possibility is also ruled out in this way.

The proof of Theorem 6 will be completed by establishing the converse. Let B be a CT finite Λ-module with resolution $0 \to F \to F' \to B \to 0$ where $F \cong F' \cong s\Lambda$ for some integer s. We may choose Λ-bases for F, F' and write

$$0 \to s\Lambda \xrightarrow{(A)} s\Lambda \to B \to 0$$

where (A) is an $s \times s$ matrix with entries in Λ. Consider the exact sequence:

$$0 \to s\wedge \oplus s\wedge \xrightarrow{\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}} s\wedge \oplus s\wedge \to B \oplus B^* \to 0.$$

This induces in the usual way a linking form on $B \oplus B^*$.

3. Structure of Tors (H_2M). The algebra of the first two sections will now be related to the geometry. If M^5 admits a free orientation-preserving (o.p.) involution then the groups H_*M become modules over $\wedge$. In particular, if we assume $w_2(M) = 0$ the finite $\wedge$-module $G =$ Tors (H_2M) admits a linking form as defined in §2. This follows from the discussion of [7, p. 250]. Note that property 2 is a result of the fact that the quadratic refinement $q(x)$ of $b(x, x)$ is identically zero in this case.

Theorem 9. Let M^5 be a closed simply-connected 5-manifold with $w_2M = 0$ and H_2M finite. If M admits a free orientation-preserving involution then in the induced $\wedge$-module structure, H_2M is CT.

Proof. Since $w_2M = 0$, the method of [7, p. 249] can be applied to construct an equivariant cobordism W^6 by surgery such that $\partial W = M \cup (-N)$ where H_2N is a finite $\wedge$-module with $|x\wedge| \leqq 2$ for all $x \in H_2N$.

Since N^5 is also a simply-connected 5-manifold with free o.p. involution, $H_2N \cong 2s(Z/2)$ with trivial $\wedge$-module structure is only possible if $s = 0$. This follows by an easy argument with the spectral sequence of the double cover. Therefore $N \approx S^5$ and from the exact sequence of $\wedge$-modules:

$$0 \to H_3W \to H_3(W, \partial W) \to H_2M \to 0$$

H_2M is CT since $H_3(W, \partial W) \cong H_3(W, M)$ is $\wedge$-free and $H_3(W) \cong$ $\mathrm{Hom}_\wedge (H_3(W, \partial W), \wedge)$.

4. Proof of Theorem 1. To prove the theorem it is only necessary to put together the results of the previous sections. Let M^5 be a 5-manifold with free o.p. involution satisfying the conditions stated. By Theorem 9, $G \cong$ Tors (H_2M) is CT in the induced $\wedge$-module structure. Since, as observed above, G supports a linking form we may use Theorem 6 to conclude that $G \cong B \oplus B$ where B has a CT $\wedge$-structure. Now Theorem 5 gives the condition on $B_{(2)}$.

Conversely, if $G \cong B \oplus B$ where $B_{(2)}$ satisfies the stated condition, then B has a CT $\wedge$-structure and so there exists a linking form on G defined by a resolution:

$$0 \to F \to \mathrm{Hom}_\wedge (F, \wedge) \to B \oplus B^* \to 0$$

where $B \oplus B^*$ has the direct sum CT structure. Any such sequence gives a prescription for surgery on the identity map $RP^5 \to RP^5$ [7, p. 256]. After performing the surgery we obtain a manifold M' with free orientation preserving involution and $w_2M' = 0$ such that $H_2M \cong H_2M'$. Therefore M is diffeomorphic to M' [4] and the proof is complete.

5. Proof of Theorem 2. Let M^5 satisfy the conditions stated and suppose rank $H_2M \geqq 1$. As before, since $w_2M = 0$, $G = \text{Tors}\ (H_2M) \cong B \oplus B$ for some group B. If B has a CT $\wedge$-structure then the same argument as in §4 gives a manifold M' with involution such that $H_2M' \cong G$ or $H_2M' \cong G \oplus Z$. Then connected sum with copies of $S^2 \times S^3$ in the orbit space gives M'' with $H_2M'' \cong H_2M$ and so $M'' \approx M$.

