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Two remarks on Wall's D2 problem

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Abstract

If a finite group G is isomorphic to a subgroup of $SO(3)$, then G has the D2-property. Let X be a finite complex satisfying Wall's D2-conditions. If $\pi_1(X) = G$ is finite, and $\chi(X) \geqslant 1 - \operatorname{def}(G)$, then $X \vee S^2$ is simple homotopy equivalent to a finite 2-complex, whose simple homotopy type depends only on G and $\chi(X)$.

1. *Introduction*

In [**32**, section 2], C. T. C. Wall initiated the study of the relations between homological and geometrical dimension conditions for finite CW-complexes. In particular, a finite complex X *satisfies Wall's* D2-*conditions* if $H_i(\widetilde{X}) = 0$, for $i > 2$, and $H^3(X; \mathcal{B}) = 0$, for all coefficient bundles $\mathcal{B}$. Here $\widetilde{X}$ denotes the universal covering of X. If these conditions hold, we will say that X is a D2-*complex*. If every D2-complex with fundamental group G is homotopy equivalent to a finite 2-complex, then we say that G *has the* D2-*property*.

In [**32**, p. 64], Wall proved that a finite complex X satisfying the D2-conditions is homotopy equivalent to a finite 3-complex. We will therefore assume that all our D2-complexes have $\dim X \leqslant 3$.

The D2 problem for a finitely-presented group G asks whether every finite complex X with fundamental group G which satisfies the D2-conditions is homotopy equivalent to a finite 2-complex. The D2 problem has been actively studied for finite groups, but answered affirmatively only in a limited number of cases (see [**18**, **21**] for references to the literature on 2-complexes and the D2-problem, and compare [**19**, **20**, **24**] for some more recent work).

In this paper, I make two remarks concerning the (stable) solution of the D2-problem and cancellation, based on my joint work with Matthias Kreck [**11**, theorem B]. I am indebted to Dr. W. H. Mannan for asking about this connection some years ago.

For G a finitely presented group, let $\operatorname{def}(G)$ denote the *deficiency of* G, defined as the maximum value of the number of generators minus the number of relations over all finite presentations of G. We note that $1 - \operatorname{def}(G)$ is the minimal Euler characteristic possible for a finite 2-complex with fundamental group G.

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Swan defined $\mu_2(G)$ as the minimum of the numbers $\mu_2(\mathcal{F}) = f_2 - f_1 + f_0$, where f_i are the ranks of the finitely generated free $\mathbb{Z}G$-modules F_i in an exact sequence

$$\mathcal{F}:\ F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

In general, Swan [**31**, proposition 1] noted that $\mu_2(G) \leqslant 1 - \mathrm{def}(G)$. For a finite D2-complex X, we have the Euler characteristic inequality $\chi(X) \geqslant \mu_2(G)$ (see Section 2 for details). In addition, $\mu_2(G) \geqslant 1$ for G a finite group by [**31**, corollary 1·3].

THEOREM A. *Let X be a finite* D2*-complex, and assume that $G := \pi_1(X)$ is a finite group. Then*:

(i) *if $\chi(X) > 1 - \mathrm{def}(G)$, X is simple homotopy equivalent to a finite* 2*-complex*;

(ii) *If $\chi(X) = 1 - \mathrm{def}(G)$, $X \vee S^2$ is simple homotopy equivalent to a finite* 2*-complex.*

In case (i) *the simple homotopy type of X depends only on $\pi_1(X)$ and $\chi(X)$.*

The uniqueness part is a direct application of [**11**, theorem B], since the resulting 2-complexes have non-minimal Euler characteristic. We remark that the unpublished work of Browning [**6**] implies the corresponding weaker statements for homotopy equivalence, rather than simple homotopy equivalance (see Corollary 2·6).

Remark 1·1. A stable solution of the problem for D2-complexes with any finitely presented fundamental group was first given by Cohen [**7**, theorem 1]: if X is a D2-complex, then there exists an integer $r \geqslant 0$ such that the stabilised complex $X \vee r(S^2)$ is homotopy equivalent to a finite 2-complex.

This result and the foundational work of J. H. C. Whitehead [**34**] shows that any two D2-complexes with isomorphic fundamental groups become stably simple homotopy equivalent after wedging on sufficiently many 2-spheres. I give a different argument in Lemma 2·1 for the stable result, and show that it holds whenever $r \geqslant b_3(X)$ (compare [**19**, proposition 3·5]). Here $b_3(X)$ denotes the number of 3-cells in X.

