FREE INVOLUTIONS ON 6-MANIFOLDS

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INTRODUCTION

In this paper, we give the diffeomorphism classification of smooth, closed, orientable manifolds M of dimension six with $\pi_1 M = Z_2$ and $\pi_2 M = 0$. This is equivalent to the classification of free differentiable orientation-preserving involutions on a connected sum of finitely many copies of $S^3 \times S^3$. In this case, it is therefore possible to carry out the program proposed in [5] for the study of involutions on (n - 1)-connected 2n-manifolds ($n \geq 3$).

The paper is organized as follows. Section 1 contains an explanation of the notation and an exposition of the results needed from [1] and [5]. In Section 2, we state the classification results, Theorems 2 and 3, and give an example. The remaining sections contain the proofs.

1. BILINEAR FORMS

Let K be a finite orientable Poincaré complex of dimension six [8] with $\pi_1 K = Z_2$ and $\pi_2 K = 0$. The generator of $\pi_1 K$ will be denoted by T. Then the integral homology and cohomology groups of the universal covering space $\tilde{K}$ are modules over the integral group ring Λ of Z_2 via the action of T. In particular, $H_3(\tilde{K}) \cong r\Lambda \oplus Z_+ \oplus Z_+$ for some integer r, where Z_+ is the group of integers with trivial action of Z_2. This can easily be shown, if it is recalled that since $H_3(\tilde{K})$ is a free abelian group it has the form $r_0 Z_+ \oplus r_1 Z_- \oplus r_2 \Lambda$ as a Λ-module. From the spectral sequence of the covering $\tilde{K} \to K$, we deduce the values $r_0 = 2$ and $r_1 = 0$.

Let us write $H = H_3(\tilde{K})$ and consider the effect of the involution on the intersection pairing λ: $H \times H \to Z$. This is a unimodular, skew-symmetric bilinear form with the further properties

(1) $\lambda(Tx, Ty) = \lambda(x, y)$ for all x, y in H, and

(2) $\lambda(x, x) = \lambda(x, Tx) = 0$ for all x in H.

Associated with λ, there is the Browder-Livesay self-intersection map ϕ: $H \otimes Z_2 \to Z_2$ (see [1] and Sections 5 and 6 below). This is related to λ by the equation

$$\phi(x + y) - \phi(x) - \phi(y) = \lambda(x, Ty) \pmod 2,$$

valid for all x, y in H. Although ϕ is actually defined on $H \otimes Z_2$, it will cause no confusion to write $\phi(x)$ for x in H, instead of $\phi(x \otimes 1)$. The geometry of K therefore gives the algebraic data (λ, ϕ, H). Any such triple, satisfying the relations listed above, will be called a Z_2-form.

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In fact, the Z_2-forms that come from Poincaré complexes have an additional structure. From obstruction theory, there is a 2-connected map $f: RP^3 \to K$, covered by $\tilde{f}: S^3 \to \tilde{K}$. In [5], it is shown that we can choose f so that $e_0 = \tilde{f}_*[S^3]$ generates a Z_+ direct summand of H. Let $(e_1, \cdots, e_r, e_0, e_\infty)$ be a set of Λ-generators of H containing e_0 such that $(e_1, \cdots, e_r)$ is a Λ-base for an $r\Lambda$ complementary summand to $Z_+ \oplus Z_+$ generated by (e_0, e_∞). Such a set will be referred to as a *basis* of H. It is not difficult to see that the only basis changes B of H that come from homotopy equivalences of K have the property

$$(*) \qquad Be_0 - e_0 = (1+T)z, \qquad \text{for some } z \text{ in } H.$$

This condition ensures that $e_0' = Be_0$ can be represented by a mapping of $RP^3 \to K$ if e_0 has such a representation. The following definitions are motivated by this geometric fact. Suppose $H = r\Lambda \oplus Z_+ \oplus Z_+$.

Definition 1. A *based* Z_2-form on H is a Z_2-form (λ, ϕ, H) together with a generator e_0 of a Z_+ direct summand of H.

Definition 2. Two based forms (λ, ϕ, e_0) and (λ', ϕ', e_0') on H are *base-equivalent* if there exists a Λ-isomorphism $B: H \to H$ such that

(1) $\lambda'(Bx, By) = \lambda(x, y)$,

(2) $\phi'(Bx) = \phi(x)$, $e_0' = Be_0$, and

(3) $Be_0 - e_0 = (1+T)z$ for some z in H.