The proof of the theorem will follow from this technique provided we can construct manifolds with involution realizing the cases $G \cong Z/4 \oplus Z/4$, $Z/2 \oplus Z/2$ and $2(Z/4) \oplus 2(Z/2)$. More precisely, we will construct manifolds X_i, Y_i with involution for $i = 1, 2, 3$ such that:

$$H_2X_1 = Z \oplus 2(Z/4)$$

$$H_2X_2 = Z \oplus 2(Z/2)$$

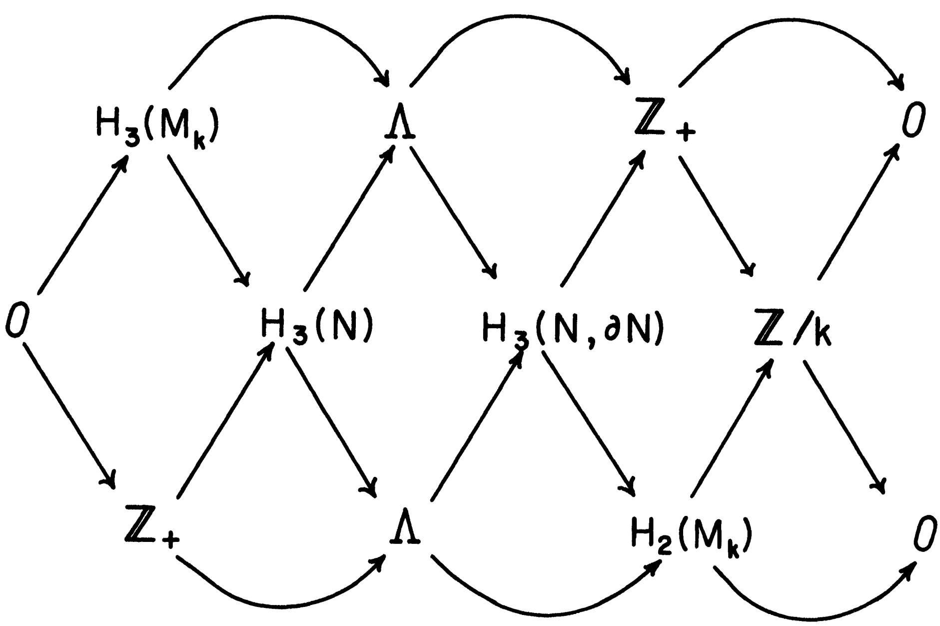
$$H_2X_3 = Z \oplus 2(Z/4) \oplus 2(Z/2)$$

and $H_2(Y_i) = H_2(X_i) \oplus Z$ for $i = 1, 2, 3$.

To construct the manifolds X_i, we start with the double cover $S^3 \times S^2 \to RP^3 \times S^2$. Let the double cover $M_k \to \bar{M}_k$ be the result of equivariant framed surgery on an embedding in $S^3 \times S^2$ representing kx where x generates $H_2(RP^3 \times S^2\,; \wedge) = Z_+$.

Lemma 10. *If k is a power of 2, then $H_2(M_k) \cong Z \oplus 2(Z/k)$.*

Proof. Let N^6 be the trace of the surgery and consider the braid:

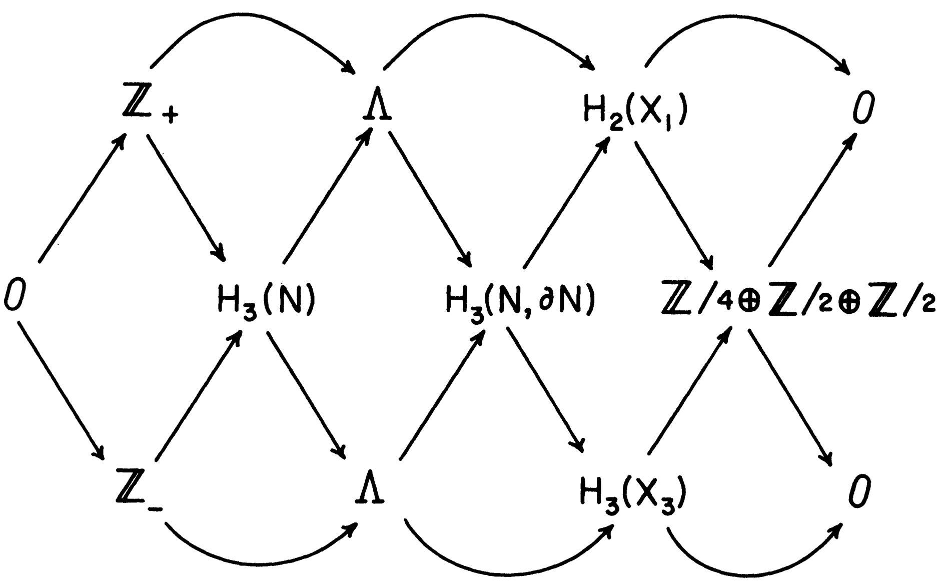


From the properties of this diagram [7, p. 247–8], $H_3(N, \partial N)$ is an extension of L by Z_+ where L is a $\wedge$-module with underlying abelian group $Z \oplus Z/k$, and $H_2(M_k)$ is an extension of L by Z/k. We obtain the short exact sequence.

$$0 \to Z/k \to \text{Tors }(H_2M_k) \to Z/l \to 0$$

where $l \leqq k$ and l is also a power of 2. However Tors $(H_2M_k) \cong B \oplus B$ for some group B so Tors $(H_2M_k) \cong Z/k \oplus Z/k$.

This lemma contains the construction of $X_1 = M_2$ and $X_2 = M_4$. To obtain X_3 we begin with X_1 and perform surgery as above on $4x$ where $x \varepsilon H_2(X_1)$ projects to a generator of $H_2(X_1)/\text{Tors }(H_2X_1) \cong Z_-$. If N again denotes the trace of the surgery, the braid in this case is:



and $H_3(X_3)$ is an extension of $Z \oplus Z/4$ by $Z/4 \oplus 2(Z/2)$. Since Tors $(H_3X_3) \cong B \oplus B$ for some B, Tors $(H_3X_3) \cong 2(Z/4) \oplus 2(Z/2)$.

Now that the manifolds X_i are constructed, the following construction produces the Y_i. For $i = 1, 2$ or 3 let $X_i(0)$ denote $X_i - f_i(S^1 \times D^4)$ where $f_i : S^1 \times D^l \to X_i$ is a tubular neighbourhood of an embedded circle invariant under the involution. Then $\partial X_i(0) = S^1 \times S^3$ with the antipodal action in each factor and we can define

$$Y_i = X_i(0) \bigcup_{\partial} (D^2 \times S^3) \ (i = 1, 2 \text{ or } 3)$$

by identifying the boundaries. The involution on $X_i(0)$ extends to Y_i in the obvious way.

From the Van Kampen Theorem, in the orbit spaces,

$$\pi_1 \bar{Y}_i = Z/2.$$

and we have the exact sequence of $\wedge$-modules:

$$0 \to H_2(X_i(0)) \to H_2(Y_i) \to H_1(S^1 \times S^3) \to 0$$
$$\| \qquad$$
$$Z \; .$$

Since $H_2(X_i(0)) \cong H_2(X_i)$, Tors $(H_2 Y_i) \cong$ Tors $(H_2 X_i)$ for $i = 1, 2, 3$.

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