A FOUR-MANIFOLD APPROACH TO THE SPHERICAL SPACE FORM PROBLEM

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Abstract. For certain 3-manifolds M with finite fundamental group, we construct smooth, negative definite 4-manifolds, with boundary containing M and some orthogonal spherical spaces forms. This allows a translation of the existence problem for finite fundamental groups of 3-manifolds into a problem in equivariant gauge theory.

1. Introduction

A well-known problem in three dimensional topology is to list all the finite groups which occur as the fundamental group of some closed 3-manifold. So far, all the known examples come from the finite subgroups $\Gamma \subset SO(4)$ which operate freely on the 3-sphere. The associated 3-manifolds S^3/Γ admit Riemannian metrics of constant positive curvature, and are known as the (orthogonal) spherical space forms. This paper is the first installment of a project whose goal is to show that these examples exhibit *all* the finite fundamental groups of closed 3-manifolds.

The classification of orthogonal spherical space forms up to isometry [21] was first proposed by Killing in 1891, and the problem attracted the attention of famous mathematicians of the time, such as Clifford, Hopf, Klein, and Poincaré. According to H. Hopf's 1925 paper [7], the following is a list of all finite fixed-point free subgroups of $SO(4)$:

(1.1) The cyclic group $C(n)$, the generalized quaternion group $Q(4n)$, the binary tetrahedral group $T^*(24)$, the binary octahedral group $O^*(48)$, and the binary icosahedral group $I^*(120)$.

(1.2) The semidirect product $C(2n+1) \rtimes C(2^k)$ of an odd order cyclic group with a cyclic 2-group. More explicitly $C(2n+1) \rtimes C(2^k)$ is given by the presentation $\{A, B : A^{2^k} = B^{2n+1} = 1, ABA^{-1} = B^{-1}\}$ where $k \geq 2, n \geq 1$.

(1.3) A semidirect product $Q(8) \rtimes C(3^k)$ of the quaternion group $Q(8)$ with a cyclic 3-group. More explicitly, $Q(8) \rtimes C(3^k)$ is given by the presentation $\{P, Q, X : P^2 = (PQ)^2 = Q^2,\ X^{3^k} = 1,\ XPX^{-1} = Q,\ XQX^{-1} = PQ\}$ where $k \geq 1$. For $k = 1$, this is the binary tetrahedral group $T^*(24)$.

(1.4) The product of any of the above groups in (1.1)-(1.3) with a cyclic group of coprime order.

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At first glance, the above list may appear to be random. In the forties and fifties, efforts were made to interpret Hopf's list using group cohomology [1] and it was discovered that all these groups have periodic Tate cohomology of period four. In general, a finite group has periodic cohomology if and only if it satisfies the p^2-conditions ("any subgroup of order p^2 is cyclic") for all primes p. From the viewpoint of group theory, this condition means that the odd Sylow subgroup is cyclic and the 2-Sylow subgroup is cyclic or generalized quaternion. If the cohomology has period four then, in addition, the pq-conditions hold ("every subgroup of order pq is cyclic") for p and q distinct odd primes.

The necessity of the $2q$-conditions was established by J. Milnor [13] in 1957, when he showed that the dihedral group of order $2q$ cannot operate freely on any $\mathbb{Z}/2$-homology sphere despite the fact that it has periodic cohomology of period 4. In [13] Milnor also compiled the following list of all finite groups, not in Hopf's list (1.1)-(1.4), but satisfying the restrictions known at the time on fundamental groups of 3-manifolds.

(1.5) The semidirect product $Q(8n, k, l)$ of the odd cyclic group $C(kl)$ with the generalized quaternion group $Q(8n)$. More explicitly, $Q(8n, k, l)$ has the presentation: $\{X, Y, Z : X^2 = Y^{2n} = (XY)^2,\ Z^{kl} = 1,\ XZX^{-1} = Z^r,\ YZY = Z^{-1}\}$. Here n, k, l are all odd integers and relatively prime to each other, $n > k > l \geq 1$, and r satisfies $r \equiv -1 \pmod{k}$, $r \equiv 1 \pmod{l}$. If $l = 1$, we set $Q(8n, k) \equiv Q(n, k, 1)$.

(1.6) The group $Q(8n, k, l)$ with the same presentation as (1.5), but with n even.

(1.7) An extension $O(48; 3^{k-1}, l)$ of the odd order cyclic group $C(3^{k-1}l)$, $3 \nmid l$, by the binary octahedral group $O^*(48)$. More precisely, $O(48; 3^{k-1}, l)$ has five generators X, P, Q, R, A and the following relations:

$$
\begin{gathered}
X^{3^k} = P^4 = A^l = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1} \\
XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP \\
RQR^{-1} = Q^{-1},\ AP = PA,\ AQ = QA,\ RAR^{-1} = A^{-1}.
\end{gathered}
$$

(1.8) The product of any of the above groups in (1.5)-(1.7) with a cyclic group of coprime order.

Thus to eliminate all the groups not on Hopf's list, it is enough to prove that groups in the above list (1.5)-(1.8) do not act freely on homotopy 3-spheres.

In the late sixties, C. T. C. Wall asked whether Milnor's result could be interpreted using the new theory of nonsimply connected surgery. Ronnie Lee [9] answered this question in 1973 by defining a "semicharacteristic" obstruction for the problem. As well as recovering the previous result of Milnor, the semicharacteristic rules out the family of groups $Q(8n, k, l)$, n even, in (1.6). Later in [17], C. B. Thomas observed that this also eliminates the family of groups $O(48, 3^{k-1}, l)$ in (1.7) because groups of this type always contain a subgroup isomorphic to $Q(16, 3^{k-1}, 1)$. These results leave undecided only the groups $Q(8n, k, l)$, n odd, in (1.5) and their products with cyclic groups of coprime order in (1.8) from Milnor's original list.

A positive answer to Hopf's question is now equivalent to settling:

Conjecture. *For any distinct odd primes p, q, the group $Q(8p, q)$ does not operate freely on any homotopy 3-sphere.*

Notice that a group $Q(8n, k, l)$ in the family (1.5) always contains a subgroup of the form $Q(8p, q)$. Hence ruling out the groups $Q(8p, q)$ also eliminates the family (1.7) in Milnor's list and the corresponding products in (1.8).

In this paper, we assume that an exotic 3-dimensional space form $\Sigma/\, Q(8p, q)$ exists, and show how to construct a smooth, negative definite 4-manifold with boundary components involving the exotic space form and associated 3-manifolds. A 4-manifold with boundary has a negative semi-definite intersection form if $b_2^+ = 0$. This is a key condition for the use of gauge theory. The main result is:

Theorem A. *Let $\Sigma/\, G$ be a nonlinear space form for $G = Q(8p, q)$. Then there exists a smooth, compact, connected, oriented 4-manifold $(Y, \partial Y)$ such that $\pi_1(Y) = G$ and the equivariant intersection form of Y is negative semi-definite. The boundary components of Y consist of two copies of $\Sigma/\, G$, together with at least two spherical space forms $S^3/\, Q(4pq)$ and some (almost) space forms associated to proper subgroups of G.*

By an *almost space form* $S'/\,\Gamma$ we mean the quotient of an integral homology sphere S' by a free action of a proper subgroup $\Gamma \subset G$ (if $\Gamma = 1$ we allow an even more general 3-manifold). For a more precise statement of the properties of $(Y, \partial Y)$, see Theorem 8.8.

The construction of the above cobordism Y starts with a framed cobordism $(U, \partial U) \to BG$ with boundary some appropriate collection of linear and nonlinear space forms $\pm\Sigma/G$, and $S^3/\,\Gamma$ for $\Gamma = Q(4pq)$, $Q(8p)$, $Q(8q)$, or $C(2pq)$. By re-attaching the top dimensional cell, we can modify U to a 4-dimensional Poincaré complex V with $\partial V = \partial U$ such that the cup product pairing on $H^2(V, \partial V; \mathbb{Z}G)$ is negative definite. In this step, we use the description of $\mathbb{Z}[Q(8p, q)]$-hermitian forms by means of the "arithmetic square" [19]. Associated to $(V, \partial V)$, there is a surgery problem whose surgery obstruction group $L_4(\mathbb{Z}G)$ has been computed by Madsen [10]. Using this result, we describe in §§7-8 how to eliminate the surgery obstruction. We modify V to construct a new Poincaré complex W, together with a new surgery problem $X \to W$ where some of the boundary components are changed to almost spherical space forms $S'/Q(8p)$, $S'/Q(8q)$, or $S'/C(2pq)$. The domain of the surgery problem is a compact, smooth, 4-manifold $(X, \partial X)$, such that $\partial X \to \partial W$ is an integral homology equivalence.

Since the surgery obstruction is zero, the intersection pairing on $H_2(X; \mathbb{Z}G)$ is the orthogonal direct sum of the pairing on W and some free hyperbolic summands. In dimension four we may not be able to complete the smooth surgeries suggested by this algebraic data. Instead, to get rid of the excess hyperbolic summands we use the techniques of Freedman [3], [4] to represent these hyperbolic summands by a suitable collection of smoothly immersed 2-spheres in the interior of X. Then we let Y' be a closed, smooth, regular neighbourhood of these immersed 2-spheres in X with $\partial Y' = N$, and define $Y = X \setminus int(Y')$ to be the complement. By construction, the manifold Y has $\partial Y = \partial_0 Y \cup N$ where $\partial_0 Y = \partial X$. In addition, the intersection pairing $H_2(Y; \mathbb{Z}G)$ (modulo its null space) is negative definite as required.

We conclude this introduction by mentioning some of the extensive work which has been done on the analogous spherical space form problem in higher dimensions: namely, the classification of finite group actions (Σ^{2n-1}, G) on homotopy spheres Σ^{2n-1} of dimension $2n - 1$, $n \geq 3$. This problem was both a motivation and an important test case for the techniques of algebraic and geometric topology developed in the period 1960–1985. P. A. Smith had already shown in 1944 that the p^2 conditions were necessary for a G-action on any homology sphere. Conversely, Swan [16] proved that every group with periodic cohomology acts freely and simplicially on a CW complex homotopy equivalent to a sphere, and asked whether there was always a *finite* simplicial action. Throughout the 1970's remarkable progress was made on the higher dimensional space form problem, culminating in the paper of Madsen, Thomas and Wall [11]. They used the surgery theory of Browder, Novikov, Sullivan and Wall to show that any finite group G satisfying the p^2 and $2p$ conditions (for all primes p) acts freely and smoothly on a homotopy sphere of *some* odd dimension $2n-1>3$. The precise dimensional bounds were not determined, although for G of period $2d$ they show that $n=2d$ is always realizable ($n=d$ is best possible).

The next big step forward was the explicit calculation by Milgram [12] in 1979 of the finiteness obstruction for some of the period 4 groups $G=Q(8p,q)$, following the method of [20]. In particular, Milgram showed that some of these groups are not fundamental groups of 3-manifolds. After this followed a sequence of papers by Milgram (see the survey in [2]), and independently by Madsen [10], aiming at the calculation of the relevant surgery obstruction. Here the problem is to determine which of the groups $Q(8p,q)$ act freely on Σ^{8k+3}, for $k>0$, since they act linearly on S^{8k+7} for all $k\geq 0$. It turned out that the answer is computable in principle, but depends sporadically on the number theory of the primes p, q. Note that the vanishing of the high-dimensional obstruction is equivalent to the existence of a free action of the corresponding group $Q(8p,q)$ on an integral homology 3-sphere.

Despite these spectacular breakthroughs in high dimensions, virtually no further progress was made using these methods on the space form problem in dimension 3. In a future paper, we hope to show how new 4-dimensional techniques from equivariant Yang-Mills gauge theory can be applied to eliminate all of the groups $Q(8p,q)$.

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2. A Framed Cobordism

We will now start to change the 3-dimensional spherical space form problem into a 4-dimensional problem. We begin by assuming the existence of a free $Q(8p,q)$-action $(\Sigma, Q(8p,q))$ on a homotopy 3-sphere Σ where p and q are two distinct odd primes.

The group $Q(8p,q)$ has the following presentation:

$$Q(8p,q)=\left\langle A,B,X,Y \;\middle|\; \begin{matrix} A^p=B^q=1, X^2=Y^2=(XY)^2, XAX^{-1}=A^{-1} \\ XBX^{-1}=B, YAY^{-1}=A, YBY^{-1}=B^{-1}, [A,B]=1 \end{matrix}\right\rangle. \tag{2.1}$$

In other words, $Q(8p,q)$ is a semidirect product $C(pq)\rtimes Q(8)$ of the cyclic group $C(pq)$ with the quaternion group $Q(8)$. Here the characteristic homomorphism

$\varphi: Q(8) \to Aut(C(pq)) = \mathbb{Z}/p-1 \times \mathbb{Z}/q-1$ is given in the following table.

(2.2)

φ	$\mathbb{Z}/p-1$	$\mathbb{Z}/q-1$
X	-1	1
Y	1	-1
XY	-1	-1

From this description, we see the following three maximal subgroups:

$$Q(8p) = \langle X, Y, A\rangle,\ Q(8q) = \langle X, Y, B\rangle,\ Q(4pq) = \langle XY, A, B\rangle.$$

Moreover, by sending the elements X, Y, XY to appropriate quaternions in $\{i, j, k\}$, we see that $Q(8p)$, $Q(8p)$, $Q(4pq)$ respectively are isomorphic to the following subgroups of the unit quaternions S^3:

$$\begin{aligned} Q(8p) &\cong \big\langle \pm 1, \pm i, \pm j, \pm k, e^{2\pi i/p}\big\rangle \\ Q(8q) &\cong \big\langle \pm 1, \pm i, \pm j, \pm k, e^{2\pi i/q}\big\rangle \\ Q(4pq) &\cong \big\langle \pm k, e^{2\pi i/pq}\big\rangle. \end{aligned}$$

In particular, there exist free linear actions $(S^3, Q(8p))$, $(S^3, Q(8q))$, $(S^3, Q(4pq))$ on the 3-sphere S^3 and hence spherical space forms $S^3/\,Q(8p)$, $S^3/\,Q(8q)$, $S^3/\,Q(4pq)$.

For our application, we also need the maximal cyclic subgroup $C(2pq)$ generated by the elements A,B, and $(XY)^2$. By identifying $C(2pq)$ with the cyclic subgroup $\langle \pm e^{2\pi i/pq}\rangle$ in $SU(2)$, we obtain the free linear action $(C(2pq), S^3)$ on S^3 which has the lens space $L(2pq, 1) = S^3/\,C(2pq)$ as quotient space.