The discussion of the preceding paragraph can be summed up: *With each Poincaré complex of our type, there is associated a based* Z_2*-form whose base-equivalence class is a homotopy invariant.*

It will be useful to observe that, given a based Z_2-form (λ, ϕ, e_0, H), we can, by a purely algebraic argument, find a 'splitting basis' of H for λ. More precisely, there exists a basis change $B: H \to H$ with property (*) such that the direct-sum splitting of H into $H_1 = r\Lambda$ and $H_0 = Z_+ \oplus Z_+$, given by the new basis, is an orthogonal splitting with respect to λ. This implies that, in the new basis (e_0', e_∞') for H_0,

$$\lambda(e_0', e_\infty') = 1 \qquad \text{and} \qquad \lambda(e_0', e_0') = \lambda(e_\infty', e_\infty') = 0.$$

The proof of this fact is an immediate consequence of the following result of [4]. In the statement, we denote $G/2G$ by $\overline{G}$, for an abelian group G. Given λ, a Z_2-form on a Λ-module N, we construct a form $\overline{\lambda}$ on $\overline{N}$ by reducing the values of λ modulo 2.

LEMMA 1. *Let* $\overline{\lambda}$ *be the reduction of a nonsingular* Z_2*-form on a* Λ*-module* $N \cong r\Lambda \oplus M$, *where* M *has no* Λ*-free direct summand. Then* $\overline{\lambda}$ *restricted to* $\overline{M}$ *is nonsingular.*

Proof. We set $Q = (1+T)r\Lambda$ and let P be the subgroup of N generated by a Λ-base for the $r\Lambda$ summand, so that as a free abelian group $N = P \oplus Q \oplus M$.

Then, if

$$\text{Ann}(\overline{Q}) = \{x \in \overline{N} \mid \overline{\lambda}(x, y) = 0 \text{ for all } y \in \overline{Q}\},$$

It is clear that $\overline{M} \oplus \overline{Q} \subseteq \text{Ann}(\overline{Q})$. Suppose $\overline{\lambda} \mid \overline{M} \times \overline{M}$ is singular. This implies that there exists a nonzero $z \in \overline{M} \cap \text{Ann}(\overline{M})$. Since $\overline{\lambda}$ is nonsingular on $\overline{N}$, there is an

$x \in \overline{P}$ with $\overline{\lambda}(x, z) = 1$. By adding suitable multiples of z to basis elements of $\overline{Q}$, we obtain $\overline{Q}'$ of the same rank (as a Z_2-vector space) with

$$\mathrm{Ann}(\overline{Q}') \supseteq \overline{Q}' \oplus \overline{M} \oplus \langle x \rangle .$$

Since $\overline{Q}'$ is also a direct summand of $\overline{N}$, there is a subgroup $\overline{T}$ of $\overline{N}$ such that $\overline{N} \cong \mathrm{Ann}(\overline{Q}') \oplus \overline{T}$. Clearly, rank $\overline{T}$ = rank $\overline{Q}'$. Now there is a contradiction: rank $\overline{N} = 2(\text{rank } \overline{Q}) + \text{rank } \overline{M} \geq 2(\text{rank } \overline{Q}) + \text{rank } \overline{M} + 1$.

We conclude this section by describing a condition the map ϕ must satisfy for K to be smoothable. Choose an embedding of $H_0 = Z_+ \oplus Z_+$ so that $H \cong H_0 \oplus H_1$. Then $\phi \mid H_0$ is an associated quadratic map to $\lambda \mid H_0$ (in the usual sense); for if x is in H_0, then $Tx = x$. Denote by $A(\phi, H_0)$ the Arf invariant of $\phi \mid H_0$. The following calculation shows that $A(\phi, H_0)$ is in fact independent of the choice of embedding of H_0.

LEMMA 2. *Let* $B: H \to H$ *be a basis change, and let* $H_0' = BH_0$. *Then* $A(\phi, H_0') = A(\phi, H_0)$.

Proof. Pick a basis (e_0, e_∞) of H_0 containing e_0, and set $e_0' = Be_0$ and $e_\infty' = Be_\infty$. Then

$$Be_0 = ae_0 + be_\infty + (1 + T)x \quad \text{for some } x \text{ in } H_1,$$

and

$$Be_\infty = ce_0 + de_\infty + (1 + T)y \quad \text{for some } y \text{ in } H_1.$$

Using the fact that $\lambda(e_0', e_\infty') \equiv 1 \pmod 2$ from Lemma 1, we deduce that $ad + bc \equiv 1 \pmod 2$. This clearly implies that $\phi(e_0')\phi(e_\infty') = \phi(e_0)\phi(e_\infty)$.

