CYCLIC GROUP ACTIONS ON CONTRACTIBLE 4-MANIFOLDS

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Abstract. There are known infinite families of Brieskorn homology 3-spheres which can be realized as boundaries of smooth contractible 4-manifolds. In this paper we show that smooth free periodic actions on these Brieskorn spheres do not extend smoothly over a contractible 4-manifold. We give a new infinite family of examples in which the actions extend locally linearly but not smoothly.

1. Introduction

The Brieskorn homology spheres $\Sigma(a,b,c)$ provide important examples of Seifert fibered 3-manifolds [29], and have been extensively studied as test cases for questions about smooth 4-manifolds and gauge theory invariants (see Anvari [1], Lawson [24], Fintushel and Stern [16, 17], Saveliev [30]). In this paper we answer a well-known question (asked by Allan Edmonds at Oberwolfach in 1988) about extending smooth free cyclic group actions on $\Sigma(a,b,c)$ to certain smooth 4-manifolds which they bound.

Kwasik and Lawson [22] found an infinite family of Brieskorn homology 3-spheres which admit free $\mathbb{Z}/p$-actions and bound smooth contractible 4-manifolds W, such that the actions extend locally linearly with one fixed point in W, but no such extended action exists smoothly. Their examples come from the list of Casson and Harer [5] of Brieskorn homology 3-spheres which bound smooth contractible 4-manifolds:

$$\begin{aligned} &\Sigma(r, rs-1, rs+1) \quad r \text{ even, } s \text{ odd} \\ &\Sigma(r, rs\pm 1, rs\pm 2) \quad r \text{ odd, } s \text{ arbitrary.} \end{aligned}$$

Necessary and sufficient conditions for a locally linear extension of a free action on an integral homology three sphere to its bounding contractible 4-manifold are contained in the work of Edmonds [10]. To show non-smoothability, Kawsik and Lawson apply the gauge theoretic results of Fintushel and Stern [15] in the orbifold setting.

In this paper we demonstrate a new technique to detect non-smoothability of these actions and apply it to obtain a complete answer:

Theorem A. *A free cyclic group action on a Brieskorn homology 3-sphere $\Sigma(a,b,c)$ does not extend to a smooth action on any contractible smooth 4-manifold W that it bounds.*

Remark 1.1. By P. A. Smith theory, any smooth or locally linear extension of a free cyclic action on $\Sigma(a,b,c)$ to a contractible manifold W must have exactly one fixed point.

Recall that the Brieskorn homology spheres for a,b,c pairwise relatively prime can be realized as the link of a complex surface singularity:

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$$\Sigma(a,b,c) = \{(x,y,z) \in \mathbb{C}^3 \,|\, x^a + y^b + z^c = 0\} \cap S^5$$

with its induced orientation. As a Seifert fibered homology sphere it admits a smooth fixed-point free circle action with three orbits of finite isotropy (see [29]).

The action of $\pi = \mathbb{Z}/p \subset S^1$ contained in the circle action will be free if and only if p is relatively prime to a, b, c. This action is referred to as the *standard π-action* on $\Sigma(a,b,c)$. Luft and Sjerve [25, Prop. 4.3] showed that any smooth free cyclic group action on $\Sigma(a,b,c)$ is conjugate to a standard action.

We give new infinite family of examples admitting locally linear extensions to a contractible W. The examples are contained in the second of the infinite families found by Stern [31]:

$$\begin{aligned}
&\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs - (\pm 1)) \quad r \text{ even, } s \text{ odd} \\
&\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs \pm 2) \quad r \text{ odd, } s \text{ arbitrary} \\
&\Sigma(r, rs \pm 2, 2r(rs \pm 2) + rs \pm 1) \quad r \text{ odd, } s \text{ arbitrary.}
\end{aligned}$$

where we take $s = kp$ for any positive integer k.

Theorem B. *Let r be odd, and let p be an integer relatively prime to $2r(r+1)$. Then for each positive integer k, the standard free action of $\pi = \mathbb{Z}/p$ on $\Sigma(r, rkp \pm 1, 2r(rkp \pm 1) + rkp \pm 2)$ extends to a locally linear π-action on a smooth contractible 4-manifold W that it bounds.*

We point out the following application to equivariant embeddings of $\Sigma(a,b,c)$ in smooth homotopy 4-spheres with semifree π-actions.