Proposition 2.3. *Assume the existence of a nonlinear space form $\Sigma/\,Q(8p, q)$. Then there exists a framed, compact, 4-manifold U with the following properties:*

- (i) $\pi_1(U) = Q(8p, q)$.
- (ii) *The boundary ∂U of U consists of two copies of $\Sigma/\,Q(8p, q)$ with opposite orientation,* **a** *copies of $S^3/\,Q(4pq)$,* **b** *copies of $S^3/\,Q(8p)$,* **c** *copies of $S^3/\,Q(8p)$, and* **d** *copies of $S^3/\,C(2pq)$ where* **a**, **b**, **c**, **d** *are all non-zero and divisible by 48.*
- (iii) *The induced homomorphism $\pi_1(\partial U) \to \pi_1(U)$ on the fundamental groups sends $\pi_1(\Sigma/\,Q(8p, q))$ or $\pi_1(S^3/\,H)$ for $H = Q(4pq)$, $Q(8p)$, $Q(8q)$, $C(2pq)$ to the corresponding subgroups $Q(8p, q)$ or $H \subset Q(8p, q)$.*

Proof. As is well-known, the tangent bundle of an oriented 3-manifold is trivial and hence can be provided with a framing. In particular, we can choose a framed manifold structure for each of the linear and nonlinear space forms: $\Sigma/\,Q(8p, q)$, $S^3/\,Q(4pq)$, $S^3/\,Q(8p)$, $S^3/\,Q(8q)$, $S^3/\,C(2pq)$. As a result, we can view the expression for ∂U in terms of these space forms as the following relation in the framed bordism group $\Omega_3^{fr}(BQ(8p, q))$:

(2.4) $$a[S^3/\,Q(4pq)] + b[S^3/\,Q(8p)] + c[S^3/\,Q(8q)] + d[S^3/\,C(2pq)] = 0$$

since the terms $[\Sigma/\, Q(8p,q)] - [\Sigma/\, Q(8p,q)]$ cancel out. If we can find a solution of (2.4) by nonzero integers a, b, c, d with $a \equiv b \equiv c \equiv d \equiv 0 \pmod{48}$, then it follows that there exists a framed 4-manifold U' satisfying:

(iv) $\partial U' = \Sigma/\, Q(8p,q) \cup -\Sigma/Q(8p,q) \cup \mathbf{a}S^3/\, Q(4pq) \cup \mathbf{b}S^3/\, Q(8p) \cup \mathbf{c}S^3/\, Q(8q) \cup \mathbf{d}S^3/\, C(2pq)$

(v) the classifying map $c\colon U' \to BQ(8p,q)$ restricted to $\partial U'$ gives the corresponding classifying map on each of the boundary components.

Note that $c_\#\colon \pi_1 U' \to Q(8p,q)$ is a surjection. By framed surgery, we can kill the kernel of $c_\#$ and obtain a framed 4-manifold U satisfying (2.3) (i)-(iii).

To solve (2.4), we compute $\Omega_3^{fr}(BG)$ using the spectral sequence with E_2 term given by

$$E^2_{i,j} = H_i(G; \Omega_j^{fr}).$$

The coefficient groups are $\Omega_i^{fr} = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24$ for $i = 0, 1, 2, 3$ respectively.

We first study the image of our relation under the Hurewicz map

$$\Omega_3^{fr}(BQ(8p,q)) \to H_3(Q(8p,q);\mathbb{Z}). \tag{2.5}$$

Since $H_3(Q(8p,q);\mathbb{Z})$ equals $\mathbb{Z}/|Q(8p,q)| = \mathbb{Z}/8pq$, we have a congruence equation in $\mathbb{Z}/8pq$. In fact, by considering $\Sigma/\, Q(8p,q)$ as the 3-skeleton of the classifying space $BQ(8p,q)$, we can deform the classifying maps for $\Sigma/Q(8p,q)$,$S^3/\, Q(8p)$, $S^3/\, Q(8q)$, $S^3/\, Q(4pq)$, and $S^3/\, C(2pq)$ to factor through $\Sigma/\, Q(8p,q)$:

$$\begin{aligned}
f_a &\colon S^3/\, Q(4pq) \to \Sigma/Q(8p,q)\\
f_b &\colon S^3/\, Q(8p) \to \Sigma/Q(8p,q)\\
f_c &\colon S^3/\, Q(8q) \to \Sigma/Q(8p,q)\\
f_d &\colon S^3/\, C(2pq) \to \Sigma/Q(8p,q).
\end{aligned}$$

Then the contribution of $[S^3/\, Q(4pq)]$, $[S^3/\, Q(8p)]$, $[S^3/\, Q(8q)]$, $[S^3/\, C(2pq)]$ to the factor $H_3(Q(8p,q);\mathbb{Z})$ amounts to counting the degrees of the mappings $\deg f_a$, $\deg f_b$, $\deg f_c$, and $\deg f_d$ modulo $8pq$.

From the theory of covering spaces, the maps f_b and f_c factor through the coverings $\Sigma/\, Q(8p) \to \Sigma/\, Q(8p,q)$, $\Sigma/\, Q(8q) \to \Sigma/\, Q(8p,q)$.

$$\begin{aligned}
f_b &\colon S^3/\, Q(8p) \xrightarrow{f_b'} \Sigma/\, Q(8p) \xrightarrow{\pi_p} \Sigma/\, Q(8p,q)\\
f_c &\colon S^3/\, Q(8q) \xrightarrow{f_c'} \Sigma/\, Q(8q) \xrightarrow{\pi_q} \Sigma/\, Q(8p,q)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\deg f_b &= \deg f_b' \cdot \deg \pi_p = q \deg f_b'\\
\deg f_c &= \deg f_c' \cdot \deg \pi_q = p \deg f_c'.
\end{aligned}$$

On the other hand, $\deg f_b'$ and $\deg f_c'$ can be taken to be units $\pmod{8pq}$ [16]. Since $(p,q) = 1$, there exist integers r and s such that $1 = rq \deg f_b' + sp \deg f_c'$. From this last equation it follows that given nonzero numbers a', d' there exist non-zero integers b' and c' such that the expression

$$a'[S^3/\,Q(4pq)] + b'[S^3/\,Q(8p)] + c'[S^3/\,Q(8q)] + d'[S^3/\,C(2pq)] = 0 \tag{2.6}$$

and so gives no contribution in $H_3(Q(8p,q))$.

The $E^2_{i,3-i}$ terms of the spectral sequence for $i = 1, 2$ are given by:

$$H_2(Q(8p,q);\Omega_1^{fr}) = H_2(Q(8p,q);\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$
$$H_1(Q(8p,q);\Omega_2^{fr}) = H_1(Q(8p,q);\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and there is a splitting $\Omega_3^{fr}(BG) = \widetilde{\Omega}_3^{fr}(BG) \oplus \Omega_3^{fr}$. Since $\Omega_3^{fr} = \mathbb{Z}/24$ and the first summand is annihilated by 16, we obtain a solution of the bordism equation (2.4) from (2.6) after multiplying the coefficients by 48. This completes the proof. □

Later we will need some information about the multisignature of our framed bordism.

Corollary 2.7. *The $\mathbb{Z}[Q(8p,q)]$-hermitian intersection pairing*

$$h\colon H_2(U;\mathbb{Z}[Q(8p,q)]) \times H_2(U;\mathbb{Z}[Q(8p,q)]) \to \mathbb{Z}[Q(8p,q)]$$

has signature divisible by 16 at each simple factor of $\mathbb{Q}G$.

Proof. We form a closed, oriented 4-manifold M by (i) identifying the copies of $\Sigma/\,G$ with opposite orientation, and (ii) attaching copies of bounding manifolds for the other space form boundary components. Let v be a characteristic element for the intersection form b_M of M. Now the ordinary signature of this 4-manifold M is divisible by 16 from the Rochlin congruence

$$sign(M) \equiv b_M(v,v) \pmod{16}$$

since the non-spin part of M comes in multiples of 16. Furthermore, since M is closed, the multisignature of M is just a multiple of the regular representation, hence is divisible by 16 at each simple factor of $\mathbb{Q}G$. By additivity of signatures, we see that each component of the multisignature of U differs from that of M by a multiple of 16. □

3. A Poincaré complex

Let $(U, \partial U)$ be a 4-dimensional, framed, cobordism satisfying Proposition 2.3 (i)-(iii). Let $G = Q(8p,q)$ and let

$$b\colon H^2(U,\partial U;\mathbb{Z}G) \times H^2(U,\partial U;\mathbb{Z}G) \to \mathbb{Z}$$

denote the non-singular, symmetric, bilinear form induced by cup product and evaluation against the fundamental class. Notice that b is a G-invariant form: $b(gx,gy) = b(x,y)$ for all $g \in G$ and all $x, y \in H^2(U,\partial U;\mathbb{Z}G)$.

In this section we show how to modify U by removing a cell e^4 in the interior of U and then re-attaching this cell e^4 by a map $f\colon \partial e^4 \to U - e^4$. The result is a CW complex

$$V = (U - e^4) \cup_f e^4$$

which contains ∂U as a subcomplex, denoted by ∂V.

Variation of the attaching map of the top cell does not change the 3-skeleton, and hence has no effect on the fundamental group and homology in dimensions ≤ 2. By Poincaré duality,

$$H^2(U, \partial U; \mathbb{Z}G) \cong H^2(V, \partial V; \mathbb{Z}G)$$

so we can identify these two groups.

The main result of this section is:

Proposition 3.1. *Let $b'\colon H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$ be a non-singular, G-invariant, symmetric bilinear form, with $b' \equiv b \pmod{|G|}$. Then there exists an attaching map f such that the pair $(V, \partial V)$ is an oriented, finite, 4-dimensional Poincaré pair with $\pi_1(V) = G$ and cup product form b'.*

We will first give a description of $H_2(U; \mathbb{Z}G)$ as a $\mathbb{Z}G$-module. Note that the framed cobordism U is not uniquely determined by (2.4) (i)-(iii). We can, for example, alter the cobordism U by taking the connected sum with copies of $S^2 \times S^2$ away from ∂U. This has the effect of changing $H_2(U; \mathbb{Z}G)$ by taking a sum with a free $\mathbb{Z}G$-module of even rank, and we will refer to this as "stabilization" of the cobordism U.

Let $(\widetilde{U}, \partial\widetilde{U})$ be the universal covering space of $(U, \partial U)$. On $\widetilde{U}$, there is a free action of $Q(8p, q)$ and hence an induced action on its homology $H_2(\widetilde{U})$. By definition, the $\mathbb{Z}G$-module structure on $H_2(\widetilde{U})$ is the same as $H_2(U; \mathbb{Z}G)$.

Note that $\partial\widetilde{U}$ consists of a collection of homotopy 3-spheres. For each of these 3-spheres, we form a cone and extend the G-action to the cone in an obvious manner. In this way, we obtain a 4-dimensional Poincaré complex $\widetilde{U}'$,

$$\widetilde{U}' = \widetilde{U} \cup \text{ (cones over boundary spheres)}$$

where the action of G is no longer free. In fact, for each of the cone points a_λ, we have an isotropy subgroup $G_\lambda \subseteq G$. The cone points, denoted by a_0, a_1, over the components (Σ, G), $(\text{-}\Sigma, G)$ are somewhat special because they are G-fixed points.

The above construction of $\widetilde{U}'$ can be compared with the following. Let $\Sigma \times I$ denote the product of Σ with the interval $I = [0, 1]$. Then on the two boundary components $\Sigma \times 0$, $\Sigma \times 1$, we can attach two cones to get the suspension $S^1 \wedge \Sigma$ of Σ. The action of G on $\Sigma \times I$ can be extended naturally to $S^1 \wedge \Sigma$ with the upper and lower cone points as fixed points. From equivariant obstruction theory, there exists a degree 1, G-equivariant map

$$\varphi\colon \widetilde{U}' \to S^1 \wedge \Sigma$$

which sends the free orbits to free orbits, a_0 to the lower cone point and all other a_λ to the upper cone point.

Let $K_*(\varphi)$ denote the kernel of the natural homorphism

$$K_*(\varphi) = Ker\{\varphi_*: H_*(\widetilde{U}') \to H_*(S^1 \wedge \Sigma)\}.$$

Then from the degree 1 property of φ there is an exact sequence

$$0 \to K_*(\varphi) \to H_*(\widetilde{U}') \to H_*(S^1 \wedge \Sigma) \to 0$$

of $\mathbb{Z}G$-modules. From this sequence it is easy to see that $K_*(\varphi) = 0$ for all but the middle homology $K_2(\varphi)$. Since adding points or deleting points does not affect the seond homology, we have

$$K_2(\varphi) = H_2(\widetilde{U}') = H_2(\widetilde{U}).$$

Thus we can shift the calculation of the homology $H_2(U; \mathbb{Z}G)$ to $K_2(\varphi)$ which has the advantage of being the only nonzero homology group of the relative chain complex $C_*(\varphi)$.

The relative chain complex

$$C_*(\varphi) = Ker\{\varphi_*: C_*(\widetilde{U}') \to C_*(S^1 \wedge \Sigma)\}$$

can be calculated by taking equivariant triangulations on $\widetilde{U}'$ and $S^1 \wedge \Sigma$ and cellular maps between them. Since the cone points can be taken to be the vertices and the action are free away form these points, we see that $C_*(\varphi)$ consists of finitely generated free $\mathbb{Z}G$-modules for $* \neq 0$ and

$$C_0(\varphi) = F \oplus \bigoplus_{\lambda \neq 0,1} \mathrm{Ind}_{G_\lambda}^G(\mathbb{Z})$$

for some finitely generated free $\mathbb{Z}G$-module F. Here $\mathrm{Ind}_{G_\lambda}^G(\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}G_\lambda} \mathbb{Z}G = \mathbb{Z}[G/G_\lambda]$ stands for the induced representation from the trivial G_λ-representation $\mathbb{Z}$ to G, and the indices in the sum go through all the cone points a_λ except for the G-fixed points a_0, a_1.