Now suppose we are given a Poincaré complex K as above, with its map ϕ defined on $H_3(\tilde{K}) \otimes Z_2$. Set $A(K) = A(\phi, H_3(\tilde{K}))$, where in view of Lemma 2, the notation for the Arf invariant has been simplified. The following restriction on ϕ was obtained in [5].

THEOREM 1. *Let* M *be a closed, smooth, oriented* 6*-manifold with* $\pi_1 M = Z_2$ *and* $\pi_2 M = 0$. *Then* $A(M) = 0$.

2. THE CLASSIFICATION

Our classification is contained in the next two results. All manifolds mentioned are smooth, closed, and oriented, and they have dimension six.

THEOREM 2. *Suppose* K *is a finite, oriented Poincaré complex that is the homotopy type of a manifold* M^6, *with* $\pi_1 M = Z_2$ *and* $\pi_2 M = 0$. *Then* K *has exactly two smoothings.*

THEOREM 3. *Homotopy types of* 6*-manifolds* M *with* $\pi_1 M = Z_2$ *and* $\pi_2 M = 0$ *are in bijective correspondence with the sets of invariants*

(1) *a* Λ*-module* $H = r\Lambda \oplus Z_+ \oplus Z_+$, *for some even integer* $r \geq 0$,

(2) *a based* Z_2*-form* (λ, ϕ, e_0, H) *on* H *with* $A(\phi, H) = 0$, *modulo the equivalence relation generated by base-equivalence of* Z_2*-forms.*

In a special case we have computed the classification also for Poincaré complexes.

PROPOSITION 1. *There are exactly ten homotopically distinct, finite, oriented Poincaré complexes* K *of dimension six, with* $\pi_1 K = Z_2$ *and* $\tilde{K} \simeq S^3 \times S^3$. *Only two are smoothable.*

This is discussed in Section 6. We remark that because of the existence of a splitting basis in each base-equivalence class, and the fact that $L_6(Z_2, +) \cong Z_2$, the classification of Theorem 3 is computable.

3. PROOF OF THEOREM 2

Suppose M is a manifold of the kind considered above. According to surgery theory, the proof of Theorem 2 amounts to computing $\mathscr{S}_{PL}(M)$ and the action of $L_7(Z_2, +)$ on it [9]. In dimension six, it is clearly enough to work in the PL category.

LEMMA 3. $[M, G/PL] \cong [M, G/PL_{(2)}] \cong Z_2 \oplus Z_2$.

Proof. It is known that $G/PL_{(odd)} = BO \otimes Z[1/2]$, and that $[M, BO \otimes Z[1/2]]$ can be computed by means of a spectral sequence with

$$E_2^{p,q} = H^p(M; KO^q(*)) \otimes Z[1/2].$$

However, $\tilde{H}^p(M, Z[1/2]) = 0$ unless $p = 3$ or $p = 6$, and $KO^q(*) \otimes Z[1/2] = 0$ unless $q \equiv 0 \pmod 4$. Therefore $E_2^{p,-p} = 0$ for all p, and $[M, G/PL_{(odd)}] = 0$.

Now it is clear that

$$[M, G/PL_{(2)}] \cong [M, Y] \oplus [M, K(Z_2, 6)],$$

where Y is the 2-stage Postnikov system occurring as a factor of $G/PL_{(2)}$ [7]. From the exact sequence

$$[M, K(Z_{(2)}, 4)] \to [M, Y] \to [M, K(Z_2, 2)] \to 0$$

and the fact that $H^4(M; Z) = 0$, we see that $[M, Y] \cong Z_2$.

We shall now prove Theorem 2 by calculating the surgery obstruction γ: $[M, G/PL] \to L_6(Z_2, +) = Z_2$, using the formula of [9, p. 178]. The map $L_7(Z_2, +) \to \mathscr{S}_{PL}(M)$ is trivial [6, p. 48]. In fact, if $g: M \to G/PL$ corresponds to the essential map $M \to K(Z_2, 6)$, then

$$\gamma(M, g) = (1 + w_2(M))\, g^*(1 + Sq^2 + Sq^2 Sq^2)\, k[M] = g^* k[M] = 1,$$

where $k = k_2 + k_6$, and k_i is in $H^i(G/PL; Z_2)$. Similarly, if g corresponds to the essential map $M \to K(Z_2, 2)$, then $\gamma(M, g) = 0$, since $Sq^2 = 0$ on $H^2(M; Z_2)$.