Corollary C. *There are infinite families of Brieskorn homology spheres $\Sigma(a,b,c)$ such that the standard free actions of $\pi = \mathbb{Z}/p$ embed equivariantly in homotopy 4-spheres with locally linear π-actions. No such smooth equivariant embeddings of $\Sigma(a,b,c)$ exist into any smooth π-action on a homotopy 4-sphere.*

Here is brief outline of the paper. The links of complex surface singularities that are integral homology three spheres are *plumbed* homology spheres; that is, they can be realized as the boundaries of smooth 4-manifolds obtained by plumbing disk bundles over 2-spheres with an intersection matrix that is negative definite. Among these is the canonical negative definite resolution in that it admits no (-1)-blowdowns. To prove non-smoothability of locally linear extensions, we extend the free action on a Brieskorn homology sphere $\Sigma = \Sigma(a,b,c)$ to its canonical negative definite resolution $M(\Gamma)$ by equivariant plumbing on the resolution graph. From this we form the closed, simply connected 4-manifold

$$X = M(\Gamma) \cup_\Sigma (-W)$$

which by Donaldson's Theorem A [6] has intersection matrix that is diagonalizable over the integers. If the action on W is smoothable, then X admits a smooth $\mathbb{Z}/p$-action which equivariantly splits along a free action on $\Sigma(a,b,c)$. The idea is that the global orientation of the moduli space prevents the configuration of invariant and fixed 2-spheres in $M(\Gamma)$ obtained from plumbing to embed equivariantly and smoothly in a connected sum of linear actions on complex projective spaces. We use equivariant Yang-Mills moduli spaces as developed in Hambleton-Lee [19, 20].

In the next section we collect results from equivariant gauge theory that we will need for the proof of Theorem A, and in Section 5 we prove that locally linear extensions exist for the infinite family in Theorem B. We work out explicit examples for the infinite family $\Sigma(3, 3s+1, 6(3s+1)+3s+2)$.

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2. Equivariant Moduli Spaces

Let (Σ, π) denote a Brieskorn integral homology 3-sphere $\Sigma = \Sigma(a, b, c)$, together with a free action of $\pi = \mathbb{Z}/p$ contained in the natural circle action of the Seifert fibration, and suppose this action extends smoothly to a contractible 4-manifold W.

Conventions. In this section, the notation π denotes a finite cyclic group of *prime* order. We also write $\pi = \mathbb{Z}/p$ to specify the order. All π-actions are smooth and orientation-preserving.

Now consider $(M(\Gamma), \pi)$ to be the canonical negative definite resolution of $\Sigma(a, b, c)$ together with the smooth free π-action extending via equivariant plumbing on the graph Γ. Then $X = M(\Gamma) \cup_\Sigma (-W)$ denotes a simply connected, smooth negative definite 4-manifold together with a homologically trivial $\mathbb{Z}/p$-action. As mentioned in the introduction, our strategy will be to study the equivariant instanton moduli spaces to obtain a contradiction to the action of π extending smoothly to W. We begin this section by collecting results about equivariant Yang-Mills moduli spaces needed to prove non-smoothability of the extension (see [8] and [19]).

Let $P \to X$ denote a principal $SU(2)$-bundle over a closed, smooth and simply connected 4-manifold X whose intersection form is odd and negative definite. By results of Donaldson and Wall, it follows that X is homotopy equivalent to a connected sum of copies of $\overline{\mathbb{C}P}^2$ (see [8]). Suppose that $\pi = \mathbb{Z}/p$ acts smoothly on X inducing the identity on homology. We fix a real analytic structure on X compatible with the group action and a real analytic π-invariant metric, so the action is given by real analytic isometries.

Let $\mathcal{A}$ denote the space of $SU(2)$ connections and $\mathcal{B} = \mathcal{A}/\mathcal{G}$ the space of connections modulo the gauge group $\mathcal{G}$. Since $SU(2)$-bundles are classified by the second Chern class $c_2(P) \in H^4(X; \mathbb{Z})$ and since the π-action on X preserves the orientation, there are lifts of the isometries $g\colon X \to X$ that are generalized bundle maps $\hat{g}\colon P \to P$. Let $\mathcal{G}(\pi)$ denote the group of all lifts, then there is an action of $\mathcal{G}(\pi)$ on the space of connections $\mathcal{A}$ which is well-defined modulo gauge and there is a short exact sequence

$$1 \to \mathcal{G} \to \mathcal{G}(\pi) \to \pi \to 1 \tag{2.1}$$

we then get a well-defined π-action on $\mathcal{B}$. The metric induces a decomposition of 2-forms $\Omega^2(\operatorname{ad} P) = \Omega^2_+(\operatorname{ad} P) \oplus \Omega^2_-(\operatorname{ad} P)$. We are interested in the "charge one" bundle, with $c_2(P) = 1$, and the Yang-Mills moduli space defined by connections modulo gauge with anti-self-dual (ASD) curvature:

$$\mathcal{M}(X) = \{[A] \in \mathcal{B}(P) \mid F_A^+ = 0\} \tag{2.2}$$

Since the curvature is gauge invariant there is a natural π-action on $\mathcal{M}(X)$. The stabilizer $\mathcal{G}_A(\pi)$ has compact isotropy subgroups; when A is irreducible $\Gamma_A = \{\pm 1\}$ and when $[A]$ is reducible, $\Gamma_A = U(1)$ and the associated complex vector bundle $E \to X$ splits as $L \oplus L^{-1}$ for a complex line bundle L over X. The stabilizer $\mathcal{G}_A(\pi)$ is an extension in the short exact sequence

$$1 \to \Gamma_A \to \mathcal{G}_A(\pi) \to \pi_A \to 1 \tag{2.3}$$

where π_A denotes the stabilizer of $[A] \in \mathcal{M}(X)$. The local finite-dimensional model of the moduli space is given by a $\mathcal{G}_A(\pi)$-equivariant Kuranishi map

$$\phi_A\colon H^1_A \to H^2_A \tag{2.4}$$

where H^1_A and H^2_A are the cohomology group of the $\mathcal{G}_A(\pi)$-equivariant fundamental elliptic complex

$$0 \to \Omega^0(X; \operatorname{ad} P) \xrightarrow{d_A} \Omega^1(X; \operatorname{ad} P) \xrightarrow{d_A^+} \Omega^2_+(X; \operatorname{ad} P) \to 0 \tag{2.5}$$

where

$$D_A^+ = d_A^* + d_A^+\colon \Omega^1(X; \operatorname{ad} P) \to \Omega^0(X; \operatorname{ad} P) \oplus \Omega^2_+(X; \operatorname{ad} P)$$

is the linearization of the ASD equation. The *formal dimension* of this moduli space $\mathcal{M}(X)$ is given by the fomula

$$\dim H^1_A - \dim H^0_A - \dim H^2_A = 5 \tag{2.6}$$

computed at any ASD connection. When $H^2_A = 0$, the origin is a regular value for ϕ_A and the infinitesimal deformations in H^1_A can be integrated, so that a neighborhood of such an irreducible ASD connection $[A]$ in the equivariant moduli space $(\mathcal{M}(X), \pi)$ is locally isomorphic to $(\phi^{-1}(0)/\Gamma_A, \pi)$ and gives 5-dimensional manifold charts on the moduli space.

However, in this equivariant setting it is known that there are obstructions to equivariant transversaliy: for example, the virtual representation $[H^1_A] - [H^2_A] \in RO(\pi)$ must be an actual representation. Moreover it may not be possible to make an equivariant perturbation of the ASD equations to get $H^2_A = 0$. Hambleton and Lee in [19] used the notion of equivariant general position as developed by Bierstone [3] and applied it to the setting of Yang-Mills moduli spaces. The idea is to make generic equivariant perturbations chart by chart giving the moduli space the structure of a equivariant stratified space. Here we list the main properties of the instanton moduli space in our setting when X is negative definite.

(i) The equivariant moduli space $(\mathcal{M}(X), \pi)$ is a Whitney stratified space which inherits an effective π-action and has open manifold strata parametrized by isotropy subgroups $\mathcal{M}^*_{(\pi')} = \{[A] \in \mathcal{M}^*(X) \mid A \text{ has isotropy subgroup } \pi_A = \pi'\}$.

(ii) An irreducible connection $[A] \in \operatorname{Fix}(\mathcal{M}^*(X), \pi)$ corresponds to an equivariant lift of the π-action on X to a $\mathcal{G}_A(\pi)$-bundle structure on P, and the connected components of the fixed set in the moduli space correspond to distinct equivalence classes of lifts [4]. In this case, $\mathcal{G}_A(\pi)$ is a (possibly non-split) extension of π by $\{\pm 1\}$. Moreover, the dimension of the fixed set can be computed from the π-fixed set of the fundamental elliptic complex:

$$0 \to \Omega^0(X; \operatorname{ad} P)^\pi \xrightarrow{d_A} \Omega^1(X; \operatorname{ad} P)^\pi \xrightarrow{d_A^+} \Omega^2_+(X; \operatorname{ad} P)^\pi \to 0$$

for a connection $[A]$ in $\mathrm{Fix}(\mathcal{M}^*(X), \pi)$. A fixed stratum is non-empty if its formal dimension is positive. In particular, the *free stratum* $\mathcal{M}^*_{(e)}$ is a 5-dimensional, smooth, noncompact manifold consisting of irreducible ASD connections.