Proposition 3.2. *After stabilization, there is an isomorphism:*

$$H_2(U; \mathbb{Z}G) \;\cong\; (\mathbb{Z}G)^r \oplus \bigoplus_{\lambda \neq 0,1} \Omega^2 \mathit{Ind}_{G_\lambda}^G(\mathbb{Z}).$$

Here we use notation $\Omega^2 L$ to denote the first term in an exact sequence:

$$0 \to \Omega^2 L \to F_2 \to F_1 \to L \to 0$$

of finitely generated $\mathbb{Z}G$-modules with F_1, F_2 free over $\mathbb{Z}G$. Since tensoring with $\mathbb{Z}G$ over $\mathbb{Z}G_\lambda$ preserves exactness, we have a stable isomorphism

$$\Omega^2 \mathrm{Ind}_{G_\lambda}^G(\mathbb{Z}) \cong \mathrm{Ind}_{G_\lambda}^G(\Omega^2 \mathbb{Z}).$$

A standard argument in homological algebra proves that the Ω-construction is well-defined up to stabilizing by free $\mathbb{Z}G$-modules.

Corollary 3.3. *After stabilization, the rank of $H_2(U;\mathbb{Q}G)$ is divisible by 16 at each simple factor of $\mathbb{Q}G$.*

Proof of (3.3). After replacing U by a connected sum with copies of $S^2 \times S^2$ if necessary, we may assume that the $r \equiv 0 \pmod{16}$ in the given expression for $H_2(U;\mathbb{Z}G)$. Since the number of boundary components is divisible by 16, the Ω^2-summands also have ranks $\equiv 0 \pmod{16}$. □

Proof of (3.2). We have an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathcal{Z}_2(\varphi) \to C_2(\varphi) \to C_1(\varphi) \to F \oplus \bigoplus_{\lambda\neq 0,1} \operatorname{Ind}_{G_\lambda}^G \mathbb{Z} \to 0 \tag{3.4}$$

so it follows that

$$\mathcal{Z}_2(\varphi) \oplus (\mathbb{Z}G)^{\ell'} \cong (\mathbb{Z}G)^{\ell} \oplus \bigoplus_{\lambda\neq 0,1} \Omega^2 \operatorname{Ind}_{G_\lambda}^G(\mathbb{Z}).$$

On the other hand, $C_*(\varphi)$ with fundamental class $[\widetilde{U}']$ can be viewed as a PD chain complex. Using the same argument as in [10, p. 199], since $K_i(\varphi) = H_i(C_*(\varphi)) = 0$ for $i \geq 3$ we can contract this complex down to a complex $C'_*(\varphi)$ concentrated in dimensions $* \leq 2$ without changing the homology. Then

$$K_2(\varphi) \oplus (\mathbb{Z}G)^{r'} \cong (\mathbb{Z}G)^{r} \oplus \bigoplus_{\lambda\neq 0,1} \Omega^2 \operatorname{Ind}_{G_\lambda}^G(\mathbb{Z}).$$

Since $K_2(\varphi) = H_2(U;\mathbb{Z}G)$, this proves (3.2). □

Proof of Proposition 3.1. We must see how the symmetric bilinear form b' leads to a suitable choice for the re-attaching map f. First we note that the conditions

$$H_3(V;\mathbb{Z}G) = H^3(V,\partial V;\mathbb{Z}G)$$
$$H_4(V;\partial V;\mathbb{Z}G) = \mathbb{Z}$$

and the non-singularity of the cup-product form are necessary for $(V,\partial V)$ to be a Poincaré complex.

Re-attaching maps may constructed as follows. First we map ∂e^4 to a wedge of two 3-spheres $\partial e^4 \to S^3 \vee S^3$ by collapsing the boundary of a 3-cell in $S^3 = \partial e^4$ to a point. Then we map $S^3 \vee S^3$ by sending the copy $S^3 \vee *$ by the inclusion map $\gamma\colon \partial e^4 \to U - e^4$ and sending the copy $* \vee S^3$ by a map $\delta\colon S^3 \to U^{(2)}$ from S^3 to the 2-skeleton $U^{(2)} = (U-e^4)^{(2)} \subseteq (U-e^4)$. In other words, f is the composite mapping

$$f\colon \partial e^4 \to S^3 \vee S^3 \xrightarrow{\gamma\vee\delta} U - e^4.$$

The choice $\delta = 0$ just gives the original complex $(U,\partial U)$.

Since $H_3(U^{(2)};\mathbb{Z}G) = 0$, it follows that δ has no effect on homology and, so as far as homology is concerned, f is the same as the original attaching map. As a result, for any such map f the complex $(V,\partial V)$ is a finite Poincaré pair provided that the cup-product form is non-singular.

Variation of the map δ has an effect on the cap product by the fundamental class $[V, \partial V]$ which in turn changes the cup product pairing $b\colon H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$. From the exact sequence in (3.4) we have

$$H_2(U^{(2)}; \mathbb{Z}G) = \mathcal{Z}_2(\varphi).$$

Comparing with the expression for $H_2(U; \mathbb{Z}G) \cong H^2(U, \partial U; \mathbb{Z}G)$ obtained in (3.2), we obtain

$$H_2(U^{(2)}; \mathbb{Z}G) = F \oplus H_2(U; \mathbb{Z}G),$$

where F is a free $\mathbb{Z}G$-module given by the image of the boundary operator from the complex $C_*(\varphi)$, $\partial\colon C_3(\varphi) \to C_2(\varphi)$. Note that

$$\pi_2(U^{(2)}) = H_2(U^{(2)}; \mathbb{Z}G) = F \oplus H_2(U; \mathbb{Z}G),$$

and by a theorem of Whitehead $\pi_3(U^{(2)})$ is just the space of symmetric pairings on $\mathrm{Hom}_{\mathbb{Z}}(\pi_2(U), \mathbb{Z})$. In particular, we can interpret δ as a symmetric pairing on $F \oplus H^2(U, \partial U; \mathbb{Z}G)$.

For any such pairing, the original cup product form

$$b\colon H^2(U, \partial U; \mathbb{Z}G) \times H^2(U, \partial U; \mathbb{Z}G) \to \mathbb{Z}$$

is changed by re-attaching the 4-cell to

$$(b + \sum g^*\delta)\colon H^2(V, \partial V; \mathbb{Z}G) \times H^2(V, \partial V; \mathbb{Z}G) \to \mathbb{Z}$$

(see [18, pp. 240-241], [5,§1]). Here $g^*\delta$ is the translate of the symmetric pairing δ by the action of the group element $g \in G$, $g^*\delta(x, y) = \delta(gx, gy)$, and $\sum g^*\delta$ is the sum of these translates as we go through all the group elements in G. Given b' in the statement of Proposition 3.1, we need to find δ so that $b' - b = \sum g^*\delta$.

Let H denote the $\mathbb{Z}G$-module $H^2(U, \partial U; \mathbb{Z}G)$, and $\mathrm{Sym}(H)$ the space of symmetric pairings on H. Then $b' - b$ is an element in $\mathrm{Sym}(H)$ which is invariant under the induced group action. However, the quotient of the group of G-invariant pairings, by those of the form $\sum g^*\delta$ is just the Tate cohomology $\widehat{H}^0(G; \mathrm{Sym}(H))$, which is a torsion group of exponent $8pq = |G|$. But $b'\colon H \times H \to \mathbb{Z}$ on H has the additional property that $b' \equiv b \pmod{|G|}$. Therefore we can write $b' = b + \sum g^*\delta$ for some symmetric pairing δ. We then use the associated map $f = \gamma \vee \delta$, to construct a Poincaré complex $(V, \partial V)$ with b' as its cup product pairing. $\square$

4. Hermitian modules

In this section we will consider the patching construction for $\mathbb{Z}[Q(8p, q)]$-hermitian modules by means of the arithmetic square:

(4.1)
$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{Q}G \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}G & \longrightarrow & \hat{\mathbb{Q}}G \end{array}$$

Here $\hat{\mathbb{Z}}G$ is the product $\prod_\ell \hat{\mathbb{Z}}_\ell G$ of the ℓ-adic group rings and $\hat{\mathbb{Q}}G$ the corresponding weak product of group algebras. Applying the homology functor $H_*(U; -)$ to the above diagram, we have

$$\dots \to H_*(U;\mathbb{Z}G) \to H_*(U;\hat{\mathbb{Z}}G) \oplus H_*(U;\mathbb{Q}G) \to H_*(U;\hat{\mathbb{Q}}G) \to \dots \tag{4.2}$$

To simplify our notation, we denote by $H(\mathbb{Z}G), H(\mathbb{Q}G), H(\hat{\mathbb{Z}}G), H(\hat{\mathbb{Q}}G)$ the degree 2 homology of U with the corresponding coefficients in $\mathbb{Z}G$, $\mathbb{Q}G$, $\hat{\mathbb{Z}}G$, or $\hat{\mathbb{Q}}G$. In particular, we can view the module $H(\mathbb{Z}G)$ as patching $H(\hat{\mathbb{Z}}G) = H(\mathbb{Z}G) \otimes \hat{\mathbb{Z}}$ and $H(\mathbb{Q}G) = H(\mathbb{Z}G) \otimes \mathbb{Q}$ together over $H(\hat{\mathbb{Q}}G) = H(\mathbb{Z}G) \otimes \hat{\mathbb{Q}}$, with some isomorphisms

$$H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \to H(\hat{\mathbb{Q}}G) \leftarrow H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}. \tag{4.3}$$

In the same manner, we can describe the $\mathbb{Z}G$-hermitian intersection pairing

$$h \colon H(U;\mathbb{Z}G) \times H(U;\mathbb{Z}G) \to \mathbb{Z}G$$

as a pull-back. There are intersection pairings over $H(\hat{\mathbb{Z}}G), H(\hat{\mathbb{Q}}G), H(\mathbb{Q}G)$ by the pull-back

$$\begin{aligned} h_{\hat{\mathbb{Z}}}&: H(\hat{\mathbb{Z}}G) \times H(\hat{\mathbb{Z}}G) \to \hat{\mathbb{Z}}G \\ h_{\hat{\mathbb{Q}}}&: H(\hat{\mathbb{Q}}G) \times H(\hat{\mathbb{Q}}G) \to \hat{\mathbb{Q}}G \\ h_{\mathbb{Q}}&: H(\mathbb{Q}G) \times H(\mathbb{Q}G) \to \mathbb{Q}G \end{aligned}$$

and they are patched together by isometries

$$(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}}) \xleftarrow{\phi} (H(\mathbb{Q}G), h_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}}. \tag{4.4}$$

We want to use this description later in Proposition 5.1 to construct a new interection pairing on the same module $H(\mathbb{Z}G)$. Our strategy is to keep the pairing and isometry

$$(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$$

on the left of (4.4) unchanged, vary the pairing $(H(\mathbb{Q}G), h_{\mathbb{Q}})$ to a negative definite one $(H(\mathbb{Q}G), h'_{\mathbb{Q}})$, and then use local classification theory to patch everything together by a new isometry ϕ'

$$(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Q} \xrightarrow{\psi} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}}) \xleftarrow{\phi'} (H(\mathbb{Q}G, h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}}. \tag{4.5}$$

The new pairing $(H(\mathbb{Z}G), h')$ on $H(\mathbb{Z}G)$ is obtained by means of the pull back diagram as in [19] or [10].

The first step involves only the rational intersection form.

Proposition 4.6. *Let $(H(\mathbb{Q}G), h_{\mathbb{Q}})$ be a non-singular form with hyperbolic rank ≥ 8, rank $H(\mathbb{Q}G) \equiv 0 \pmod{16}$, and sign $h_{\mathbb{Q}} \equiv 0 \pmod{16}$ at every simple factor of $\mathbb{Q}G$. Then there exists a hermitian pairing $(H(\mathbb{Q}G), h'_{\mathbb{Q}})$ such that*

(i) *$h'_{\mathbb{Q}}$ is negative definite at all of the real representations of $\mathbb{Q}G$,*
(ii) *$(H(\mathbb{Q}G), h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \cong (H(\mathbb{Q}G), h_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}}$ over $\hat{\mathbb{Q}}G$,*
(iii) *$\det h'_{\mathbb{Q}} = \det h_{\mathbb{Q}}$ at each simple factor of $\mathbb{Q}G$, and*
(iv) *$(H(\mathbb{Q}G), h'_{\mathbb{Q}})$ contains $\langle -1 \rangle$ as an orthogonal summand.*

The assumptions of Proposition 4.6 are satisfied for the intersection form of U by Corollary 2.7 and Corollary 3.3, after stabilization again with $S^2 \times S^2$'s if necessary to increase the hyperbolic rank. The proof follows from well-known techniques in quadratic forms (see [15, Ch. 10] for the existence of global forms with prescribed local invariants). First, we recall that $S = \mathbb{Q}[Q(8p, q)]$ is a semi-simple algebra and hence can be decomposed into a product $\prod_\chi (\mathbb{Q}G)_\chi$ of simple algebras $(\mathbb{Q}G)_\chi$ where χ goes through all the irreducibles of G. Since

$$\mathbb{Q}[C(pq)] = \mathbb{Q} \times \mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_q) \times \mathbb{Q}(\zeta_{pq})$$

it follows that $S = \prod_{d|pq} S(d)$ where

$$S(d) = \mathbb{Q}(\zeta_d)^t[X, Y \mid X^2 = Y^2 = (XY)^2]$$

is a twisted group algebra. From the presentation of $G = Q(8p, q)$ given in (2.1) we see that the element $X^2 = (XY)^2 = Y^2$ is central of order two, so the group algebra $S = \mathbb{Q}[Q(8p, q)]$ contains the central idempotent $\frac{1}{2}(1+X^2)$ and splits into a product of two simple algebras $S = S_+ \times S_-$. The first factor $S_+ = \mathbb{Q}[D(2p) \times D(2q)]$ is the group algebra of the product of the two dihedral groups subgroups $D(2p) = \langle A, X\rangle$ and $D(2q) = \langle B, Y\rangle$. From the representation theory of these groups, it follows that

$$\begin{array}{c}\mathbb{Q}[D(2p)] = \mathbb{Q}_+ \times \mathbb{Q}_- \times M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \\ \mathbb{Q}[D(2q)] = \mathbb{Q}_+ \times \mathbb{Q}_- \times M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})]\end{array}. \tag{4.7}$$

Therefore

$$S(1)_+ = \mathbb{Q}_{++} \times \mathbb{Q}_{+-} \times \mathbb{Q}_{-+} \times \mathbb{Q}_{--}$$

while

$$\begin{aligned} S(p)_+ &= M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \otimes \mathbb{Q}_{++} \times M_2[\mathbb{Q}(\zeta_p + \zeta_p^{-1})] \otimes \mathbb{Q}_{+-} \\ S(q)_+ &= M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})] \otimes \mathbb{Q}_{++} \times M_2[\mathbb{Q}(\zeta_q + \zeta_q^{-1})] \otimes \mathbb{Q}_{-+} \end{aligned}$$

and

$$S(pq)_+ = M_4[\mathbb{Q}(\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1})]. \tag{4.8}$$

The subscripts $+, -$, indicate the appropriate sign representations of $Q(8p, q)$ and $(\zeta_p, \zeta_q$ are respectively primitive p^{th} roots and q^{th} roots of unity (see [10, p. 211]). There is a similar decomposition for the second factor S_- into simple algebras which are *non-split* at all the real places:

$$\begin{array}{ll} S(1)_- = \mathbb{Q}[i, j, k], & S(p)_- \ = \mathbb{Q}(\zeta_p)^t[i, j, k] \\ S(q)_- = \mathbb{Q}(\zeta_q)^t[i, j, k] & S(pq)_- = M_2(\mathbb{Q}(\zeta_{pq} + \zeta_{pq}^{-1})^t[i, j, k]) \end{array}. \tag{4.9}$$

It is easy to see that all the factors in the above decomposition are preserved under the canonical involution $\alpha\colon \overline{\sum a_g g} \mapsto \sum a_g g^{-1}$ of the group algebra $\mathbb{Q}G$. As an algebra with involution, all the factors in S_+ belong to the type $OK(\mathbb{R})$ while the factors in S_- belong to the type $SpD(\mathbb{H})$. Here we use the classification of [6, p. 549]. A simple algebra (D, α) of dimension n^2 over its centre E has type O (resp. Sp) if E is fixed by α and the fixed set of α on D has dimension $\frac{1}{2}(n^2 + n)$ (resp. $\frac{1}{2}(n^2 - n)$) over E. We further divide into

(i) type $OK(\mathbb{R})$ if (D, α) has type O, $D = E$ and E has a real imbedding, or
(ii) type $SpD(\mathbb{H})$ if (D, α) has type Sp, $D \neq E$, and D is nonsplit at infinite primes.