4. CONSTRUCTION OF 6-MANIFOLDS

In this section, we prove half of Theorem 3 by constructing a one-to-one map from the equivalence classes of invariants to the homotopy classes of manifolds. This is a special case of a construction in [5].

Suppose we are given a based form (λ, ϕ, e_0, H), and set $w_2 = 1 + \phi(e_0)$ in Z_2. Let ξ be the orientable 3-plane bundle over RP^3, with second Stiefel-Whitney class

w_2. Then ξ is either 3ε or $\varepsilon \oplus 2\eta$, where ε (respectively, η) is the trivial (respectively, nontrivial) line bundle over RP^3. After changing the Z_2-form, if necessary, within its base-equivalence class, we may assume that $H \cong H_0 \oplus H_1$ is a split decomposition of H for λ with e_0 in H_0. Let $(e_1, \cdots, e_r, e_0, e_\infty)$ be the splitting basis, and denote by $D(\xi)$ and $S(\xi)$ the disk and sphere bundles, respectively, associated with ξ.

We begin the construction by forming $W_0 = D(\xi) \cup_f D^3 \times D^3$ and using an embedding $f: S^2 \times D^3 \to S(\xi)$, obtained from trivializing $S(\xi)$ over a disk $D^3 \subset RP^3$. Observe that $\partial W_0 \approx S(\xi_0)$, where ξ_0 is $i^*(\xi \oplus \eta)$, the bundle $\xi \oplus \eta$ pulled back over $i: RP^2 \subset RP^3$.

If $r = 0$, we finish by attaching $D(\xi_0)$ to W_0 along ∂W_0 with some diffeomorphism. If $r > 0$, we attach r disjoint handles $D^3 \times D^3$ to W_0 along ∂W_0 to obtain W. We construct the required embeddings $f_i: S^2 \times D^3 \to \partial W_0$ by first picking r unknotted and unlinked embeddings $f_i^0: S^2 \times D^3 \to \partial W_0$ $(i = 1, \cdots, r)$ inside disjoint embedded disks $D_i^5 \subset \partial W_0$. These embeddings are then moved by regular homotopies $\eta_i: S^2 \times D^3 \times I \to \partial W_0$ $(i = 1, \cdots, r)$ (with both ends embedded) whose intersections and self-intersection numbers are prescribed by λ and ϕ as in [9, p. 53]. For f_i we take $\eta_i \mid S^2 \times D^3 \times 1$.

An easy surgery argument (see [5]) now shows that $\partial W \approx S(\xi_0)$ also, and we finish as before by attaching $D(\xi_0)$ with a diffeomorphism $h: \partial W \to S(\xi_0)$. Denote $M = W \cup_h D(\xi_0)$ by $\Gamma(\theta)$, where $\theta = (\lambda, \phi, e_0, H)$. The following lemma shows that we may omit the map h from our notation.

LEMMA 4. *Any two choices of the diffeomorphism* $h: \partial W \to S(\xi_0)$ *result in homotopy-equivalent manifolds* M.

This is the main step in showing that Γ is surjective. It will be carried out in the next two sections. First we apply the result to conclude that Γ is well-defined on equivalence classes and is one-to-one.

LEMMA 5. *Let* $\theta = (\lambda, \phi, e_0, H)$ *and* $\theta' = (\lambda', \phi', e_0', H)$. *Then* $M = \Gamma(\theta)$ *and* $M' = \Gamma(\theta')$ *are homotopy-equivalent if and only if the forms* θ *and* θ' *are base-equivalent.*

Proof. Since any homotopy equivalence induces a base-equivalence of the forms and w_2 is a homotopy invariant, the necessity is clear.