(iii) The strata have topologically locally trivial equivariant cone bundle neighborhoods.

(iv) There is an ideal boundary in the moduli space leading to π-equivariant Uhlenbeck-Taubes compactification $(\overline{\mathcal{M}(X)}, \pi)$ consisting of highly-concentrated ASD connections parametrized by a copy of X:

$$\overline{\mathcal{M}(X)} = \mathcal{M}(X) \cup X$$

where $\mathcal{M}(X)$ has a π-equivariant collar neighborhood diffeomorphic to $X \times (0, \lambda)$ for small λ with the product action being trivial on $(0, \lambda)$.

(v) There are equivariantly transverse charts at each reducible connection; that is $H^2_A = 0$ for each reducible connection $[A]$, and there exists a π-invariant neighborhood which is equivariantly diffeomorphic to a cone over some linear action on complex projective space $\overline{\mathbb{C}P}^2$.

By equivariant general position, the closures of singular strata of dimension ≥ 5 are disjoint from the closure of the free stratum. Moreover, there is a connected component of the free stratum containing the collar and the set of reducible connections. The fixed sets that occur in $\mathcal{M}_{(e)}(X)$ have even codimension. Since the disjoint high-dimensional singular strata play no role in our arguments, *from now on the notation* $\overline{\mathcal{M}(X)}$ *will just mean the closure of the free stratum.*

A final ingredient in the Yang-Mills setting is the map

$$\mu\colon H_2(X;\mathbb{Z}) \to H^2(\mathcal{B}^*;\mathbb{Z})$$

defined in [8, §5.2]. If $\tau\colon X \to \overline{\mathcal{M}(X)}$ denotes the inclusion of the Taubes boundary, then $\tau^*(\mu(\alpha)) = PD(\alpha)$, for any class $\alpha \in H_2(X;\mathbb{Z})$. Furthermore, for the restriction of $\mu(\alpha)$ to the copy of $\mathbb{C}P^\infty$ which links the reducible connection A, we have

$$\mu(\alpha)\,|_{\,\mathrm{lk}[A]} = -\langle c_1(L), \alpha\rangle h$$

where $h \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the positive generator, and $L \to X$ is the complex line bundle in the splitting $E = L \oplus L^{-1}$ induced by A (see [8, 5.1.21]).

Remark 2.7. Interchanging the roles of L and L^{-1} leaves the gauge equivalence class $[A]$ unchanged, but the identifications $\psi_\pm\colon \mathrm{lk}[A] \cong \mathbb{C}P^\infty$ differ by complex conjugation. Hence the right-hand side of this formula is independent of the ordering L, L^{-1} (see [6, Remark 2.29]).

The Yang-Mills moduli spaces inherit a canonical orientation from that of X. The top exterior power of the tangent space $T_{[A]}\mathcal{M} = \mathrm{Ker}\, D^+_A$ can be identified with the determinant line bundle of the elliptic complex

$$\det D^+_A = \Lambda^{\max}(\mathrm{Ker}\, D^+_A) \otimes_{\mathbb{R}} \Lambda^{\max}(\mathrm{Coker}\, D^+_A) \tag{2.8}$$

when $H^2_A = 0$ and in [8] it is shown that the determinant line bundle $\Lambda(P)$ is independent of deformations of A and extends to $\mathcal{B}^*$. Moreover, $\Lambda(P)$ admits a canonical trivialization giving an orientation on the free stratum $\mathcal{M}_{(e)}$ and inducing the orientation

$$\text{(inward pointing normal)} \times \text{(given orientation on } X)$$

on the end of the moduli space as a collar on the equivariant Taubes embedding of (X, π) in $(\mathcal{M}(X), \pi)$ (see [7, p. 426]). The canonical orientation of $\mathcal{M}$ near a link of a reducible connection agrees with that $\mathbb{C}P^2$ (see [7, Example 4.3]).

In the equivariant setting, we will show that there is a preferred generator $h \in H^2(\mathbb{C}P^2; \mathbb{Z})$ at the link of each reducible. An action (X, π), where $\pi = \mathbb{Z}/p$, is *oriented* by fixing a negative definite orientation on X, and a π-equivariant $Spin^c$-structure on X for $p = 2$.