We wish to reconstruct the pairing on $(H(\mathbb{Q}G), h_{\mathbb{Q}})$ so that it becomes negative definite. In view of the decomposition above, it is enough to construct a negative definite pairing over each of the simple factors $(H(\mathbb{Q}G)_\chi, h'_\chi)$ with the prescribed local data $(H(\hat{\mathbb{Q}}G)_\chi, h_\chi)$.

For simple factors of type OK we will use the Hasse-Minkowski Theorem. Its proof can be found in many textbooks on quadratic forms (e.g. [15, p. 225]).

Theorem 4.10. *Let E be a global field. For each prime spot ℓ of E let an n-dimensional form ψ_ℓ over E_ℓ be given. Then there exists a form ϕ over E with $\phi_\ell \cong \psi_\ell$ for all ℓ if and only if the following conditions are satisfied:*

(i) *There exists $d \in E^\times$ with $d = \det(\psi_\ell)$ in $E_\ell^\times / E_\ell^{\times 2}$ for all ℓ.*
(ii) *The number of ℓ for which $s(\psi_\ell) = -1$ is finite and even.*

For the remaining simple factors, of type $SpD(\mathbb{H})$, we have the following version of the local to global correspondence:

Theorem 4.11. *Let D be an quaternion skew field with centre E, and let $(D, *)$ be the canonical involution which fixes exactly the elements of E. Given a $(D, *)$-hermitian form $h\colon V \times V \to D$ over the vector space V, the formula $x \longmapsto h(x, x)$ defines a quadratic form known as the trace form $q_h\colon V \to K$ of h*

(i) *Two hermitian forms over $(D, *)$ are isometric if and only if their trace forms are isometric.*
(ii) *If E is a ℓ-adic field, then non-degenerate hermitian forms over D are classified by their dimension.*
(iii) *If E is an algebraic number field then non-degenerate hermitian forms over D are classified by their dimension and their signatures at the real places where D is definite.*

Proof. The proof of (i) is in [15, Thm 10.1.7] and [15, 10.1.8(iii)]. Recall that the canonical involution on $\mathbb{Q}[i, j, k]$ is the one which is type Sp (see [15, p.75]). As is well known, a nondegenerate quadratic form q over an algebraic number field E is completely determined by its rank, $\dim(q)$, determinant $\det(q)$, Hasse symbols $s(q)$, and signatures $\operatorname{sign}(q_\ell)$ at all real places. For $h = \langle \alpha_1, \dots, \alpha_n\rangle$, its trace form q_h is of the form

$$q_h = \oplus\langle \alpha_i, -\alpha_i a, -\alpha_i b, \alpha_i ab\rangle$$

where a, b are elements in E with $D = (a, b)$. From this it is easy to see that

$$\dim(q_h) = 4\dim h, \quad \det(q_h) = 1, \quad \operatorname{sign}(q_h) = 4\operatorname{sign}(h).$$

These invariants are determined by the dimension and signature, and a short computation shows that

$$s_\ell(q_h) = \left(\frac{a, b}{\ell}\right)^n \left(\frac{-1, (-1)^n}{\ell}\right)$$

so the Hasse invariants are also determined . □

Proof of Proposition 4.6. We will begin with the type OK factors $(\mathbb{Q}G)_\chi$ and explain the method by working out the simplest case. Let χ be the trivial representation and $(\mathbb{Q}G)_\chi = \mathbb{Q}_{++} = \mathbb{Q}$. Since the involution is trivial, the hermitian pairing

$(H(\mathbb{Q}_{++}), h_{\mathbb{Q}_{++}}) = (H(\mathbb{Q}), b)$ is nothing but a non-singular symmetric bilinear form over the rational vector space $H(\mathbb{Q})$.

We will construct a new bilinear form $(H(\mathbb{Q}), b')$ with the same localizations $(H(\mathbb{Q}_\ell), b_\ell)$, $\ell = 2, 3, \dots, \infty$ as the given form $(H(\mathbb{Q}), b)$. Over the real place, the form $(H(\mathbb{Q}_\infty), b_\infty) = (H(\mathbb{R}), b_\mathbb{R})$ is not necessarily negative definite but its rank and signature are multiples of 16. As a result, we see that

$$s(b_\mathbb{R}) = (-1)^{s(s-1)/2} = 1$$

since $s \equiv 0 \pmod 8$ is the number of negatives in a diagonal form equivalent to $b_\mathbb{R}$. It follows that

$$\det(b_\mathbb{R}) = 1 \quad \text{in} \quad \mathbb{R}^\times/\mathbb{R}^{\times 2}.$$

If we replace $b_\mathbb{R}$ by a negative definite form $b'_\mathbb{R}$, then the same equations are satisfied:

$$\det(b'_\mathbb{R}) = \det(b_\mathbb{R}), \quad s(b'_\mathbb{R}) = s(b_\mathbb{R}).$$

For the rest of the primes, we let b'_ℓ equal b_ℓ. Then the collection $\{b'_2, \dots, b'_\infty\}$ with $d = \det b$ satisfies the conditions of Theorem 4.10 to be the local data for a global form. It follows that we have a bilinear form $(H(\mathbb{Q}), b')$ which is negative definite at the real place and is the same as $(H(\mathbb{Q}_\ell), b_\ell)$ for all other primes.

For the other simple factors $(\mathbb{Q}G)_\chi$ of type OK , the modification of the hermitian pairing $(H(\mathbb{Q}G)_\chi, h'_\chi)$ to a negative definite one can be achieved in the same manner, after applying Morita equivalence to translate from forms over $M_2(E)$ to forms over E.

Next we consider the case of simple factor of type $SpD(\mathbb{H})$, and we begin again with the simplest case when $(\mathbb{Q}G)_\chi$ is a division ring. To reconstruct $(H(\mathbb{Q}G_\chi), h)$ we first express h as a diagonal form $\langle a_1, \dots, a_n\rangle$ over the division ring $D = \mathbb{Q}G_\chi$ and define $h' = \langle -1, \dots, -1\rangle$ where h' has the same rank as h. By Theorem 4.11(ii), the forms $h_\ell \cong h'_\ell$ at all finite primes ℓ. On the other hand, h' is negative definite at the real places.

For a general type $SpD(\mathbb{H})$-factors, we have a matrix ring $M_2(D_\chi)$ over a division algebra $(D_\chi, *)$ with an involution defined by the transpose-conjugation operation:

$$(a_{ij}) \longmapsto (a^*_{ji}).$$

By Morita equivalence, the classification of hermitian forms over such simple factors can be reduced to the classification over D_χ. As a result the reconstruction problem of $(H(\mathbb{Q}G_\chi), h_\chi)$ can be treated as the corresponding problem over D_χ, which we have just considered. We complete the proof of parts (i)-(iii) by putting all the modified hermitian forms $(H(\mathbb{Q}G_\chi), h'_\chi)$ together. For part (iv), we use the assumption that form $(H(\mathbb{Q}G), h)$ contains a hyperbolic form of rank ≥ 8, and a special case of the above construction: let $L = (\mathbb{Q}G)^{16}$, take b the hyperbolic form, and b' the diagonal $\langle -1\rangle$ form of rank 16. Then $(L, b) \otimes \hat{\mathbb{Z}} \cong (L, b') \otimes \hat{\mathbb{Z}}$. $\square$

5. Strong Approximation

In Proposition 4.6, we constructed a negative definite hermitian form $(H(\mathbb{Q}G), h'_\mathbb{Q})$ such that its completion $(H(\mathbb{Q}G), h'_\mathbb{Q}) \otimes \hat{\mathbb{Z}}$ is isometric to the adelic completion

$(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$ of the original hermitian form. In particular, this implies $\det h'_{\mathbb{Q}} = \det h_{\mathbb{Q}} \in K_1(\mathbb{Q}G)$. Each choice of isometry

$$(H(\mathbb{Q}G, h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \xrightarrow{\phi'} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$$

gives rise to a form (H', h') on some module over $\mathbb{Z}G$ by pull-back, but there are many possible choices.

Proposition 5.1. *Let $H = H^2(U; \mathbb{Z}G)$ and h denote the $\mathbb{Z}G$-hermitian cup product pairing $h\colon H \times H \to \mathbb{Z}G$. Then there exists an isometry $\phi'\colon (H(\mathbb{Q}G, h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \to (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$ and a hermitian pairing $h'\colon H \times H \to \mathbb{Z}G$ such that*

(i) *h' is the pull-back $(h_{\hat{\mathbb{Z}}}, \phi', h'_{\mathbb{Q}})$*
(ii) *$h' \equiv h \pmod{|G|}$, and*
(iii) *h' is negative definite at all of the real representations of $\mathbb{Q}G$.*

When h' has the properties listed in Proposition 5.1, we can use the method explained in Section 3 to construct a finite Poincaré pair $(V, \partial V)$ with negative definite intersection form.

Proposition 5.2. *There exists an attaching map f for $V = (U - e^4) \cup_f e^4$ such that the pair $(V, \partial V)$ is an oriented, finite, weakly simple, 4-dimensional Poincaré pair with $\pi_1(V) = G$ and orientation class $[V] \in H_4(V, \partial V; \mathbb{Z}G)$. Moreover the non-singular $\mathbb{Z}G$-hermitian pairing*

$$H^2(V, \partial V; \mathbb{Z}G) \times H^2(V, \partial V; \mathbb{Z}G) \to \mathbb{Z}G$$

induced by cup product and the evaluation against the fundamental cycle $[V]$ is negative definite.

The condition "weakly simple" means that the Whitehead torsion of the Poincaré duality map is zero measured in $Wh'(\mathbb{Z}G) \cong \operatorname{Im}(Wh(\mathbb{Z}G) \to Wh(\hat{\mathbb{Q}}G))$. This is automatically true for manifolds and we will preserve this property in our construction of V from U using (4.6)(iv).

Over each simple factor of $\mathbb{Q}G$ or $\hat{\mathbb{Q}}G$, every module is a direct sum of copies of an irreducible simple module, so we can choose a basis (see [10, §2]), and then compute the determinant of an isometry. Over non-commutative factors, the determinant must be interpreted as the reduced norm. An isometry of based forms with determinant 1 is called a *simple* isometry, and such forms are then called SU-equivalent.

The manifold $(U, \partial U)$ has a basis for its chain complex given by its associated piecewise smooth triangulation. To express the Whitehead torsion of its simple Poincaré duality map in terms of Reidemeister torsions, it is necessary to base the homology groups. Let $\underline{b} = \{e_i\}$ denote a basis of $H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q}$. Using the given isomorphism

$$\Phi\colon H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\psi} H(\hat{\mathbb{Q}}G) \xrightarrow{\phi^{-1}} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$$

we have a corresponding basis $\Phi(\underline{b}) = \{\Phi(e_i)\}$ on $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}})$ under Φ. In particular, Φ is a simple isometry of the given hermitian forms with respect to these bases.

Lemma 5.3. *There exists an isometry*

$$\phi' \colon (H(\mathbb{Q}G), h'_{\mathbb{Q}}) \otimes \hat{\mathbb{Z}} \xrightarrow{\approx} (H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$$

such that the composite

$$\Phi' \colon H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\psi} H(\hat{\mathbb{Q}}G) \xrightarrow{(\phi')^{-1}} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$$

is a simple isometry with respect to the bases $\underline{b}$ *and* $\Phi(\underline{b})$.

Proof. It follows from Proposition 4.6 (iv) that $(H(\mathbb{Q}G), h'_{\mathbb{Q}})$ contains the form $\langle -1 \rangle$ on some basis element $e \in H(\mathbb{Q}G)$, in the given basis. This allows us to pre-compose any ϕ' with an isometry of the form $e \mapsto ue$, where $u \in \hat{\mathbb{Q}}G$ and $u\bar{u} = 1$. This realizes all possible values of the reduced norm for an isometry since $\det h_{\mathbb{Q}} = \det h'_{\mathbb{Q}}$. □

Proof of Proposition 5.2. Our new form $(H(\mathbb{Z}G), h')$ is constructed in Proposition 5.1 by pull-back using the simple isometry ϕ' in Lemma 5.3. We then apply Proposition 3.1 to construct V from U. It follows that the based chain complex used to compute the adelic Reidemeister torsion of $(V, \partial V)$ is simple chain homotopy equivalent to the one for $(U, \partial U)$. Therefore the image of the Whitehead torsion $\tau(V, \partial V)$ is zero in $Wh(\hat{\mathbb{Q}}G)$ and the Poincaré complex $(V, \partial V)$ is weakly simple. □

To prove Proposition 5.1 we will need the following:

Lemma 5.4. *There exist isomorphisms*

$$\psi_1 \colon H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G)$$
$$\psi_2 \colon H(\mathbb{Q}G) \to H(\mathbb{Q}G)$$

such that $\Phi = (\psi_2 \otimes id)^{-1} \circ \Phi' \circ (\psi_1 \otimes id)$.