If the group ring $\mathbb{Z}G$ is noetherian, then there exists a uniform bound for this stable range, depending only on the fundamental group (see Proposition 2·7). This remark applies for example to polycyclic-by-finite fundamental groups.

THEOREM B. *Let G be a finite subgroup of $SO(3)$. Then any finite* D2*-complex with fundamental group G is simple homotopy equivalent to a finite* 2*-complex, and G has the* D2*-property.*

This result is an application of [**11**, theorem 2·1]. The result was known for cyclic and dihedral groups (see [**23**, **26**, **28**]), but the argument given here is more uniform and the tetrahedral, octahedral and isosahedral groups do not seem to have been covered before.

Remark 1·2. Brown and Kahn [**5**, theorem 2·1] proved that that a D2-complex which is a nilpotent space is homotopy equivalent to a 2-complex, but this does not appear to settle the D2 problem for nilpotent fundamental groups.

Remark 1·3. A result essentially contained in the proof of Wall [**33**, theorem 4] shows that there exist finite D2-complexes X, with $\pi_1(X) = G$ and $\chi(X) = \mu_2(G)$ realizing this minimum value, for every finitely presented group G. Since $\mu_2(G) \leqslant 1 - \mathrm{def}(G)$ by Swan [**31**, proposition 1], a *necessary* condition for any group G to have the D2-property is that $\mu_2(G) = 1 - \mathrm{def}(G)$.

2. *Cancellation and the D2 Problem*

We assume that X is a finite, connected 3-complex, with fundamental group $G = \pi_1(X)$, satisfying the D2-conditions. We use the following notation for the chain complex $C(\widetilde{X};\mathbb{Z})$ of the universal covering:

$$C(X) : 0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

considered as a chain complex of finitely-generated, free Λ-modules relative to a single 0-cell as base-point, where $\Lambda = \mathbb{Z}G$ is the integral group ring.

The boundary map ∂_3 is injective because $H_3(\widetilde{X}) = 0$. Let $B_3 = \operatorname{im}(\partial_3)$, with $j\colon B_3 \to C_2$ the inclusion map, and consider the boundary map $\partial_3\colon C_3 \to B_3$ as defining a 3-cocycle. Since $H^3(X; B_3) = 0$, there is a Λ-module homomorphism $\phi\colon C_2 \to B_3$ such that $\phi \circ j =$ id. We have an exact sequence

$$0 \longrightarrow C_3 \longrightarrow \pi_2(K) \longrightarrow \pi_2(X) \longrightarrow 0,$$

where $K \subset X$ denotes the 2-skeleton (since $\pi_2(K) = Z_2 = \ker\partial_2$). It follows that C_3 is a direct summand of $\pi_2(K)$, and hence $\pi_2(X)$ is a representative of the stable class $\Omega^3(\mathbb{Z})$. More explicitly, the map ϕ induces a direct sum splitting $C_2 = \operatorname{im}(\partial_3) \oplus P$, and $P \cong C_2/\operatorname{im}(\partial_3)$ is a finitely-generated, stably-free Λ-module since $C_3 \cong \operatorname{im}(\partial_3)$ is a finitely-generated, free Λ-module. This gives a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C_3 & \overset{\partial_3}{=\!=\!=} & B_3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Z_2 & \longrightarrow & C_2 & \longrightarrow & B_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_2(X) & \longrightarrow & C_2/B_3 & \longrightarrow & B_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the vertical sequences are split exact, and hence a resolution

$$0 \longrightarrow \pi_2(X) \longrightarrow P \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0\,.$$

By a sequence of elementary expansions (on the chain complex these are just the direct sum with copies of $\Lambda = \Lambda$ in dimensions 1 and 2), we may assume that P is a finitely-generated, free Λ-module. This operation doesn't change the (simple) homotopy type of X. The following result has also been observed in [**7**], [**19**, theorem 3·5]. Our proof uses the techniques of [**11**, section 2].

LEMMA 2·1. *The stabilised complex $X \vee r(S^2)$, with $r = b_3(X)$, is simple homotopy equivalent to a finite 2-complex K.*

Proof. Let $u\colon K \subset X$ denote the inclusion of the 2-skeleton of X, so that we have the identification $\pi_2(K) \cong \pi(X) \oplus C_3$ discussed above. We further identify

$$\pi_2(K \vee r(S^2)) \cong \pi_2(K) \oplus \Lambda^r \cong \pi_2(X) \oplus C_3 \oplus F \qquad (2{\cdot}2)$$

and fix free Λ-bases $\{e_1, \dots, e_r\}$ for $C_3 \cong \Lambda^r$, and $\{f_1, \dots, f_r\}$ for $F \cong \Lambda^r$. The same notation $\{e_i\}$ and $\{f_j\}$ will also be used for continuous maps $S^2 \to K \vee r(S^2)$ in the homotopy classes of $\pi_2(K \vee r(S^2))$ defined by these basis elements. Notice that the maps $f_j \colon S^2 \to K \vee r(S^2)$ may be chosen to represent the inclusions of the S^2 wedge factors.