Theorem 2.9 ([20], [21])**.** *Let (X, π) be an oriented action of $\pi = \mathbb{Z}/p$ on X. The fixed set $\operatorname{Fix}(\overline{\mathcal{M}(X)}, \pi)$ is path connected, and inherits a preferred orientation from the π-action on the moduli space.*

Proof. The first statement follows from [20, Theorem C], but for convenience we include an outline of the proof. First we suppose that $\pi = \mathbb{Z}/p$, for p an *odd* prime. In this case, the π-action induces a complex structure on the fibres of any π-equivariant $SO(2)$ bundle. This implies, for example, that a 2-dimensional component $F \subset \operatorname{Fix}(X, \pi)$ inherits a preferred orientation from the given orientation on X, and the complex structure induced by the π-action on the normal bundle of F in X.

In [20, Lemma 8] it is shown that the $\mathbb{Z}/p$-action on the moduli space, for an odd prime p, induces a preferred orientation on the fixed set. The idea is that for any π-fixed ASD connection $[A]$ there is a splitting of the fundamental elliptic complex $\Omega^* = (\Omega^*)^\pi \oplus (\Omega^*)^\perp$ and

$$\Lambda(P) = \Lambda((\Omega^*)^\pi) \otimes \Lambda((\Omega^*)^\perp). \tag{2.10}$$

Since the fixed set has even codimension and p is odd, the action induces a complex structure on $\Lambda((\Omega^*)^\perp)$ and hence a preferred orientation. Together with the canonical orientation of the moduli space, this induces a preferred orientation on the fixed set of any connected component containing $[A]$. The path connectedness of $\operatorname{Fix}(\overline{\mathcal{M}(X)}, \pi)$ is proved in [21, Theorem 3.11], where one key step is to show that every 1-dimensional fixed set in $\mathcal{M}(X)^*$ has at least one reducible connection as a limit point (see [20, Lemma 17]). A counting argument (see [20, p. 729]) now completes the proof.

For involutions ($\pi = \mathbb{Z}/2$), the basic ingredient is a generalization due to Ono [28] of a result of Edmonds [9], namely that a π-equivariant $Spin^c$ structure on (X, π) induces a preferred orientation on each 2-dimensional component of $\operatorname{Fix}(X, \pi)$.

Since our actions are homologically trivial, the existence of a π-equivariant $Spin^c$ structure on (X, π) follows by combining [11, 5.2] and [19, Theorem 6.2]. Finally, recall that the fixed set of a linear involution on $\overline{\mathbb{C}P}^2$ always contains a fixed 2-sphere. The fixed 2-spheres in the links of the reducible connections in the moduli space are part of the 3-dimensional strata in $\mathcal{M}(X)^*$, and by [20, Theorem 16] the closure of each component of these strata must intersect the Taubes boundary in a fixed 2-sphere in $\operatorname{Fix}(X, \pi)$.

This means that the fixed 2-spheres in the links of the reducible connections all inherit a preferred orientation from the choice of a π-equivariant $Spin^c$ structure on (X, π). It follows that any two reducibles are in the closure of exactly one component of the fixed set in $\mathcal{M}(X)^*$. A counting argument again shows that $\operatorname{Fix}(\overline{\mathcal{M}(X)}, \pi)$ is path connected. $\square$

Corollary 2.11. *Let $\{[A_1], [A_2], \dots, [A_n]\}$ denote the set of reducible connections in $\mathcal{M}(X)$. If (X, π) is oriented, then there is a preferred choice of generator $h_i \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ in the link $\mathrm{lk}[A_i] \cong \mathbb{C}P^\infty$ of each reducible. Equivalently, there is a preferred choice of line bundles $\{L_1, L_2, \dots, L_n\}$ such that $E = L_i \oplus L_i^{-1}$ is the splitting induced by A_i, for $1 \le i \le n$.*

Proof. An oriented action (X, π) has a preferred orientation on $\mathrm{Fix}(\overline{\mathcal{M}(X)}, \pi)$, and hence a preferred orientation on the π-fixed or π-invariant 2-spheres in the linear actions on

$$\mathbb{C}P^2 = \mathrm{lk}[A_i] \cap \mathcal{M}(X) \subset \mathbb{C}P^\infty$$

in the link of each reducible. The Poincaré duals of these oriented 2-spheres provide the preferred generators $h_i \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$, for $1 \le i \le n$. We have seen in Remark 2.7 that the choice of generator $\pm h_i$ corresponds to a choice of line bundle $L_i^\pm$. □

The μ-map also provides π-invariant strata in the moduli space, since our actions (X, π) are homologically trivial. The following construction will be used in the next section.