Lemma 5.5. *For every divisor* ℓ *of* $|G|$*, the reduction of* ψ_1 *modulo* ℓ

$$\bar{\psi}_1 \colon H(\hat{\mathbb{Z}}G) \otimes \mathbb{Z}/\ell \to H(\hat{\mathbb{Z}}G) \otimes \mathbb{Z}/\ell$$

is an isometry of the hermitian module $(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}}) \otimes \mathbb{Z}/\ell$.

Proof of Proposition 5.1. Assuming these two assertions (5.4) and (5.5), we can complete the proof of Proposition 5.1. Let (H', h') be the pull-back of our original ℓ-adic form $(H(\mathbb{Z}G), h) \otimes \hat{\mathbb{Z}} = (H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}})$ and the new rational form $(H(\mathbb{Q}G), h'_{\mathbb{Q}})$ given by Proposition 4.6, pulled back using the isometry ϕ' of Lemma 5.3. This form will satisfy (5.1)(i) and (5.1)(iii) once we prove that $H' \cong H(\mathbb{Z}G)$ as a $\mathbb{Z}G$-module. The remaining property (5.1)(ii) will follow from Lemma 5.5.

Recall from (4.3) that the module $H(\mathbb{Z}G)$ is obtained by forming the pull-back of the diagram:

$$H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q} \xrightarrow{\Phi} H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}} \longleftarrow H(\mathbb{Q}G).$$

Lemma 5.4 gives us a commutative diagram:

$$(5.6)\qquad \begin{tikzcd} H(\hat{\mathbb{Z}}G) \arrow[r] \arrow[d, "\psi_1"] & H(\hat{\mathbb{Z}}G)\otimes\mathbb{Q} \arrow[r, "\Phi"] \arrow[d, "\psi_1\otimes id"] & H(\mathbb{Q}G)\otimes\hat{\mathbb{Z}} \arrow[d, "\psi_2\otimes id"] & H(\mathbb{Q}G) \arrow[l] \arrow[d, "\psi_2"] \\ H(\hat{\mathbb{Z}}G) \arrow[r] & H(\hat{\mathbb{Z}}G)\otimes\mathbb{Q} \arrow[r, "\Phi'"] & H(\mathbb{Q}G)\otimes\hat{\mathbb{Z}} & H(\mathbb{Q}G) \arrow[l] \end{tikzcd}$$

From this, it follows that there exists an isomorphism

$$\Psi\colon H(\mathbb{Z}G)\to H(\mathbb{Z}G)'$$

between the pullback $H(\mathbb{Z}G)$ of the top row in (5.6) and the corresponding pullback $H(\mathbb{Z}G)'$ of the bottom row. Furthermore, this isomorphism Ψ is compatible with ψ_1 after taking the completion

$$(5.7)\qquad \begin{tikzcd} H(\mathbb{Z}G) \arrow[r] \arrow[d, "\Psi"] & H(\hat{\mathbb{Z}}G) \arrow[d, "\psi_1"] \\ H(\mathbb{Z}G)' \arrow[r] & H(\hat{\mathbb{Z}}G) \end{tikzcd}$$

Now the pullback diagram

$$(H(\hat{\mathbb{Z}}G),h_{\hat{\mathbb{Z}}})\to(H(\hat{\mathbb{Z}}G),h_{\hat{\mathbb{Z}}})\otimes\mathbb{Q}\xrightarrow{\Phi'}(H(\mathbb{Q}G),h'_{\mathbb{Q}})\otimes\hat{\mathbb{Z}}\leftarrow(H(\mathbb{Q}G),h'_{Q})$$

gives rise to the desired hermitian pairing $(H(\mathbb{Z}G)',h')$ over $H(\mathbb{Z}G)'$. In addition, we have a hermitian pairing $(H(\mathbb{Z}G)',h')\otimes\mathbb{Z}/|G|$ after taking the tensor product with $\mathbb{Z}/|G|$.

From (5.7), we have a commutative diagram:

$$\begin{tikzcd} (H(\mathbb{Z}G),h)\otimes\mathbb{Z}/|G| \arrow[r, "\approx"] \arrow[d, "\Psi\ (\mathrm{mod}\ |G|)"] & (H(\hat{\mathbb{Z}}G),h_{\hat{\mathbb{Z}}})\otimes\mathbb{Z}/|G| \arrow[d, "\psi_1\ (\mathrm{mod}\ |G|)"] \\ (H(\mathbb{Z}G)',h')\otimes\mathbb{Z}/|G| \arrow[r, "\approx"] & (H(\hat{\mathbb{Z}}G),h_{\hat{\mathbb{Z}}})\otimes\mathbb{Z}/|G| \end{tikzcd}$$

where the two horizontal arrows are isometries. Since $\psi_1 \pmod{|G|}$ is an isometry by Lemma 5.5, it follows that the isomorphism Ψ is an isometry after reduction modulo $|G|$. Or in other words, the pullback hermitian pairing $\Psi^*(h')\equiv h \pmod{|G|}$. This proves (5.1) (ii) and the proof of (5.1) is complete. □

To prove (5.4) and (5.5), we need the following version of the Strong Approximation Theorem for special linear groups due to Eichler and Kneser (see [15, 10.5.1]). Let R be a Dedekind domain with the global field K as quotient field. Let D be a finite-dimensional skew field with centre K and $A=M_n(D)$ and let Γ be an R-order on A. The special linear group $SL(\Gamma)$ is the subgroup of $SL(n,D)$ preserving Γ.

Theorem 5.8. *Let $\mathfrak{P}$ be a finite set of non-archimedean primes , $\mathcal{T} \in SL(n, \hat{D})$ and $\epsilon > 0$. Then there exists $T \in SL(n, D)$ and $\mathcal{S} \in SL(\hat{\Gamma})$ such that $\mathcal{T} = T \circ \mathcal{S}^{-1}$, and $\|S_{\mathfrak{p}} - Id\|_{\mathfrak{p}} < \epsilon$ for all $\mathfrak{p} \in \mathfrak{P}$.*

Proof. Consider the element $\mathcal{T} = \{T_{\mathfrak{p}} : S_{\mathfrak{p}} \in SL(n, \hat{D}_{\mathfrak{p}})\}$ in the adelic special linear group $SL(n, \hat{D})$. Then, by definition, for all but finitely many primes $\mathfrak{P}_0 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ the component $T_{\mathfrak{p}} \in SL(\hat{\Gamma}_{\mathfrak{p}})$ for $\mathfrak{p} \neq \mathfrak{p}_1, \dots, \mathfrak{p}_k$. We enlarge $\mathfrak{P}$ if necessary to assume that it contains all primes $\mathfrak{p} \in \mathfrak{P}_0$.

Using [15, 10.5.1] (with $\mathfrak{q}$ one of the infinite primes), and any given $\delta > 0$ we have $T \in SL(n, D)$ such that

$$\|T - T_{\mathfrak{p}_i}\| < \delta$$

for $\mathfrak{p}_i \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ and $T \in SL(\hat{\Gamma}_{\mathfrak{p}})$ elsewhere. In particular, by choosing δ small enough we can ensure that $T_{\mathfrak{p}_i}^{-1} \circ T$ is in any given ϵ-neighborhood of the identity. Since $SL(\hat{\Gamma}_{\mathfrak{p}_i})$ is open in $SL(n, \hat{D}_{\mathfrak{p}}))$, it follows that for $\delta > 0$ sufficiently small $T_{\mathfrak{p}_i}^{-1} \circ T = S_{\mathfrak{p}_i}$ is in $SL(\hat{\Gamma}_{\mathfrak{p}_i})$. Define $S_{\mathfrak{p}} = T_{\mathfrak{p}}^{-1} \cdot T$ for the other primes as well. Then $\mathcal{S} = \{S_p\}$ is an integral adele, $\mathcal{S} \in SL(\hat{\Gamma})$, and $\mathcal{T} = T \circ \mathcal{S}^{-1}$. □

Proof of Lemma 5.4. To apply the above, we recall that $\Phi' \circ \Phi^{-1}$ is a simple automorphism of the vector space $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$. Note that $H(\hat{\mathbb{Q}}G) \otimes \hat{\mathbb{Z}}$ is decomposed as a product $\prod H(\mathbb{Q}G_{\chi}) \otimes \hat{\mathbb{Z}}$ of simple modules over each of the simple factors $(\mathbb{Q}G)_{\chi}$ of $\mathbb{Q}G$. In each factor we will take Γ_{χ} to be the image of $\mathbb{Z}G$ in $(\mathbb{Q}G)_{\chi}$. Since we can apply the above theorem to each of these factors and multiply them together, we will not distinguish between $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$ and its factors.

For all but a finite set of primes $\mathfrak{P} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, the automorphism $\Phi' \circ \Phi^{-1}$ preserves the lattice $H(\hat{\mathbb{Z}}G)$. By enlarging this set $\mathfrak{P}$ if necessary, we can assume that it contains all the prime divisors of $|G|$. By (5.8), there exist simple automorphisms ψ_1 of $H(\hat{\mathbb{Z}}G)$ and ψ_2 of $H(\mathbb{Q}G)$ such that

$$\Phi' \circ \Phi^{-1} = (\psi_2 \otimes id) \circ \Phi \circ (\psi_1 \otimes id)^{-1} \circ \Phi^{-1}.$$

As usual "simplicity" of ψ_1, ψ_2 is measured by reduced norms with respect to our fixed bases, after tensoring to $H(\hat{\mathbb{Z}}G) \otimes \mathbb{Q}$ or $H(\mathbb{Q}G) \otimes \hat{\mathbb{Z}}$. This establishes (5.4). □

Proof of Lemma 5.5. Since Φ and Ψ' are isometries between the hermitian forms, by choosing $\epsilon > 0$ small enough we can conclude that $\psi_1 \colon H(\hat{\mathbb{Z}}G) \to H(\hat{\mathbb{Z}}G)$ induces an isometry of the hermitian form $(H(\hat{\mathbb{Z}}G), h_{\hat{\mathbb{Z}}})$ modulo $|G|$. Thus condition (5.5) is also satisfied. □

In Section 8, we will need to vary the construction of $(V, \partial V)$. Recall that the attaching map f for $V = (U - e^4) \cup_f e^4$ is determined by the hermitian form $(H(\mathbb{Z}G), h')$, which is a pull-back of forms over $H(\mathbb{Q}G)$ and $H(\hat{\mathbb{Z}}G)$ identified by the simple isometry ϕ' given in Lemma 5.3.

Proposition 5.9. *Let ϕ' be a simple isometry as in (5.3). For any unitary automorphism $\beta \in SU(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$, the Poincaré complex $(V_\beta, \partial V_\beta)$ constructed from $\phi'_\beta = \beta \circ \phi'$ is also weakly simple and has negative definite intersection form.*

Proof. The image of the Whitehead torsion $\tau(V, \partial V)$ in $Wh(\hat{\mathbb{Q}}G)$ is computed by reduced norms. By construction, these values are the same as those for $\tau(U, \partial U)$.

Now we can repeat the proof of Proposition 5.2 using $\Phi'_\beta = \psi \circ (\phi')^{-1} \circ \beta^{-1}$ to construct a hermitian form h'_β, and then re-attach the top cell to get V_β. $\square$

6. Four-Dimensional Surgery

In Sections 2-5 we constructed a collection of weakly simple Poincaré complexes $(V, \partial V)$ with $\pi_1(V) = G$ and negative definite intersection forms. The boundary $\partial V = \partial U$ is the disjoint union of linear and non-linear space forms. These complexes are parametrized by elements $\beta \in SU(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$, but this dependence will be suppressed for the moment. In this section, we will show that each of these Poincaré complexes admits a degree 1 normal map from a smooth 4-manifold. We then begin to study the surgery obstruction.

Proposition 6.1. *The Spivak normal fiber space $\xi \to V$ is trivial. There exists a trivialization $p\colon E(\xi) \to \mathbb{R}^\ell$, $\ell = \dim \xi + 1$, and an associated degree 1 normal map $f\colon (X, \partial X) \to (V, \partial V)$, $b\colon \nu_X \to \xi$, such that*

(i) *$(X, \partial X)$ is a compact, smooth, oriented 4-manifold with $\pi_1(X) = G$,*
(ii) *$\partial X = \partial V$ and $(f, b)\,|\,\partial X = id$,*
(iii) *$sign(X) = sign(V)$, and*
(iv) *the surgery obstruction $\lambda(f, b)$ lies in the "weakly simple" surgery obstruction group $L'_0(\mathbb{Z}G)$.*

Proof. Note that $\partial V = \partial U$ is a union of framed manifolds. Hence $\xi\,|\,\partial V$ has a vector bundle reduction and in fact a framed structure over ∂V. This smooth structure can be extended to give a vector bundle reduction for ξ over V since the first exotic spherical characteristic class is zero on oriented 4-dimensional Poincaré complexes. Then since $w_2(V) = w_3(V) = 0$ and V is homotopy equivalent to a 3-complex, we conclude that ξ is the trivial bundle. We fix a trivialization of ξ to serve as a base-point for normal invariants.

Let $p : E(\xi) \to \mathbb{R}^\ell$, $\ell = \dim \xi + 1 >> 0$, be any fibre homotopy trivialization of ξ extending the given trivialization of $\xi|\partial V$. By making p transverse to $0 \in \mathbb{R}^\ell$, we obtain a compact, smooth 4-manifold X_ξ with $\partial X_\xi = \partial V$ and a degree 1 normal map $f_\xi : X_\xi \to V$ covered by a bundle map $b_\xi : \nu(X_\xi) \to \xi$

$$(6.2)\qquad \begin{array}{ccc} \nu(X_\xi) & \xrightarrow{b_\xi} & \xi \\ \downarrow & & \downarrow \\ X_\xi & \xrightarrow{f_\xi} & V \end{array}$$

and $(f_\xi, b_\xi)|\partial X_\xi = id$.