We first claim that there exists a (simple) self-homotopy equivalence

$$h\colon K \vee r(S^2) \longrightarrow K \vee r(S^2)$$

such that the induced isomorphism

$$h_*\colon \pi_2(K \vee r(S^2)) \xrightarrow{\cong} \pi_2(K \vee r(S^2))$$

has the property $h_*(e_i) = f_i$, for $1 \leqslant i \leqslant r$, with respect to the chosen bases in the right-hand side of (2·2), and induces the identity on the summand $\pi_2(X)$.

The construction of the required self-homotopy equivalences is given in [**11**, p. 101], where the realization of the group of elementary automorphisms $E(P_1, L \oplus P_0)$ is studied. In this notation P_0, P_1 are free modules of rank one, and L is an arbitrary Λ-module. The basic construction is to realise automorphisms of the form $1+f$ and $1+g$, where $f\colon L \oplus P_0 \to P_1$ and $g\colon P_1 \to L \oplus P_0$ are arbitrary Λ-homomorphisms. We apply this to the sub-module $L \oplus \Lambda \cdot e_1 \oplus \Lambda \cdot f_1$, where $L = \pi_2(X)$, and realise the automorphism $\mathrm{id}_L \oplus \alpha$ with $\alpha(e_1) = -f_1$ and $\alpha(f_1) = e_1$ via the composition

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We can now construct a homotopy equivalence $f\colon X \vee r(S^2) \to K$, by extending the simple homotopy equivalence $h\colon K \vee r(S^2) \to K \vee r(S^2)$ over the (stabilised) inclusion

$$u \vee \mathrm{id}\colon K \vee r(S^2) \longrightarrow X \vee r(S^2)$$

by attaching the 3-cells of X in domain, and 3-cells in the range which cancel the S^2 wedge factors. For the attaching maps $[\partial D^3_i] = e_i$, $1 \leqslant i \leqslant r$, of the 3-cells of X we have $h \circ [\partial D^3_i] = f_i$. Hence we can extend by the identity to 3-cells attached along the maps $\{f_i\colon S^2 \to K \vee r(S^2)\}$. We obtain a map

$$h'\colon X \vee r(S^2) \longrightarrow K \vee r(S^2) \bigcup \{D^3_i : [\partial D^3_i] = f_i, 1 \leqslant i \leqslant r\} \simeq K$$

extending h. It is easy to check that h' is a (simple) homotopy equivalence.

An *algebraic* 2-*complex over the group ring* $\Lambda := \mathbb{Z}G$ is a chain complex (F_*, ∂_*) of the form

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

consisting of an exact sequence of finitely-generated, stably-free Λ-modules, such that $H_0(F_*) = \mathbb{Z}$. An r*-stabilisation* of an algebraic 2-complex is the result of direct sum with a complex (E_*, ∂_*), where $E_2 = \Lambda^r$ for some $r \geqslant 0$, $\partial_* = 0$ and $E_i = 0$ for $i \neq 2$. We say that an algebraic 2-complex is *geometrically realisable* if it is chain homotopy equivalent to the cellular chain complex $C(X)$ of a (geometric) finite 2-complex X with fundamental group $\pi_1(X) = G$.

LEMMA 2·3. *Any algebraic* 2-*complex* (F_*, ∂_*) *over* $\Lambda = \mathbb{Z}G$ *is geometrically realisable after an* r*-stablisation, for some* $r \geqslant 0$.

Proof. We compare the resolution

$$0 \longrightarrow L \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $L = \ker \partial_2$, to one obtained from the chain complex

$$0 \longrightarrow \pi_2(K) \longrightarrow C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K) \longrightarrow \mathbb{Z} \longrightarrow 0$$

of any finite 2-complex K with fundamental group G. Then Schanuel's Lemma shows that these two resolutions of Λ-modules (regarded as connected 3-dimensional chain complexes) are stably chain isomorphic after direct sum with elementary complexes of the form $\Lambda = \Lambda$ in degrees $(i, i-1)$ for $1 \leqslant i \leqslant 3$ (compare [**33**, lemma 3B], or [**12**, p. 415]).