Lemma 2.12. *For any $\alpha \in H_2(X; \mathbb{Z})$, the class $\mu(\alpha) \in H^2(\mathcal{B}^*; \mathbb{Z})$ corresponds to a π-equivariant line bundle $\mathcal{L}_\alpha \to \mathcal{B}^*$. Moreover, there exists an equivariant section s of $\mathcal{L}_\alpha$ restricted to $\mathcal{M}^*(X)$, so that the zero set $V_\alpha = s^{-1}(0)$ is in equivariant general position in the moduli space.*

Proof. For π a finite cyclic group, equivariant line bundles L over a space (Y, π) are classified by a cohomology class $[L] \in H^2(Y \times_\pi E\pi; \mathbb{Z})$. The natural map $H^2(Y \times_\pi E\pi; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ sends $[L] \mapsto c_1(L)$, and a spectral sequence calculation shows that this map is surjective. This shows that there exists a π-equivariant line bundle $\mathcal{L}_\alpha \to \mathcal{B}^*$ with $c_1(\mathcal{L}_\alpha) = \mu(\alpha)$. We may now restrict this line bundle to $\mathcal{M}^*(X)$ and perturb the zero section into equivariant general position by the method of [19]. The perturbation may be chosen so that V_α also intersects the links of the reducible connections in equivariant general position. □

For a class $\alpha \in H_2(X; \mathbb{Z})$ represented by an invariant 2-sphere $F \subset X$, the zero section V_α of $\mathcal{L}_\alpha$ is a stratified codimension two cobordism whose intersection with the Taubes collar may be chosen to be $F = \tau(X) \cap V_\alpha$. The other boundary components provide surfaces in the links of the reducible connections, that are π-invariant under the linear actions on complex projective spaces.

3. Smooth actions on negative definite 4-manifolds

The equivariant moduli space provides an equivariant stratified cobordism that relates a smooth π-action on a negative definite 4-manifold to an equivariant connected sum of linear actions on complex projective spaces.

Example 3.1 (Linear Models)**.** The complex projective plane $\mathbb{C}P^2$ admits linear actions of any finite cyclic group $\pi = \mathbb{Z}/m$, given in homogeneous coordinates by the formula

$$t \cdot [z_0 : z_1 : z_2] = [z_0 : \zeta^a z_1 : \zeta^b z_2], \tag{3.2}$$

where $t \in \pi$ is a generator, $\zeta = e^{2\pi i/m}$ is a primitive root of unity, and a and b are integers such that the greatest common divisor $(a, b, m) = 1$. For these actions, π induces the identity on homology, and the singular set always contains the three fixed points

$x_1 = [1 : 0 : 0]$, $x_2 = [0 : 1 : 0]$, and $x_3 = [0 : 0 : 1]$. In addition, the three projective lines through the points x_i and x_j, for $i \neq j$, are smoothly embedded π-invariant or π-fixed 2-spheres with various isotropy subgroups depending on the values of a and b (see [21, §1]).

Remark 3.3. Let (X, π) denote an orientation-preserving smooth action of a cyclic group $\pi = \mathbb{Z}/m$ on a closed oriented 4-manifold X. If m is odd, then the π-action induces an orientation at a fixed point $x_0 \in F$ for the normal bundle to a smoothly embedded surface $F \subset X$. In this way, any connected π-invariant surface containing a fixed point inherits a preferred orientation. For $m = 2$ we need an extra ingredient, namely the existence of a π-equivariant $Spin^c$ structure, as discussed in the proof of Theorem 2.9.

For example, in the above linear actions on $\mathbb{C}P^2$, the orientation induced by the action on the projective lines $\mathbb{C}P^1 \subset \mathbb{C}P^2$, each representing a primitive generator of $H_2(\mathbb{C}P^2) = \mathbb{Z}$, is the complex orientation.

From now on, we will work with smooth actions of cyclic groups $\pi = \mathbb{Z}/p$ of *prime* order on *negative* definite 4-manifolds $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$. If $p = 2$ there exists a π-equivariant $Spin^c$-structure on (X, π).

Definition 3.4. Let (X, π) be an oriented action on $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$. The *standard* orientation on a connected π-invariant surface containing a fixed point is the orientation induced by the action, if p is an odd prime, or the orientation induced by the π-equivariant $Spin^c$-structure, if $p = 2$.

For an oriented action (X, π) of $\pi = \mathbb{Z}/p$ on $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$, Corollary 2.11 provides a preferred generator $h_i \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$, for $1 \leq i \leq n$, at the link $\mathrm{lk}[A_i]$ of each reducible. This is used in the following important definition.