By varying within the normal cobordism class of (f_ξ, b_ξ) if necessary, we may assume that f_ξ induces an isomorphism of fundamental groups, so $\pi_1(X_\xi) = G$. Furthermore, since $(V, \partial V)$ is a weakly simple Poincaré pair, the surgery obstruction $\lambda(f_\xi, b_\xi)$ for (6.2) lies in group $L'_0(\mathbb{Z}G)$ computed in [19]. As a simply connected surgery problem, (6.2) has an obstruction given by the difference $\mathrm{sign}(X_\xi) - \mathrm{sign}(V)$ of two signatures. However we can get rid of this obstruction by the following modification.

Consider a degree 1 map $\varphi : V/\partial V \to S^4$. From [8], it is known that $\pi_4(G/PL) = \mathbb{Z}$ and its generator is represented by a vector bundle η over S^4 with a homotopy trivialization $p : E(\eta) \to \mathbb{R}^\ell$ and $\frac{1}{3}p_1(\eta)[S^4] = -16$. Pulling back this G/PL-structure to V via φ, we can add this to ξ to get a new G/PL-structure $\xi \sharp \varphi^*\eta$. Note that the relative Pontrjagin class $\frac{1}{3}p_1(\xi \sharp \varphi^*\eta)[V/\partial V] = \frac{1}{3}p_1(\eta)[S^4] = -16$ with respect to our base-point trivialization on ξ. Therefore by repeating this construction k-times, $k = \mathrm{sign}(V)/16$, we arrive at a G/PL-structure ξ' over $V/\partial V$ with $\frac{1}{3}p_1(\xi') = -\mathrm{sign}(V)$. Using ξ' instead of ξ, we obtain a corresponding surgery problem:

$$\begin{tikzcd} \nu(X_{\xi'}) \arrow[r, "b_{\xi'}"] \arrow[d] & \xi' \arrow[d] \\ X_{\xi'} \arrow[r, "f_{\xi'}"] & V \end{tikzcd}$$

Since we have

$$\mathrm{sign}(X_{\xi'}) = \frac{1}{3}p_1(\tau(X_{\xi'}))[X_{\xi'}/\partial X_{\xi'}] = -\frac{1}{3}p_1(\xi')[V/\partial V] = \mathrm{sign}(V)$$

it follows that the simply connected surgery obstruction equals zero. $\square$

There are other surgery obstructions for our problem $(f, b) : X \to V$, independent of the simply-connected signature obstruction. In fact, the relevant surgery obstruction group $L'_0(\mathbb{Z}Q(8p, q))$ has been computed by Madsen in [10] following the methods of Wall [19].

As in [10, p. 208], let

$$L'_n(\mathbb{Z}G) = \begin{cases} L^Y_n(\mathbb{Z}G, \alpha, 1) & \text{for } n \equiv 0 \pmod 2, \text{and} \\ L^Y_n(\mathbb{Z}G, \alpha, 1)/\langle \begin{smallmatrix} 0 & 1 \\ \pm 1 & 0 \end{smallmatrix} \rangle & \text{for } n \equiv 1 \pmod 2, \end{cases}$$

where the decoration $Y = SK_1(\mathbb{Z}G) \oplus \langle \pm g \mid g \in G \rangle$.

Theorem 6.3. *There is a natural splitting:*

$$L^Y_n(\mathbb{Z}G) = \sum\nolimits^{\oplus} \{L^Y_n(\mathbb{Z}G)(d) : d \mid pq\}$$

such that

(i) *for* $d \neq 1, L^Y_n(\mathbb{Z}G)(d) = L^X_n(\mathbb{Z}G)(d)$ *where the decoration* X *stands for* $SK_1(\mathbb{Z}G)$.

(ii) $L^X_n(\mathbb{Z}G)(d) \cong L^X_n(\mathbb{Z}[\mathbb{Z}/d \rtimes Q(8)])(d)$

(iii) *for each* $d \mid pq$, *there is a long exact sequence:*

$$\ldots \to CL^X_{n+1}(S(d)) \to L^Y_n(\mathbb{Z}G)(d) \to L^X_n(T(d)) \oplus \prod_{\ell \nmid d} L^X_n(\hat{R}_\ell(d)) \ldots$$

where

$$R(d) = \mathbb{Z}[\zeta_d]^t Q(8) \qquad S(d) = \mathbb{Q}[\zeta_d]^t Q(8)$$
$$T(d) = \mathbb{R} \otimes S(d) \qquad \hat{R}_\ell(d) = R(d) \otimes \hat{\mathbb{Z}}_\ell$$

and

$$CL_n^X(S(d)) = L_n^X(S(d) \to \hat{S}(d) \oplus T(d))$$

(iv) *The K-theory decorations are given by*

$$X(S(d)) = X(T(d)) = X(\hat{R}_\ell(d)) = \{0\}, \ (\ell \ odd).$$

Since the calculations of $L_*^Y(\mathbb{Z}G))(d)$ for different $d \mid pq$ are quite similar, Madsen concentrated on the most difficult case when $d = pq$. For this he proved the following [10, Thm. 4.16]:

Theorem 6.4. *There is an exact sequence*

$$0 \to Coker\,\psi_1^F \to L_0^X(\mathbb{Z}G)(pq) \to Ker\,\psi_0^F \to 0,$$

where $Ker\,\psi_0^F$ is the free abelian group detected by the signature invariants corresponding to real places of $F = \mathbb{Q}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$. The term $Coker\,\psi_1^F$ is determined by the exact sequence

$$(6.5) \quad 0 \to Ker\,\tilde{\psi}_1^F \to F^{(2)}/F^{\times 2} \to H^0((A/pq)^\times) \to Coker\,\psi_1^F \to H^0(\Gamma(F)) \to 0,$$

where $A = \mathbb{Z}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$, $F = \mathbb{Q}[\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1}]$, $\Gamma(F) = I(F)/F^\times$ is the ideal class group of F, $I(F) = F_A^\times/F_\infty^\times \cdot \hat{A}^\times$ is the ideal class group, and $F^{(2)} \subset F^\times$ consists of elements with even evaluation at all finite primes.

For a geometric surgery problem (f, b), the image of the surgery obstruction $\lambda(f, b)$ in the group $\operatorname{Ker}\psi_0$ can be interpreted as the difference $sign_\alpha(V) - sign_\alpha(X)$ between the multi-signatures of X and V.

Proof of Theorem 6.4. The exact sequence of (6.4) comes from the calculation of L-groups [19]: we substitute

$$\prod_{\ell \nmid pq} L_1^X(\hat{A}_\ell) = \prod_{\ell \nmid pq} H^0(A_\ell^\times) \times A/2,$$

and $L_1^X(F_\infty) = H^0(F_\infty^\times)$ together with $CL_1^X(F) = H^0(C(F))$ into the commutative diagram

$$\begin{array}{ccc}
\prod_{\ell \nmid pq} H^0(A_\ell^\times) \times A/2 \times H^0(F_\infty^\times) & \xrightarrow{\psi_1^F} & H^0(C(F)) \\
\downarrow \approx & & \downarrow \approx \\
\prod_{\ell \nmid pq} L_1^X(\hat{A}_\ell) \times L_1^X(F_\infty) & \longrightarrow & CL_1^X(F)
\end{array}$$

where $C(F) = F_A^\times/F^\times$ is the idèle class group and the vertical maps are induced by the "change of decoration" Rothenberg sequences in L-theory comparing L^X with L^K. In describing the cokernel of ψ_1^F, it is convenient to compare with the natural homomorphism

$$H^0(\hat{A}^\times) \times H^0(F_\infty^\times) \times H^0(F^\times) \to H^0(F_A^\times)$$

which has kernel $F^{(2)}/F^{\times 2}$ and cokernel $H^0(\Gamma(F))$. Putting this information together we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Ker}\tilde{\psi}_1^F & \to H^0(\hat{A}_{d'}^{\times}) \times H^0(F_\infty^{\times}) \times H^0(F^{\times}) & \to & H^0(\hat{F}_A^{\times}) & \to & \operatorname{Coker}\tilde{\psi}_1^F & \to 0 \\
& \downarrow & \downarrow & & \| & & \downarrow & \\
0 \to & F^{(2)}/F^{\times 2} & \to H^0(\hat{A}^{\times}) \times H^0(F_\infty^{\times}) \times H^0(F^{\times}) & \to & H^0(\hat{F}_A^{\times}) & \to & H^0(\Gamma(F)) & \to 0 \\
& \downarrow & \downarrow & & & & & \\
0 \to & H^0(\hat{A}_d^{\times}) = & H^0(\hat{A}_d^{\times}) & & & & &
\end{array} \tag{6.6}$$

Here $H^0(\hat{A}_d^{\times}) = \prod_{\ell | pq} H^0(\hat{A}_\ell^{\times})$ and $H^0(\hat{A}_{d'}^{\times}) = \prod_{\ell \nmid pq} H^0(\hat{A}_\ell^{\times})$. The snake lemma and the isomorphism $\operatorname{Coker}\tilde{\psi}_1^F = \operatorname{Coker}\psi_1^F$ yields the exact sequence in (6.5). $\square$

We will now apply these calculations to study the surgery obstructions which lie in $\operatorname{Coker}\psi_1$. Let $SU_r(\hat{\mathbb{Q}}G)$ denote the group of unitary automorphisms of the hyperbolic form of rank r over $\hat{\mathbb{Q}}G$.

Lemma 6.7. *There is a natural projection $SU_r(\hat{\mathbb{Q}}G) \to \operatorname{Coker}\psi_1$, for $r \geq 3$.*

We will denote the image of $\beta \in SU_r(\hat{\mathbb{Q}}G)$ under this projection by $[\beta]$.

Proof. Since
$$CL_n^X(S(d)) = L_n^X(S(d) \to \hat{S}(d) \oplus T(d))$$
is actually a quotient of $L_n^S(\hat{S}(d) \oplus T(d))$ by [19,1.2] and $L_1^S(T(d)) = 0$, we see that $CL_1^S(S(d))$ and hence $\operatorname{Coker}\psi_1$ is a quotient of $L_1^S(\hat{S}(d))$. However, by definition $L_1^S(\hat{\mathbb{Q}}G)$ is a quotient of the stablized special unitary group $SU(\hat{\mathbb{Q}}G)$ and the projection
$$SU_r(\hat{\mathbb{Q}}G)/RU_r(\hat{\mathbb{Q}}G) \to L_1^S(\hat{\mathbb{Q}}G)$$
is an epimorphism for $r \geq 3$. $\square$

Recall from (5.9) that we can vary our Poincaré complex $(V, \partial V)$ to $(V_\beta, \partial V_\beta)$ for any $\beta \in SU(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$. Exactly the same process can be used to vary any algebraic quadratic Poincaré complex (as defined in [14]).

Any 4-dimensional quadratic $\mathbb{Z}G$ Poincaré complex $C(\psi)$ can be stabilized by adding a hyperbolic form $\mathbb{H}(q(r)) = \mathbb{H}(\mathbb{Z}G)^r$ of rank r over $\mathbb{Z}G$ (considered as a 4-complex concentrated in the middle dimension). This is just the algebraic analogue of adding copies of $S^2 \times S^2$ to the domain of a geometric surgery problem. Now if $C(\psi(r)) = C(\psi) \oplus \mathbb{H}(q(r))$ is the r-stabilization of $C(\psi)$ and $\beta \in SU_r(\hat{\mathbb{Q}}G)$, we can construct a new quadratic Poincaré complex $C(\psi_\beta(r))$ by pulling back using the same rational and ℓ-adic pieces as $C(\psi) \oplus \mathbb{H}(q(r))$. The identification over $\hat{\mathbb{Q}}G$ is altered by composing with β (just as in Proposition 5.9).

Lemma 6.8. *For any $\beta \in SU_r(\hat{\mathbb{Q}}G)$, $r \geq 3$, and any 4-dimensional quadratic Poincaré complex $C(\psi)$ over $\mathbb{Z}G$, the surgery obstruction $\lambda(C(\psi_\beta(r))) \in L_0'(\mathbb{Z}G)$ is independent of r and given by $\lambda(C(\psi_\beta(r))) = \lambda(C(\psi))) + [\beta]$.*

Proof. Stabilization does not change the surgery obstruction of $C(\psi)$ so
$$\lambda(C(\psi(r))) = \lambda(C(\psi)).$$

Similarly, $\lambda(C(\psi_\beta(r)))$ is independent of r since $r \geq 3$. We can also assume that the patching over $\hat{\mathbb{Q}}G$ used to construct $C(\psi)$, and the action of β, take place in orthogonal direct summands of $C(\psi(r))$. Therefore

$$C(\psi_\beta(r)) = C(\psi(r)) \oplus \mathbb{H}(q_\beta(r)).$$

Since the surgery obstruction is just the algebraic Poincaré cobordism class of $C(\psi_\beta(r))$, and $\lambda(\mathbb{H}(q_\beta(r))) = [\beta]$ by definition, the given formula holds. $\square$

7. Induction Maps

This section contains an algebraic result we will need to handle the multisignature surgery obstruction. Let $R(G)$ denote the real representation ring of G, and recall that there is a natural transformation [19, 2.2] of Mackey functors

$$\sigma_G \colon L'_0(\mathbb{Z}G) \to R(G)$$

given by diagonalizing a hermitian form (H, h) over each irreducible representation α of $\mathbb{R}G$ (see [19, §2.2]) and then taking the formal difference $\operatorname{sign}_\alpha(H, h)$ of the maximal positive and negative definite G-invariant subspaces.