The stabilisations in degrees $(i, i-1)$ for $i < 3$ produce a complex (F'_*, ∂'_*) of finitely generated free Λ-modules, and a chain homotopy equivalence $(F'_*, \partial'_*) \simeq (F_*, \partial_*)$. The additional degree $(3, 2)$ stabilisations produce a complex (F''_*, ∂''_*), and a chain homotopy equivalence $(F''_*, \partial''_*) \simeq (F_*, \partial_*) \oplus (E_*, \partial_*)$, where (E_*, ∂_*) is a complex concentrated in degree 2 (as defined above).

In other words, the resulting stabilised complex $(F_*, \partial_*) \oplus (E_*, \partial_*)$ is an r-stabilisation of (F_*, ∂_*). The chain homotopy equivalence

$$(F_*, \partial_*) \oplus (E_*, \partial_*) \simeq C_*(K \vee r(S^2))$$

shows that the algebraic 2-complex (F_*, ∂_*) is geometrically realisable after r-stabilisation.

COROLLARY 2·4 (Wall). *Every algebraic* 2*-complex* F_* *is chain homotopy equivalent to the chain complex* $C_*(X)$ *of a* D2*-complex.*

Proof. The construction produces a chain homotopy equivalence

$$(F_*, \partial_*) \oplus (E_*, \partial_*) \simeq C_*(K \vee r(S^2))$$

after an r-stabilisation of F_*, and in particular an isomorphism $L \oplus E_2 = \pi_2(K) \oplus \Lambda^r$, for some $r \geqslant 0$. Then one can attach 3-cells to $K \vee r(S^2)$, using the images in $\pi_2(K \vee r(S^2))$ of a free basis of the summand $E_2 \cong \Lambda^r$, to produce a D2-complex X and a chain homotopy equivalence $C(X) \simeq F_*$.

Remark 2·5. The ingredients in the proof of Lemma 2·3 are essentially the same as those used by Wall to prove [**33**, theorem 4]. Similar ideas appear in [**21**, appendix B], [**25**, theorem 2·1].

Proof of Theorem A. Let X be a finite 3-complex which satisfies the D2-conditions. By Lemma 2·1, there exists a finite 2-complex K and a simple homotopy equivalence $f: X' := X \vee r(S^2) \to K$, for any $r \geqslant b_3(X)$. Now let $G = \pi_1(X)$ be a finite group, and let K_0 denote a minimal finite 2-complex K_0 with fundamental group G. Then $\chi(K_0) = 1 - \text{def}(G)$, and, after perhaps stabilising further, we can assume that K is simple homotopy equivalent to a stabilisation of K_0. We then obtain a simple homotopy equivalence of the form

$$X \vee r(S^2) \simeq K_0 \vee t(S^2) \vee r(S^2),$$

where $t \geqslant 0$ provided that $\chi(X) \geqslant 1 - \text{def}(G) = \chi(K_0)$. We note that the arguments in [**11**, section 2] are at first completely algebraic (to obtain cancellation of the π_2 modules via elementary automorphisms). This step depends on the assumption of finite fundamental group for the application of [**11**, corollary 1·2 and lemma 1·16] (compare the proof of

[**11**, theorem B]). Then we show (following [**11**, p. 101]) how to realize the necessary elementary automorphisms by simple homotopy equivalences.

If $\chi(X) > \chi(K_0)$, then $t \geqslant 1$ and we can construct simple self-equivalences of $K_0 \vee t(S^2) \vee r(S^2)$ to cancel the extra r wedge summands of $X \vee r(S^2)$. The resulting 2-complex will be $K' \simeq K_0 \vee t(S^2)$.

If $\chi(X) = \chi(K_0)$, then $t = 0$ but we can perform the same operations after replacing X by $X \vee S^2$, and the resulting 2-complex will be $K' \simeq K_0 \vee S^2$. In either case, the resulting 2-complex K' has non-minimal Euler characteristic $\chi(K') > \chi(K_0)$, so its simple homotopy type is uniquely determined by G and $\chi(X)$ (see [**11**, theorem B]).

The techniques used in this proof also give a version for algebraic 2-complexes (answering a question of Browning [**6**, section 5·6]). We recall that an *s-basis* for a stably free Λ-module M is a free Λ-basis for some stabilisation $M \oplus \Lambda^r$ by a free module.

COROLLARY 2·6. *Let F and F' be s-based algebraic 2-complexes over $\Lambda = \mathbb{Z}G$, where G is a finite group. If $\chi(F) = \chi(F') > \mu_2(G)$, then F and F' are simple chain homotopy equivalent.*

Proof. We apply Corollary 2·4 and the method of proof for Theorem A.