Now suppose that e_0 and e_0' are contained in splitting bases and that $B: H \to H$ gives a base-equivalence of θ and θ'. Then $\phi(e_0) = \phi'(e_0')$. Because B is based, there is a map $RP^3 \to M'$ representing Be_0 that is a 2-connected and can be taken to be an embedding, by Haefliger's theorem [3]. In fact, by general position, we can assume that this embedding lies in W'; therefore we let $N \subset W'$ be a small tubular neighborhood. One can use the basis Be_i of H to attach handles to N inside W' and thus to produce an embedding of $W \subset$ interior W'. It is easy to see that W' - W is an h-cobordism between $\partial W'$ and ∂W, so that $W \approx W'$. From Lemma 4, we conclude that $M \simeq M'$.

5. REDUCTION TO INVOLUTIONS ON $S^3 \times S^3$

We shall prove Lemma 4 by listing the possible homotopy types of oriented Poincaré complexes K^6, then proving that the smoothable homotopy types can be specified by our invariants.

The first step is to reduce the problem to the case where $r = 0$. Consider a normal cell decomposition [8] of K induced by a splitting basis of $H_3(\tilde{K})$ with respect to λ. It will be necessary to have a notation for the skeleta K^i:

$$K^3 = RP^3 \vee S^3_\infty \vee L_r, \qquad \text{where } L_r = \bigvee_{k=1}^{r} S^3_k,$$

$$K^{i+1} = K^i \cup D^{i+1} \qquad \text{for } 3 \leq i \leq 5.$$

One can show [5] that for $i = 4$ and $i = 5$,

$$\tilde{K}^i \simeq S^i \vee N_r, \qquad \text{where } N_r = S^3_0 \vee S^3_\infty \vee L_r \vee L^*_r.$$

As the notation indicates, the obvious inclusion $j: S^3_\infty \vee L_r \subset \tilde{K}^4 \subset \tilde{K}$ has the property that $j_*[S^3_i] = e_i$ for $i = 1, \cdots, r$ and $i = \infty$, while the inclusion $RP^3 \subset K^4 \subset K$ is covered by $S^3_0 \subset \tilde{K}^4 \subset \tilde{K}$ and represents e_0. Finally, L^*_r is another copy of L_r, the image of L_r under the covering transformation T in $\tilde{K}$.

We ask what complexes K' have the same homology and cup-product as K. Clearly, K^4 is determined by homology. However, the attaching map of the 5-cell has homotopy class in $\pi_4 \tilde{K} \cong \pi_4 S^4 \oplus \pi_4 N_r$. Let α be the summand from $\pi_4 N_r$ (which is a direct sum of copies of $Z_2 \cong \pi_4 S^3$). This element α must have the property that $(1 - T)\alpha = 0$. It is not detected by homology or cup-product. Similarly, the summand of the homotopy class of the attaching map for the 6-cell that is not detected by this means is β, in

$$(Z_2)^{2r+2} \oplus \bigoplus_{k=1}^{r} \pi_5(S^3 \vee S^3)_{(k)}.$$

We shall use e_k for the inclusions $S^3_k \subset \tilde{K}$ as well as for the homology classes they represent.

LEMMA 6. *Let* $(e_1, \cdots, e_r, e_0, e_\infty)$ *be a splitting basis for* K. *There exists a normal cell decomposition of* K *induced by the basis, as above, such that*

(1) $\alpha = e_0 \circ \alpha_0 + e_\infty \circ \alpha_\infty$, *where* α_0 *and* α_∞ *are in* $\pi_4 S^3$, *and*

(2) $$\beta = \sum_{k=1}^{r} e_k \circ \beta_k + e_0 \circ \beta_0 + e_\infty \circ \beta_\infty + \sum_{k=1}^{r} m_k [e_k, Te_k],$$

where the element β_k *is in* $\pi_5 S^3$, *the coefficient* m_k *is in* Z_2, *and* $[e_k, Te_k]$ *generates* $\pi_5(S^3 \vee S^3)_{(k)}$.

LEMMA 7. *The covering space* $\tilde{K}$ *is smoothable if and only if* $\beta_k = 0$ *for* $1 \leq k \leq r$.

Proof. In the notation established at the beginning of the section, $\tilde{K} \simeq N_r \cup_{(1+T)\beta} D^6$. Clearly, $\tilde{K}$ is smoothable if and only if it is the homotopy type of a connected sum of copies of $S^3 \times S^3$. In that case, there exists for each $k = 1, \cdots, r$ a projection $p_k: \tilde{K} \to S^3_k$ such that $p_k \circ e_k$ is the identity. Hence β_k, the obstruction to the existence of p_k, is zero.