In particular, the homomorphisms $\sigma_H \colon L'_0(\mathbb{Z}H) \to R(H)$ for $H = Q(8p)$,$Q(8q)$, and $C(2pq)$ are compatible with the induction maps between the surgery obstruction groups, and the corresponding induction homomorphisms:

$$\begin{aligned} i_{1*} &: R(Q(8p)) \;\;\to R(Q(8p,q)) \\ i_{2*} &: R(Q(8q)) \;\;\to R(Q(8p,q)) \\ i_{3*} &: R(C(2pq)) \to R(Q(8p,q)). \end{aligned}$$

The reduced representation ring $\widetilde{R}(G) = \ker(R(G) \to R(1))$ is generated by elements of the form $(\alpha - \dim\alpha \cdot \mathbf{1})$ for all real G-representations α. This ideal of $R(G)$ is closed under induction and restriction. The transformation σ_G induces

$$\sigma \colon \tilde{L}'_0(\mathbb{Z}G) \to \widetilde{R}(G),$$

which is again compatible with the Mackey structure, and we have the commutative diagram

$$\begin{array}{ccc} \tilde{L}'_0(Q(8p)) \oplus \tilde{L}'_0(Q(8q)) \oplus \tilde{L}'_0(C(2pq)) & \xrightarrow{I_{1*}\oplus I_{2*}\oplus I_{3*}} & \tilde{L}'_0(\mathbb{Z}G) \\ \big\downarrow \sigma & & \big\downarrow \sigma \\ \widetilde{R}(Q(8p)) \oplus \widetilde{R}(Q(8q)) \oplus \widetilde{R}(C(2pq)) & \xrightarrow{i_{1*}\oplus i_{2*}\oplus i_{3*}} & \widetilde{R}(G). \end{array}$$

The main result of this section is:

Proposition 7.1. *The image of* $\sigma \colon \tilde{L}'_0(\mathbb{Z}G) \to \widetilde{R}(G)$ *restricted to* $Im\,(I_{1*} \oplus I_{2*} \oplus I_{3*})$ *contains the subgroup* $16 \cdot \widetilde{R}(G)$.

We will describe the splitting used in [19, §4] and Theorem 6.3, in order to study the induction homomorphisms between these groups.

Lemma 7.2. *The map σ_G has a direct sum decomposition $\sigma = \oplus_{d|pq}\sigma(d)$ where $\sigma(d)\colon L'_0(\mathbb{Z}G)(d) \to R(G)(d)$. A similar splitting exists for the subgroup $C(2pq)$, and the induction map $I_3\colon L'_0(\mathbb{Z}C(2pq)) \to L'_0(\mathbb{Z}G)$ preserves the components.*

Proof. Note that the group algebra $\mathbb{Q}[C(2pq)]$ decomposes into the product of four different fields $\mathbb{Q}$, $\mathbb{Q}(\zeta_p)$, $\mathbb{Q}(\zeta_q)$, and $\mathbb{Q}(\zeta_{pq})$. This induces a corresponding decomposition on $\mathbb{Q}G = \mathbb{Q}[C(2pq)]^t Q(8)$ and hence on every functor of $\mathbb{Q}G$. In fact, for every covariant functor $A(-)$ from finite subgroups of G to abelian groups, an analogous decomposition exists for $A(G)$. Let $f_p, f_q : C(2pq) \to C(2pq)$ denote the endomorphisms which project onto $C(p)$ and $C(q)$ respectively. They extend to endomorphisms $\hat{f}_p, \hat{f}_q$ of $Q(8p,q)$ by setting $\hat{f}_p \,|\, Q(8) = \hat{f}_q \,|\, Q(8) = id$. Since $\hat{f}_p^2 = \hat{f}_p, \hat{f}_q^2 = \hat{f}_q$, we obtain idempotent endomorphisms $F_p = (\hat{f}_p)_*$ and $F_q = (\hat{f}_q)_*$ of $A(G)$. Hence there is a decomposition

$$A(G) = A(G)(1) \oplus A(G)(p) \oplus A(G)(q) \oplus A(G)(pq)$$

where

$$\begin{aligned}
A(G)(1) &= F_pF_q(A(G))\\
A(G)(p) &= F_p(1-F_q)(A(G))\\
A(G)(q) &= F_q(1-F_p)(A(G))\\
A(G)(pq) &= (1-F_p)(1-F_q)(A(G)).
\end{aligned}$$

Applying this splitting to the surgery obstruction group $L'_0(\mathbb{Z}G)$, we have

$$L'_0(\mathbb{Z}G) = L'_0(\mathbb{Z}G)(1) \oplus L'_0(\mathbb{Z}G)(p) \oplus L'_0(\mathbb{Z}G)(q) \oplus L'_0(\mathbb{Z}G)(pq).$$

Similarly for $R(G)$, we have

$$R(G) = R(G)(1) \oplus R(G)(p) \oplus R(G)(q) \oplus R(G)(pq).$$

Since the splittings are given by idempotents, we get a corresponding direct sum decomposition for σ.

The idempotent endomorphisms $\hat{f}_p, \hat{f}_q$ also exist on the subgroup $C(2pq)$ and hence give the corresponding decompositions on $L'_0(C(2pq))$ and $R(C(2pq))$. The commutativity of the following diagram $(d \mid pq)$

$$\begin{array}{ccc}
L'_0(\mathbb{Z}C(2pq))(d) & \xrightarrow{\sigma} & R(C(2pq))(d)\\
\big\downarrow{\scriptstyle I_{3*}} & & \big\downarrow{\scriptstyle i_{3*}}\\
L'_0(\mathbb{Z}G)(d) & \xrightarrow{\sigma} & R(G)(d)
\end{array} \tag{7.3}$$

means that the induction maps from $C(2pq)$ preserve the components. □

There is one more functorial fact which simplifies our computation. Both $Q(8p,q)$ and $C(2pq)$ contain a unique order 2 element $X^2 = Y^2 = (XY)^2$ in the centre. By Schur's lemma the action of this element on the irreducible are either $+1$ or -1. Accordingly the representation rings decompose into two components:

$$\begin{aligned}
R(Q(8p,q))(pq) &= R(Q(8p,q))(pq)_+ \oplus R(Q(8p,q))(pq)_-\\
R(C(2pq))(pq) &= R(C(2pq))(pq)_+ \oplus R(C(2pq))(pq)_-
\end{aligned}$$

and the homomorphism i_{3*} preserves these decompositions.

Proposition 7.4. *On the* (-1)*-component the homomorphism*

$$(i_3)_* : R(C(2pq))(pq)_- \to R(Q(8p,q))(pq)_-$$

is surjective, and on the $(+1)$*-component the image of the homomorphism*

$$(i_3)_* : R(C(2pq))(pq)_+ \to R(Q(8p,q))(pq)_+$$

equals $2 \cdot R(Q(8p,q))(pq)_+$

Proof. Recall that the splittings on $R(C(2pq))$ and $R(Q(8p,q))$ can be achieved by first applying the splittings to the group algebras $\mathbb{Q}[C(2pq)], \mathbb{Q}[Q(8p,q)]$. By (4.8) and (4.9) we see that the rational representations in the top components $S(pq)_+$ and $S(pq)_-$ are induced up from $\mathbb{Q}[C(2pq)]$, and become the twisted group algebras $\mathbb{Q}(\zeta_{pq})^t[X,Y]_\pm$ which have dimension $4(p-1)(q-1)$. In the regular representation of $\mathbb{Q}G$, this factor decomposes as the direct sum of 4 copies (respectively 2 copies) of the simple module for $S(pq)_+ = M_4(F)$ (resp. $S(pq)_- = M_2(D)$). Note that $\mathbb{Q}(\zeta_{pq})\otimes\mathbb{R}$ splits into $(p-1)(q-1)/2$ copies of the complex numbers $\mathbb{C}$, and the centre field $F = \mathbb{Q}(\zeta_p + \zeta_p^{-1}, \zeta_q + \zeta_q^{-1})$ splits into $(p-1)(q-1)/4$ copies of $\mathbb{R}$. By counting dimensions (over $\mathbb{R}$), we see that each one of these irreducible representations $\mathbb{C}$ of $C(2pq)$ induces up to a real 8-dimensional representation. Since the irreducible module $L_\pm$ for a simple factor of $M_4(F)\otimes\mathbb{R}$ (resp. $M_2(D)\otimes\mathbb{R}$) has real dimension 4 (resp. 8), we conclude that the induced real representations is $L_+ \oplus L_+$ in the $(+1)$-component and L_- in the (-1)-component. □

Proof of Proposition 7.1. The endomorphism $\hat{f}_p$ factors through the subgroup $Q(8p)$, and we have the inclusion $\operatorname{Im} F_p \subseteq \operatorname{Im} i_{1*}$. On the other hand, because $\hat{f}_p \circ i_1 = i_1$, we have $(F_p - 1)\operatorname{Im} i_{1*} = 0$. Similar relations hold for F_q and i_{2*}. From the definition of the summands $R(G)(d)$ in terms of these idempotents, it follows that

$$R(G)(1) \oplus R(G)(p) \oplus R(G)(q) = \operatorname{Im}(i_{1*}) + \operatorname{Im}(i_{2*}). \tag{7.5}$$

On the other hand, by Proposition 7.4, we have

$$2 \cdot R(G)(pq) \subseteq \operatorname{Im}(i_{3*})$$

and we can conclude that

$$2 \cdot R(G) \subseteq \operatorname{Im}(i_{1*} \oplus i_{2*} \oplus i_{3*}) \subseteq R(G).$$

Moveover, it follows from the results of [6], [19,2.2.1] on the divisibility of the signature invariants, that

$$\sigma_H \colon L'_0(\mathbb{Z}H) \to R(H)$$

has image containing the subgroup $8 \cdot R(H)$ for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$. By naturality of σ,

$$16 \cdot R(G) \subseteq \sigma_G(\operatorname{Im}(I_{1*} \oplus I_{2*} \oplus I_{3*})) \subseteq R(G). \quad \square$$

8. Almost Spherical Space Forms

We are now ready to consider the surgery obstructions of the degree 1 normal maps constructed in Proposition 6.1.

Proposition 8.1. *Let $f_\xi\colon(X,\partial X)\to(V,\partial V)$, $b_\xi\colon\nu_X\to\xi$ be a degree 1 normal map satisfying the conditions in (6.1). Then there exists an element $\beta\in SU_r(\hat{\mathbb{Q}}G)$, $r\geq 3$, and a degree 1 normal map $f'_{\xi,\beta}\colon(X',\partial X')\to(V_\beta,\partial V_\beta)$, $b'_{\xi,\beta}\colon\nu_{X'}\to\xi$ such that*

(i) *$f'_{\xi,\beta}\,|\,\partial X'$ is an integral homology equivalence, and*
(ii) *$\lambda(f'_{\xi,\beta},b'_{\xi,\beta})=0\in L'_0(\mathbb{Z}G)$.*

After giving the proof of this result, we will use it to construct the smooth 4-manifold $(Y,\partial Y)$ described in the Introduction.

Proof. We will first consider the multisignature obstruction $\operatorname{sign}_\alpha(V)-\operatorname{sign}_\alpha(X_\xi)$ given by irreducible real representations α, $\alpha\neq 1$, of $G=Q(8p,q)$. Note that this set of surgery obstructions generates the group $\operatorname{Ker}\psi_0$ and so if $\operatorname{sign}_\alpha(V)-\operatorname{sign}_\alpha(X_\xi)=0$ for all α then $\lambda(f_\xi,b_\xi)\in\operatorname{Coker}\psi_1$.

We begin with the ρ-invariant $\rho_\alpha(N)$ of a 3-manifold N with a unitary representation $\alpha:\pi_1(N)\to U(n)$. Suppose $N=\partial M$ and α extends to a representation of $\pi_1(M)$. Then
$$\rho_\alpha(N)=n\cdot\operatorname{sign}(M)-\operatorname{sign}_\alpha(M).$$
As a consequence of this formula, we have
$$\begin{aligned}\operatorname{sign}_\alpha(V)-\operatorname{sign}_\alpha(X)&=\operatorname{sign}_\alpha(V)-[n\cdot\operatorname{sign}(X)-\rho_\alpha(\partial X)]\\&=\operatorname{sign}_\alpha(V)-[n\cdot\operatorname{sign}(V)-\rho_\alpha(\partial X)]\end{aligned}$$
or, in other words, the vanishing of the obstruction $\operatorname{sign}_\alpha(V)-\operatorname{sign}_\alpha(X)$ is the same as requiring that the following equation

$$\rho_\alpha(\partial X)=n\cdot\operatorname{sign}(V)-\operatorname{sign}_\alpha(V)\tag{8.2}$$

be satisfied by the domain $(X,\partial X)$ of our degree one normal map. Note that this equation, and the fact that $\operatorname{sign}X=\operatorname{sign}V$, implies that the multisignature difference depends only on $\partial X=\partial V$.

In general, equation (8.2) may not be satisfied and so these are nontrivial obstructions for our surgery problem. To get rid of these obstructions, the idea is to replace copies of the spherical space forms $S^3/\,Q(8p)$, $S^3/\,Q(8q)$, or $S^3/\,C(pq)$ on the boundary ∂X by almost spherical space forms $S'/Q(8p)$, $S'/Q(8q)$, $S'/C(2pq)$ and therefore change the ρ-invariants. After this process, our new normal map will no longer restrict to the identity on the boundary, but just to an integral homology equivalence.

One way to construct an almost space form S'/H is to start with an element $\sigma\in L'_0(\mathbb{Z}H)$ and apply the Wall realization theorem to construct a degree 1 normal map
$$(f,b)\colon(M^4,\partial_0M^4,\partial_1M^4)\to(S^3/\,H\times I,S^3/\,H\times 0,S^3/\,H\times 1)$$
such that $\lambda(f,b)=\sigma$. More explicitly, this surgery problem is constructed by representing σ by a quadratic form on a free $\mathbb{Z}H$-module and using this algebraic data

as a prescription for attaching 2-handles to $S^3/H \times I$. By construction, the lower boundary component $\partial_0 M^4 = S^3/H$ and the restriction of (f,b) is the identity. The upper boundary component $\partial_1 M^4 = S'/H$ is an almost space form. On this boundary component the restriction $f\colon S'/H \to S^3/H$ is just an integral homology equivalence, and a surjection on fundamental groups. The fact that we have lost some control of $\pi_1(S'/H)$ is a typical problem with surgery in dimension 3, but at least S' is an integral homology sphere.

Now suppose that we start with σ_1, σ_2, and σ_3 in $L'_0(Q(8q))$, $L'_0(Q(8q))$, and $L'_0(C(2pq))$ respectively. Then we construct 4-manifolds M_1, M_2, M_3 whose boundary components are the spherical space forms S^3/H_i and the almost space forms S'/H_i with $H_i = Q(8p)$, $Q(8q)$, or $C(2pq)$ for $i = 1, 2$ or 3. Let $g_i\colon S'/H_i \to S^3/H_i$, $1 \le i \le 3$, denote the integral homology equivalence obtained by restricting the degree 1 normal map (f_i, b_i) used to construct $(M_i, \partial M_i)$ to the top boundary component.

