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ORIENTABLE 4-DIMENSIONAL POINCARÉ COMPLEXES HAVE REDUCIBLE SPIVAK FIBRATIONS

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ABSTRACT. We show that the Spivak normal fibration of an orientable 4-dimensional Poincaré complex has a vector bundle reduction.

1. Introduction

A Poincaré complex (*PD-complex*), as introduced by Wall [10, p. 214], is a (connected) finitely dominated CW complex X equipped with:

(1) a homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ defining a twisted $\Lambda := \mathbb{Z}\pi_1(X)$ module structure $\mathbb{Z}^t$ on $\mathbb{Z}$,
(2) an integer n and a class $[X] \in H_n(X; \mathbb{Z}^t)$ such that
(3) for all integers $r \geq 0$, cap product with $[X]$ induces an isomorphism

$$[X] \frown\colon H^r(X; \Lambda) \to H_{n-r}(X; \Lambda \otimes \mathbb{Z}^t) \ .$$

The integer $n = \dim X$ is called the *dimension* of X. It follows from the foundational results of Kirby and Siebenmann [5, Annex 3] that every closed topological n-manifold has the homotopy type of a Poincaré complex of dimension n (see the discussion in Wall [11, §17B]). In the manifold case, the homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ is given by the first Stiefel-Whitney class. Accordingly, a PD-complex X is called *orientable* if its homomorphism w is trivial.

Spivak [9] discovered that every simply-connected PD-complex X with $\dim X = n$ has an associated spherical fibration, denoted ν_X, which is unique up to stable fibre homotopy equivalence. It is constructed by embedding X in a high-dimensional Euclidean space $\mathbb{R}^{n+k}$ ($k \gg n$) and considering the fibration homotopic to the projection map $p\colon \partial N \to X$ from the boundary of a regular neighbourhood $N \subset \mathbb{R}^{n+k}$. The duality properties of X imply that the fibres of p are homotopy equivalent to S^{k-1}. The definition and the uniqueness statement were generalized by Wall [10, §3] to all PD-complexes, and ν_X is now called the *Spivak normal fibration* of X.

In the smooth manifold case, ν_X is the spherical fibration associated to the sphere bundle of the (stable) normal k-vector bundle of X. For topological manifolds, the corresponding notion is the (stable) normal $\mathbb{R}^k$-bundle ($k \gg n$) and its sub-bundle with fibres $\mathbb{R}^k - \{0\} \simeq S^{k-1}$.

After the further development of geometric surgery theory, due to Browder, Milnor, Novikov, Sullivan, and Wall, the normal structures on PD-spaces and manifolds

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were re-expressed via classifying spaces (see [11, §§10 and 17B], [5], [8], [6]). One outcome was the construction of a sequence of classifying spaces

$$BO \to BPL \to BTOP \to BG$$

relating smooth, PL, and topological bundles to spherical fibrations. In particular, the (stable) Spivak normal fibre space ν_X is classified by a map $\nu_X : X \to BG$.

Definition 1.1. We say that PD-complex X has a *reducible Spivak normal fibration* if the classifying map $\nu_X : X \to BG$ lifts to a map $\tilde{\nu}_X : X \to BTOP$.

Similarly, we say that the Spivak normal fibre space is reducible to a vector bundle if ν_X lifts to a map $\tilde{\nu}_X : X \to BO$. The lifting obstruction is given by the image of ν_X in $[X, B(G/TOP)]$ or $[X, B(G/O)]$, respectively. In dimensions ≥ 5, these are different problems, but if $\dim X \leq 4$ these two obstruction groups are the same because

$$[X, B(G/O)] = [X, B(G/PL)] = [X, B(G/O)] \cong H^3(X; \mathbb{Z}/2) \quad \text{if } \dim X \leq 4.$$

This is explained in Kirby-Taylor [6, §2]. In other words, the obstruction to reducibility for the Spivak normal fibration of a PD-complex X in dimensions ≤ 4 is a single characteristic class $k_3(X) \in H^3(X; \mathbb{Z}/2)$.

Theorem A. *Let X be a Poincaré complex. If* $\dim X \leq 3$ *or* $\dim X = 4$ *and X is orientable, then the Spivak normal fibration of X is reducible to a vector bundle.*

Remark 1.2. The dimension 4 case was known to the experts (see the statements in Spivak [9, p. 95] and Kirby-Taylor [6, p. 399]), but Land [7] pointed out the lack of a proof in the literature and provided his own argument. For dimensions ≤ 2 the result is immediate, and the dimension 3 cases follow easily from the dimension 4 statement. In general, non-oriented PD-complexes in dimensions ≥ 4 do not have reducible Spivak normal fibrations (see Hambleton and Milgram [4] for explicit examples in every even dimension ≥ 4). The first non-reducible *orientable* example occurs in dimension 5 (see Gitler and Stasheff [3]).

2. The proof of Theorem A

Here is a short argument to show that an orientable 4-dimensional Poincaré complex has a reducible Spivak normal fibration. The proof is essentially contained in [4].

1. Suppose that X is an orientable 4-dimensional PD-complex such that ν_X is not reducible. Then by Poincaré duality there is a class $e \in H^1(X.\mathbb{Z}/2)$ such that

$$\langle k_3(X) \cup e, [X] \rangle \neq 0,$$

where $k_3(X)$ denotes the pullback to X of the first exotic characteristic class.

2. Let $f : X \to RP^\infty$ represent the cohomology class $e \in H^1(X; \mathbb{Z}/2)$. Then the element $0 \neq (X, f) \in \mathcal{N}_4^{PD}(RP^\infty)$ has Arf invariant $A(X, f) = 1$ (see [4], Corollaries 4.2 and 5.3, and Theorem 5.6).

3. By low-dimensional surgery, we may assume that $\pi_1(X) = \mathbb{Z}/2$ and that $f : X \to RP^\infty$ classifies its universal covering $\widetilde{X} \to X$ (see Wall [10, Corollary 2.3.2] to justify this much Poincaré surgery).

4. The form $B(a, b) = \langle a \cup T^* b, [\widetilde{X}] \rangle$ is a symmetric unimodular bilinear form on $H^2(\widetilde{X}, \mathbb{Z})$, where T denotes the non-trivial covering involution. The form B is even (see Bredon [1, Chapter VII, Theorem 7.4]).

5. The invariant $A(X, f)$ is the Arf invariant associated to the Browder-Livesay quadratic map q (see [2, §4] and [4, Theorem 1.4]), which refines the mod 2 reductions of B. By [2, Lemma 4.6], we have

$$q(a) \equiv \frac{B(a,a)}{2} \pmod 2$$

since $T\colon \widetilde{X} \to \widetilde{X}$ is orientation preserving. But B is an even unimodular symmetric bilinear form, so the Arf invariant obtained in this way is zero, and we have a contradiction. □

Remark 2.1. To obtain the reducibility results for 3-dimensional PD-complexes, one can make an appropriate circle bundle construction (which does not affect reducibility) resulting in orientable 4-dimensional PD-complexes and then apply Theorem A.

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