Proof of Theorem B. The same remarks as above apply to the proof of [**11**, theorem 2·1]. In addition, we note that $\mu_2(G) = 1 - \mathrm{def}(G)$ for all of the finite subgroups of $SO(3)$. For these groups, $\mathrm{def}(G) \geqslant -1$ (see Coxeter [**8**, section 6·4]), and $\mu_2(G)$ can be estimated by group cohomology using Swan [**31**, theorem 1·1]. We can now apply cancellation down to $r = 0$ for fundamental groups which are finite subgroups of $SO(3)$. This proves that every algebraic 2-complex with one of these fundamental groups is geometrically realisable.

The uniform stability bound for D2-complexes in Theorem A is a special result for finite fundamental groups, based initially on the fact that their integral group rings are finite algebras over the integers. Here is a sample stability result which applies to certain infinite fundamental groups (compare Brown [**4**]).

PROPOSITION 2·7. *Let G be a finitely presented group such that the integral group ring $\mathbb{Z}G$ is noetherian of Krull dimension d_G. If X is a finite complex with $\pi_1(X) = G$ satisfying the* D2*-conditions, then $X \vee r(S^2)$ is simple homotopy equivalent to a finite 2-complex, for $r \geqslant d_G + 1$, whose simple homotopy type is uniquely determined by G and $\chi(X)$.*

Proof. (Sketch) The arguments follow the same outline as those used by Bass [**1**, chapter IV·3·5] to prove a cancellation theorem for modules using elementary automorphisms. The ingredients in these arguments were generalised to apply to non-commutative noetherian rings by Magurn, van der Kallen and Vaserstein [**22**], and Stafford [**29**, **30**] (see also McConnell and Robson [**27**, chapter 11]). The application to 2-complexes follows by realising elementary automorphisms by simple homotopy self-equivalences, as in [**11**, section 2].

Remark 2·8. For G finite, the integral group ring $\mathbb{Z}G$ has Krull dimension $d_G = 1$, so the Bass stability bound would be $d_G + 1 = 2$. If G is a polycyclic-by-finite group, the group ring $\mathbb{Z}G$ is again noetherian and $d_G = h_G + 1$, where h_G denotes the *Hirsch length* of G (see [**27**, 6·6·1]). The examples of [**9**, **15**, **16**, **17**] show that for general infinite fundamental groups (for example, the fundamental group of the trefoil knot), there can be (infinitely) many distinct 2-complexes with the same Euler characteristic.

3. *The relation gap problem*

We will conclude by mentioning a related problem. If F/R is a finite presentation for a group G, then the action of the free group F by conjugation on the normal subgroup R induces an action of G on the quotient abelian group $R_{ab} := R/[R.R]$. This $\mathbb{Z}G$-module R_{ab} is called the *relation module* for G.

Let $d(\Gamma)$ denote the minimum number of elements needed to generate a group Γ, and if a group Q acts on Γ, then let $d_Q(\Gamma)$ denote the minimum number of Q-orbits needed to generate Γ. Note that $d(\Gamma) \geqslant d_G(\Gamma)$.

In this notation, $d_F(R)$ is the minimum number of normal generators for R, and $d_G(R/[R.R])$ is the minimum number of $\mathbb{Z}G$-module generators for the module R_{ab}.

Definition 3·1. For a finite presentation F/R of a group G, the *relation gap* is the difference $d_F(R) - d_G(R/[R, R])$. The *relation gap problem* is to decide whether there exists a finite presentation with a positive relation gap.

The survey articles of Harlander [**13**, **14**] provide some key examples (such as those constructed by Bridson and Tweedale [**3**]), and a guide to the literature. A connection to the D2 problem is provided by the following result:

THEOREM 3·2 (Dyer [**13**, theorem 3·5]). *Let G be a group with $H^3(G; \mathbb{Z}G) = 0$. If there exists a finite presentation F/R with a positive relation gap, realizing the deficiency of G, then the* D2 *property does not hold for G.*

The D2 problem can be considered a generalisation of the Eilenberg–Ganea conjecture [**10**], which states that a group G with cohomological dimension 2 also has geometric dimension 2. If $\mathrm{cd}(G) = 2$ and the classifying space BG is homotopy equivalent to a finite complex, then G will satisfy the Eilenberg–Ganea conjecture if G has the D2 property.

A striking result of Bestvina and Brady [**2**, theorem 8·7] shows that either the Eilenberg–Ganea conjecture is false, or there is a counterexample to the Whitehead conjecture, which states that every connected subcomplex of an aspherical 2-complex is aspherical.

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