Next we attach these surgery problems (f_i, b_i) to our degree 1 normal map $X \to V$ along the appropriate boundary components of $\partial X = \partial V$. More precisely, we attach the 4-manifold M_i to a component of ∂X with boundary S^3/H_i and extend the degree 1 map by using the normal maps (f_i, b_i) to collars $S^3/H_i \times I$ on the same component of ∂V. This produces a new degree 1 normal map (which we will consider to be a relative surgery problem):

$$(f'_\xi, b'_\xi)\colon (X', \partial X') \to (V, \partial V), \tag{8.3}$$

where the domain is

$$X' = X \cup M_1 \cup M_2 \cup M_3,$$

and f'_ξ restricted to $\partial X'$ is an integral homology equivalence.

Moreover, if we choose $\sigma, \sigma_2, \sigma_3$ to lie in the reduced surgery obstruction groups $\tilde{L}'_0(Q(8p))$, $\tilde{L}'_0(Q(8q))$, or $\tilde{L}'_0(C(2pq))$, then the simply-connected signature invariants $\operatorname{sign}(M_i) = 0$ for $i = 1, 2$ and 3. It follows that $\operatorname{sign}(X') = \operatorname{sign}(V)$.

We can now compute the effect on the multi-signature obstruction

$$\operatorname{sign}_\alpha(V) - \operatorname{sign}_\alpha(X') = \rho_\alpha(\partial X') - n \cdot \operatorname{sign}(V) + \operatorname{sign}_\alpha(V). \tag{8.4}$$

We have changed the ρ-invariants on the boundary by the formula:

$$\rho_\alpha(\partial X') = \rho_\alpha(\partial X) + \operatorname{sign}_\alpha[I_{1*}(\sigma_1) + I_{2*}(\sigma_2) + (I_{3*}(\sigma_3)]. \tag{8.5}$$

Here I_{1*}, I_{2*}, I_{3*} are the induction homomorphisms between the surgery obstruction groups:

$$\begin{aligned} I_{1*} &: L'_0(Q(8p)) \ \to L'_0(Q(8p,q)) \\ I_{2*} &: L'_0(Q(8q)) \ \to L'_0(Q(8p,q)) \\ I_{3*} &: L'_0(C(2pq)) \to L'_0(Q(8p,q)) \end{aligned}$$

already used in Section 7. Substituting (8.5) into (8.4), we have the equation

$$\rho_\alpha(\partial X) + \sum_{1 \le k \le 3} \operatorname{sign}_\alpha[(I_{k*}(\sigma_k)] = n \cdot \operatorname{sign}(V) - \operatorname{sign}_\alpha(V) \tag{8.6}$$

as the requirement for vanishing of the multisignature obstruction for the surgery problem of (f'_ξ, b'_ξ). Therefore our goal is to choose $\sigma_1, \sigma_2, \sigma_3$ in such a manner that the expression (8.6) is satisfied.

The nonsingular hermitian pairing (H, h) for $H = H_2(V; \mathbb{Z}G)$ gives us an element in $R(G)$, whose α-component is $\mathrm{sign}_\alpha(V)$. Therefore we can interpret the expression $n \cdot \mathrm{sign}(V) - \mathrm{sign}_\alpha(V)$, $n = \dim \alpha$, in (8.2) as the α-component of an element $\sigma(V) \in \widetilde{R}(G)$. Similarly, we have $\sigma(X) \in \widetilde{R}(G)$. In addition, from the construction of $(V, \partial V)$ both $\sigma(V)$ and $\sigma(X)$ are divisible by 16, or in other words

$$\sigma(V) - \sigma(X) \in 16 \cdot \widetilde{R}(G) \subseteq \widetilde{R}(G).$$

Since $\rho_\alpha(\partial X) = n \cdot \mathrm{sign}(X) - \mathrm{sign}_\alpha(X)$, we can rewrite (8.6) as an equation in the reduced representation ring:

$$\sigma_G(I_{1*}(\sigma_1) + I_{2*}(\sigma_2) + I_{3*}(\sigma_3)) = \sigma(V) - \sigma(X) \in 16 \cdot \widetilde{R}(G) \subseteq \widetilde{R}(G) \tag{8.7}$$

where

$$\sigma_G : L'_0(\mathbb{Z}G) \to R(G)$$

is the multisignature natural transformation from Section 7. But the main result of that section, Proposition 7.1, states that equation (8.6) has a solution σ_1, σ_2, σ_3. We may therefore use these elements to construct a degree 1 normal map $(f'_\xi, b'_\xi)\colon (X', \partial X') \to (V, \partial V)$ as in (8.3) with $\lambda(f'_\xi, b'_\xi) \in \mathrm{Coker}\, \psi_1$. Since the multisignature vanishes, this surgery obstruction is independent of the choice of normal map (i.e. depends only on the range $(V, \partial V)$).

To complete the proof of Proposition 8.1, we pick $\beta \in SU_r(\hat{\mathbb{Q}}G)$, for $r \geq 3$, projecting to $\lambda(f'_\xi, b'_\xi) \in \mathrm{Coker}\, \psi_1$. This is possible by Lemma 6.7 (note that the obstructions are now 2-torsion). We then consider the new Poincaré pair $(V_\beta, \partial V_\beta)$ as constructed in Proposition 5.9. By construction, the boundary $\partial V_\beta = \partial V$ and the multisignature of $(V_\beta, \partial V_\beta)$ equals that of $(V, \partial V)$. It follows from Proposition 6.1 that there is a degree 1 normal map $(f_{\xi,\beta}, b_{\xi,\beta})$ onto $(V_\beta, \partial V_\beta)$ which is the identity on the boundary. Since $\partial V_\beta = \partial V$

$$\mathrm{sign}_\alpha(f_{\xi,\beta}, b_{\xi,\beta}) = \mathrm{sign}_\alpha(f_\xi, b_\xi)$$

and we may use the same elements $\sigma_1, \sigma_2, \sigma_3$ to construct a modified normal map $(f'_{\xi,\beta}, b'_{\xi,\beta})$, inducing an integral homology equivalence on the boundary, with zero multisignature obstruction

The surgery obstruction $\lambda(f'_{\xi,\beta}, b'_{\xi,\beta})$ is determined by the induced quadratic structure [14] on the mapping cone complex $C_*(f'_{\xi,\beta})$:

$$0 \to C_*(X'_{\xi,\beta}) \to C_*(V_{\xi,\beta}) \to C_*(f'_{\xi,\beta}) \to 0.$$

This sequence can be analysed as in §4 by means of the arithmetic square. Again by stabilizing our Poincaré complexes, we can assume that the new identification over $\hat{\mathbb{Q}}G$ given by β takes place on some hyperbolic factors of $(H(\hat{\mathbb{Q}}G), h_{\hat{\mathbb{Q}}})$ orthogonal to those summand used in constructing the map $f'_{\xi,\beta}$. The new Poincaré complex is normally Poincaré bordant to the original (relative to the boundary) with bundle data induced by a trivialization of the Spivak normal fibre space (6.1).

Now by taking a transverse pre-image we obtain a degree one normal map (F, B) with domain a smooth manifold triad, whose boundaries are just (f'_ξ, b'_ξ) and $(f_{\xi,\beta}, b_{\xi,\beta})$. From the usual cobordism interpretation of the Ranicki-Rothenberg sequences, the difference of surgery obstructions

$$\lambda(f'_{\xi,\beta}, b'_{\xi,\beta}) - \lambda(f'_\xi, b'_\xi)$$

is given by the Whitehead torsion of the quadratic mapping cone $C_*(F, B)$. However, the quadratic Poincaré complex $C_*(f'_{\xi,\beta})$ can be constructed from the exact sequence of chain complexes

$$0 \to C_*(X') \to C_*(V_\xi) \to C_*(f'_\xi) \to 0$$

by re-mixing the complexes $C_*(V_\xi)$ and $C_*(f'_\xi)$ simulaneously with β to produce $C_*(V_{\xi,\beta}) \to C_*(f'_{\xi,\beta})$. From Lemma 6.8, it follows that

$$\lambda(f'_{\xi,\beta}, b_{\xi,\beta}) = \lambda(f'_\xi, b_\xi) + [\beta] = 0$$

and the proof is complete. $\square$

The final result of our construction is a precise version of Theorem A, which was stated in the Introduction. The proof is an application of (8.1) and topological 4-manifold techniques due to Freedman [3], [4].

Theorem 8.8. *Let Σ/G be a nonlinear space form for $G = Q(8p, q)$, and let k, l be non-zero integers such that $kp + lq = -1$. Choose a spherical space form S^3/Γ for each of the subgroups $\Gamma = Q(4pq)$, $Q(8p)$, $Q(8q)$, and $C(2pq)$. Then there is a framed, compact, connected, oriented 4-manifold Y with the following properties:*

(i) *The boundary $\partial Y = N \cup \partial_0 Y$, where N is a connected 3-manifold with $H_1(N; \mathbb{Z})$ torsion-free.*

(ii) *There is a reference map $c: Y \to BG$ so that $c_\#: \pi_1(Y) \xrightarrow{\approx} G$ is an isomorphism and the composite $N \hookrightarrow Y \xrightarrow{c} BG$ is null-homotopic.*

(iii) *The boundary components $\partial_0 Y$ of Y consists of two copies of Σ/G with opposite orientation, a positive number of copies of the spherical space form $S^3/Q(4pq)$, at least $|k|$ copies of $S^3/Q(8q)$, at least $|l|$ copies of $S^3/Q(8p)$, and some almost space forms S'/H for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$.*

(iv) *The induced homomorphism $\pi_1(\partial_0 Y) \to \pi_1(Y) \xrightarrow{c_\#} G$ on the fundamental groups sends $\pi_1(\Sigma/G)$, $\pi_1(S^3/Q(4pq))$, and $\pi_1(S'/H)$ for $H = Q(8p)$, $Q(8q)$, or $C(2pq)$ onto the corresponding subgroups $Q(8p, q)$, $Q(4pq)$ or $H \subseteq Q(8p, q)$.*

(v) *the induced $Q(8p, q)$-hermitian intersection pairing*

$$h: H_2(Y; \mathbb{Z}G) \times H_2(Y; \mathbb{Z}G) \to \mathbb{Z}G$$

has radical Im $(H_2(N; \mathbb{Z}G) \rightarrowtail H_2(Y; \mathbb{Z}G))$ and is negative definite on the orthogonal complement of this submodule.

Proof of (8.8). In Proposition 8.1 we have constructed a surgery problem

$$(f, b)\colon (X, \partial X) \to (W, \partial W)$$

with $\lambda(f, b) = 0$, where we take $(W, \partial W) = (V_\beta, \partial V_\beta)$, $X = X'_{\beta,\xi}$ and $f = f'_{\beta,\xi}$. The assertions in (iii) about the boundary components hold for ∂X by making appropriate choices for U in Section 2. Certain congruences must be satisfied but the numbers of boundary components can be made as large as necessary. The manifold Y will be a closed submanifold of X with $\partial_0 Y = \partial X$. Let $c\colon X \to BG$ denote the classifying map of the universal covering of X, so that $c_\#\colon \pi_1(X) \xrightarrow{\approx} G$ fixes an identification of the fundamental group. We will use the same notation c for its restriction to $Y \subset X$.

Since the surgery obstruction (f, b) is zero, the intersection form on $H_2(X; \mathbb{Z}G)$ may be identified with the orthogonal direct sum of the negative definite form $(H_2(W; \mathbb{Z}G), h)$ and a hyperbolic form $H((\mathbb{Z}G)^r)$ on a free $\mathbb{Z}G$ module of some rank $2r$. By [3], [4, 2.9] we can represent this hyperbolic form geometrically by a smooth immersion of 2-spheres in the interior of X such (i) that the algebraic intersection numbers between different spheres are realized geometrically, and (ii) the spheres are immersed transversely (with only ordinary double points) and π_1-null in the sense of [4, p.50]. In particular, these immersed 2-spheres do not intersect those giving a basis for the complementary summand $H_2(W; \mathbb{Z}G)$. Let Y denote the complement of a smooth, open regular neighbourhood Y' of these immersed 2-spheres, so that $X = Y \cup Y'$ and $\partial Y = \partial_0 Y \cup N$ where $\partial_0 Y = \partial X$ and $N = \partial Y'$. Note that $\pi_1(N) \to \pi_1(Y)$ is trivial by construction of the immersion, and hence $\pi_1(Y) = \pi_1(X)$, so that $c_\#\colon \pi_1(Y) \xrightarrow{\approx} G$ is an isomorphism.

We also have a surjection from $H_2(Y; \mathbb{Z}G)$ onto the summand $H_2(W; \mathbb{Z}G) \subset H_2(X; \mathbb{Z}G)$, and the Mayer-Vietoris sequence then gives

$$H_1(N; \mathbb{Z}G) = H_1(Y; \mathbb{Z}G) \oplus H_1(Y'; \mathbb{Z}G).$$

Since $\pi_1(N) \to \pi_1(Y')$ is surjective, the classifying map c composed with the the inclusion of Y in X is also null-homotopic. Therefore, the induced G-covering over Y' is just a disjoint union of copies of the base, and $H_1(Y'; \mathbb{Z}G) = ZG \otimes_{\mathbb{Z}} H_1(Y'; \mathbb{Z})$. But $H_1(Y; \mathbb{Z}G) = 0$ so

$$H_1(N; \mathbb{Z}G) = H_1(Y'; \mathbb{Z}G) = \mathbb{Z}G \otimes_{\mathbb{Z}} H_1(Y'; \mathbb{Z}).$$

Since Y' is a regular neighbourhood of a collection of immersed 2-spheres with at most double point singularities, the homology $H_1(Y'; \mathbb{Z})$ is a torsion-free abelian group.

Finally, note that the inclusion induces an injection $H_2(N; \mathbb{Z}G) \to H_2(Y; \mathbb{Z}G)$. The subspace $\mathcal{N} = \operatorname{Im}(H_2(N; \mathbb{Z}G) \to H_2(Y; \mathbb{Z}G))$ is the null space or radical of the intersection form on $H_2(Y; \mathbb{Z}G)$ and the induced form on the quotient $H_2(Y; \mathbb{Z}G)/\mathcal{N}$ is isometric to $(H_2(W; \mathbb{Z}G), h)$, hence is negative definite. $\square$

Remark 8.9. We are indebted to Peter Teichner for the above argument. He pointed out how to improve our original construction of Y to obtain $\pi_1(Y) = G$ (instead of $H_1(Y; \mathbb{R}G) = 0$) by a more sophisticated use of Freedman's work.

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