Next we identify the coefficients m_k occurring in the expression for β of Lemma 6. Recall that in [1] ϕ is defined by means of a cohomology operation ψ: $H^3(\tilde{K}, Z_2) \to H^6(K, Z_2) \cong Z_2$ (see also Section 6 below). In fact, if $\bar{x}$ is the Poincaré dual to x in $H_3(\tilde{K}, Z_2)$, then

$$\phi(x) = \psi(\bar{x})[K], \quad \text{where } [K] \text{ generates } H_6(K: Z_2).$$

An easy calculation using the definition of ψ yields the following result.

LEMMA 8. *Let* e_k^* *be reduction modulo* 2 *of the class in* $H^3(\tilde{K})$ *dual to* e_k. *Then* $m_k = \psi(e_k^*)$ *for* $k = 1, \cdots, r$.

By the construction of Section 4, each value of $\phi(e_k)$ (and therefore each value of m_k) is possible for $k = 1, \cdots, r$ in smoothable complexes. Note that $K_0 = K/L_r$ has the homotopy type of a Poincaré complex with $\pi_1 K_0 = Z_2$ and $\tilde{K}_0 \simeq S^3 \times S^3$. The following result is a consequence of Lemmas 6 to 8.

LEMMA 9. K *is smoothable if and only if* K_0 *and* $\tilde{K}$ *are smoothable.*

6. INVOLUTIONS ON $S^3 \times S^3$

First we shall prove part of Proposition 1.

LEMMA 10. *There are ten distinct complexes* K *of our type with* $\tilde{K} \simeq S^3 \times S^3$.

Proof. The possibilities for $(\alpha_0, \alpha_\infty)$ may be written (00), (10), (01), and (11). Similarly for (β_0, β_∞). For constructing K^5, given K^4, we have three choices: $K^5(00)$, $K^5(10)$, and $K^5(01)$, since $K^5(11)$ is homotopy-equivalent to $K^5(01)$. For constructing K^6, given K^5, we have three complexes based on $K^5(00)$ or $K^5(10)$, but four on $K^5(01)$. These must all be shown to be distinct.

Suppose that K and K_1 are two of the complexes above and that f: $K \to K_1$ is a cellular homotopy equivalence. Since both complexes are orientable, f maps the top cell with degree ± 1 and induces a homotopy equivalence $\bar{f}$: $K^5 \to K_1^5$. This proves that $f_*\beta = \beta'$. Moreover, because K^5 is nonorientable, $\bar{f}$ maps the 5-cell with degree ℓ (odd). Therefore $f_*\alpha = \ell\alpha' = \alpha'$. The only variation is therefore caused by basis changes in $H_3(\tilde{K})$. The only one that affects the attaching maps, namely setting $e_\infty' = e_0 + e_\infty$ and $e_0' = e_0$, is allowed for in our list. (Recall here that the isomorphism defined by $e_0' = e_0 + e_\infty$ and $e_\infty' = e_\infty$ is not a base-equivalence.)

Remark. The complex corresponding to the choice $\alpha = \beta = 0$ is $RP^3 \times S^3$; the choice $\alpha = (01)$ and $\beta = 0$ gives $K = S(2\varepsilon \oplus 2\eta)$. These are distinguished by $w_2(K)$ or Sq^2.

The remainder of the proof of Proposition 1 is contained in the three lemmas below. These show how the homotopy description of Section 5 can be given in terms of the Z_2-form, at least for smoothable complexes, and enable us to identify the other invariants $w_2(K)$ and Sq^2.

In the statement that follows, recall that e_0^*, e_∞^* is the (cohomology) dual basis to e_0, e_∞. By $\bar{e}_\infty^*$ we mean the class in $H^3(K)$ dual to that represented by $S_\infty^3 \subset K$, reduced modulo 2. It should also be noted that the Poincaré duals of e_0 and e_∞ are e_∞^* and e_0^*, respectively.

LEMMA 11. $\alpha_0 \neq 0$ *if and only if* $\psi(e_0^*) = \phi(e_\infty) \neq 0$.

LEMMA 12. (1) $\alpha_\infty \neq 0$ *if and only if* $\psi(e_\infty^*) = \phi(e_0) = 0$.

(2) $\alpha_\infty \neq 0$ *if and only if* $Sq^2 \bar{e}_\infty^* \neq 0$ (*or* $w_2 \neq 0$).

LEMMA 13. *If* K *is smoothable, then* $\beta_0 = \beta_\infty = 0$.