Definition 3.5. Let (X, π) be a smooth, oriented action on $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$. A diagonal basis $\{e_1, e_2, \ldots, e_n\}$ for $H_2(X; \mathbb{Z})$, with $e_i \cdot e_j = -\delta_{ij}$, is called a *standard* basis if

$$\mu(e_i) \,|\, _{\mathrm{lk}[A_i]} = -\langle c_1(L_i), e_i \rangle h_i = h_i \in H^2(\mathbb{C}P^\infty; \mathbb{Z}),$$

for $1 \leq i \leq n$, where $[A_i]$ denotes the reducible connection in $\overline{\mathcal{M}(X)}$ determined by $\{\pm e_i\}$ and h_i denotes the preferred generator. A standard basis is unique up to re-ordering of the basis elements.

We can construct examples by equivariant connected sums at fixed points. The building blocks use the smooth, oriented π-actions on $\overline{\mathbb{C}P}^2$, given by the formula (3.2). Note that the induced standard orientation on the projective lines is opposite to the complex orientation. We will use the notation $\overline{\mathbb{C}P}^1 \subset \overline{\mathbb{C}P}^2$ for this oriented embedded surface.

The linear models of smooth homologically trivial π-actions on a connected sum $X = \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$ are then obtained by a *tree* of equivariant connected sums, where we connect linear actions on $\overline{\mathbb{C}P}^2$ at fixed points. In order to preserve orientation, the tangential rotation numbers at the attaching points must be of the form (c, d) and $(c, -d)$.

The equivariant moduli space shows that every smooth π-action on an odd negative definite 4-manifold strongly resembles an equivariant connected sum of linear actions.

Theorem 3.6 ([20, Theorem C])**.** *Let (X, π) be a smooth cyclic group action on $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$ inducing the identity on homology. Then there exists an equivariant connected* *sum of linear actions on $\overline{\mathbb{C}P}^2$ with the same fixed point data and tangential isotropy representations.*

Let F denote a fixed 2-sphere for the π-action on an equivariant connected sum of linear actions on $\#_1^{\mathrm{n}}\overline{\mathbb{C}P}^2$. We give F the standard orientation, and then it is clear that the homology class $[F]$ can be written as $\sum_i a_i e_i$ for $a_i \in \{0,1\}$ in the diagonal basis e_i represented as $\overline{\mathbb{C}P^1} \subset \overline{\mathbb{C}P^2}$. The same statement holds for smooth π-actions on $X \simeq \#_1^{\mathrm{n}}\overline{\mathbb{C}P}^2$. If (X,π) is a homologically trivial action, then the fixed set consists of a disjoint union of isolated points and smoothly embedded 2-spheres (see [11, Proposition 2.4]).

Theorem 3.7 ([20, Thm. 16])**.** *Let $\pi = \mathbb{Z}/p$, for p a prime, and (X,π) be an oriented, smooth, homologically trivial action on a smooth 4-manifold $X \simeq \#_1^{\mathrm{n}}\overline{\mathbb{C}P}^2$. Then the integral homology class for each standardly oriented fixed 2-sphere $F \subset X$ is given by an expression:*

$$[F] = \sum_i a_i e_i$$

where $\{e_i\}$ is a standard diagonal basis and $a_i \in \{0,1\}$.

Proof. Since $X \simeq \#_1^{\mathrm{n}}\overline{\mathbb{C}P}^2$, we have a standard diagonal basis $\{e_1,\dots,e_n\}$ for the intersection form on $H_2(X;\mathbb{Z})$. We can express $[F] = \sum_i a_i e_i$, for some integers a_i. Let $\hat{e}_i = PD(e_i)$ be the Poincaré dual to e_i, so that $\langle \hat{e}_i, e_j\rangle = \langle \hat{e}_i \cup \hat{e}_j, [X]\rangle = -\delta_{ij}$. Let L_i denote the corresponding line bundle over X, with $c_1(L_i) = \hat{e}_i$, which provides the reduction $E = L \oplus L^{-1}$ and a reducible ASD connection $[A_i]$ on L_i.

In the compactified equivariant moduli space $\overline{\mathcal{M}(X)}$, the fixed set of the π-action is path connected, by Theorem 2.9. It follows that the links of the reducible connections all inherit the same standard orientation as $\overline{\mathbb{C}P}^2$.

If V denotes the 3-dimensional π-fixed stratum which is the zero set in Bierstone general position for $\mu([F]) = \sum a_i \mu(e_i)$, then V inherits a preferred orientation from the free stratum, and the induced orientation on each component $\partial V_i = V \cap \mathrm{lk}[A_i]$ depends only on its homology class.