Application of Lemmas 11 to 13, together with Theorem 1, which eliminates the case $\alpha = (10)$, reduces the list of ten complexes to two possible smooth ones. These are precisely $RP^3 \times S^3$ and $S(2\varepsilon \oplus 2\eta)$, and clearly they are smoothable. This proves Proposition 1.

Combined with Lemma 9, the result evidently establishes that the homotopy type of a smoothable complex is completely determined by the base-equivalence class of its based Z_2-form, and Lemma 4 follows. As we observed earlier, we can now conclude that the map Γ is one-to-one and surjective. This proves Theorem 3.

Proof of Lemma 11. We recall the definition of ψ in [1]. Let $T: \tilde{K} \to \tilde{K}$ be a simplicial, free involution and z a cocycle in $Z^3(\tilde{K}, Z_2)$. Then there exist cochains v^{3+i} for $0 \le i \le 3$ in $C^{3+i}(\tilde{K}, Z_2)$ such that

$$z \cup_{3-i} Tz + \delta v^{3+i-1} = (1+T)v^{3+i} \qquad (0 \le i \le 3),$$

where $v^2 = 0$. It turns out that cocycle $(1+T)v^6$ represents a class in $H_T^6(\tilde{K}, Z_2) \cong H^6(K, Z_2)$, which depends only on the cohomology class of z. Set $\psi(z) = \text{cls}((1+T)v^6)$.

This operation can be evaluated on a complex L obtained from K by forming $K/S_\infty^3 = (RP^3 \vee S^4) \cup D^5 \cup D^6$ and then collapsing the resulting S^4. Let $j: K \to L$ be the quotient map, and u the generator of $H^3(\tilde{L}, Z_2)$. Evidently,

$$L \simeq (RP^3 \cup_{\alpha_0} D^5) \cup D^6,$$

and $j^* u = e_0^*$. Using the fact that Sq^2 detects the generator of $\pi_4 S^3$, we see that

$$Sq^2\, u \neq 0 \text{ as a cochain if and only if } \alpha_0 \neq 0;$$

therefore $\psi(u) \neq 0$ if and only if $\alpha_0 \neq 0$. The result now follows by naturality.

Proof of Lemma 12. For this argument, let $L = K/RP^3$, let $j: K \to L$ be the quotient map, and let u be the generator of $H^3(L, Z_2)$. Clearly,

$$L \simeq (S_\infty^3 \cup_{\alpha_\infty} D^5 \cup_{\beta_\infty} D^6) \vee S^4 \qquad \text{and} \qquad j^* u = e_\infty^*.$$

Part (2) now follows by naturality of Sq^2. Since $e_\infty^* = (1+T)\sigma^3$, where σ^3 is in $C^3(\tilde{K}, Z_2)$, it is easy to compute $\psi(e_\infty^*)$ and obtain (1).

Proof of Lemma 13. Suppose that M is a smoothing of K and that ξ is the normal bundle of an embedded RP^3 in M. It follows from the decomposition

$$M \approx D(\xi) \cup D(\xi)$$

that $K/RP^3 \simeq M/RP^3 \simeq$ (Thom space of ξ). But if $\beta_\infty \neq 0$, then $T(\xi)$ carries the nonzero secondary cohomology operation on the Thom class U described in [2]. This implies that the Gitler-Stasheff characteristic class of ξ is nonzero, so that ξ is not a vector bundle.

For the other part, consider the decomposition

$$M \approx W \cup_h D(\xi_0)$$

of Section 4, where ξ_0 is the normal bundle to a 2-connected embedding of RP^2 in M. By Theorem 1 and Lemma 11, there is a basis of $H_3(\tilde{K})$ in which $\phi(e_\infty) = 0$. Therefore β_0 is the only obstruction to a map $p: M \to RP^3$ with the property that $p \mid RP^3$ is the identity. However, $W \simeq S^3_\infty \vee RP^3$; hence we can try to extend the projection $W \to S^3_\infty \vee RP^3 \to RP^3$ to all of M. Since the homotopy class of the attachment of a cell in a cell decomposition of M modulo W factors through the map

$$h_*: \pi_i(S(\xi_0)) \to \pi_i(\partial W),$$

and since the composition

$$\pi_i(\partial W) \to \pi_i(W) \to \pi_i(RP^3) \qquad (i > 1)$$

is zero, the extension is possible.

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