Since the fixed strata in the links arise from a linear π-action on complex projective space, we see that $\partial V = F \cup \bigcup \partial V_i$, where each non-empty component ∂V_i in the link $\mathrm{lk}[A_i]$ is a fixed 2-sphere representing the homology class of $\mathbb{C}P^1 \subset \mathbb{C}P^2$. We now evaluate

$$0 = \langle \mu(e_k), [\partial V]\rangle = \langle \mu(e_k), \tau_*[F]\rangle + \sum \langle \mu(e_k), [\partial V_i]\rangle \tag{3.8}$$

But $\langle \mu(e_k), \tau_*[F]\rangle = \langle PD(e_k), [F]\rangle = -a_k$, and

$$\langle \mu(e_k), [\partial V_i]\rangle = -\langle c_1(L_k), e_k\rangle \langle h_k, [\partial V_i]\rangle = \delta_{ik}.$$

since h_k is the positive generator. It follows that the coefficients in $[F] = \sum a_i e_i$ all have values in $\{0,1\}$. □

We will now generalize the statement of Theorem 3.7 to handle smoothly embedded π-invariant 2-spheres. Note that such a 2-sphere is either fixed by π or contains exactly two π-fixed points. In either case, the standard orientation is defined.

Theorem 3.9. *Let $\pi = \mathbb{Z}/p$, for p a prime, and (X, π) be an oriented, smooth, homologically trivial action on a smooth 4-manifold $X \simeq \#_1^n \overline{\mathbb{C}P}^2$. Let $F \subset X$ be a smoothly embedded π-invariant 2-sphere with the standard orientation. Then the homology class $[F] \in H_2(X; \mathbb{Z})$ is given by the formula*

$$[F] = \sum_i a_i e_i$$

where $\{e_i\}$ is a standard diagonal basis and each $a_i \geq 0$.

Remark 3.10. If F does not have the standard orientation, then each $a_i \leq 0$.

Proof. Let F be a smoothly embedded π-invariant 2-sphere in the action (X, π). We assume that F is not π-fixed, hence it contains exactly two isolated fixed points $x_0, x_1 \in F$. Let $\alpha = [F] \in H_2(X; \mathbb{Z})$ and let $V \subset \overline{\mathcal{M}(X)}^*$ be the zero set of an equivariant section (in Bierstone general position) of the line bundle $\mathcal{L}_\alpha$ given by $\mu(\alpha) \in H^2(\mathcal{B}^*; \mathbb{Z})$. We may assume that $V \cap X = F$ at the Taubes boundary, and that $\partial V_i := V \cap \operatorname{lk}[A_i]$ is a π-invariant surface in a linear action $(\overline{\mathbb{C}P}^2, \pi)$ for each reducible connection $[A_i]$.

Note that without additional information, we can only conclude that the π-invariant surfaces ∂V_i are smoothly embedded in $\overline{\mathbb{C}P}^2$ except possibly in small neighbourhoods around the fixed points, where the surfaces might contain cones over π-invariant knots in (S^3, π). At these points the embeddings are only topological (and not locally flat).

However, we observe that the compactification $\overline{V}$ contains two 1-dimensional π-fixed strata of $\overline{\mathcal{M}(X)}^*$ joining each of the isolated fixed points on F to reducible connections, and passing through isolated fixed points on two components, say on ∂V_0 and ∂V_1. By Bierstone general position, the intersections $V \cap \operatorname{lk}[A_i]$ are equivariantly transverse at these points (for $i = 0, 1$). Moreover, the fixed set

$$Z := \operatorname{Fix}(\overline{\mathcal{M}(X)}, \pi) \cap \overline{V}$$

is a tree by [21, Theorem 3.11]. Since each link $(\overline{\mathbb{C}P}^2, \pi)$ has at most three isolated fixed points, and there is a unique path in Z from x_0 to x_1 (up to homotopy), it follows that $\operatorname{Fix}(\partial V_i, \pi)$ contains exactly two fixed points for each non-empty π-invariant surface ∂V_i. At the "initial" component ∂V_0, that contains a fixed point connected to $x_0 \in F$, we also see that the standard orientation on ∂V_0 agrees with the complex "positive" orientation on $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Since Z is connected, each non-empty component ∂V_i also inherits the positive orientation. It follows that $\langle h_k, [\partial V_i] \rangle \geq 0$, and the same calculation given in (3.8) completes the proof. □

Remark 3.11. Note that if $F \subset X$ is π-invariant but not π-fixed, and there exists a π-fixed 2-sphere S, standardly oriented and with $[S]^2 = -1$, then F has the standard orientation if $[F] \cdot [S] = -1$.

4. Proof of Theorem A

The minimal negative definite resolution for a Brieskorn homology sphere is obtained from the dual resolution graph of the singularity whose link is the Brieskorn homology 3-sphere $\Sigma(a_1, a_2, a_3)$ (see Saveliev [30, Ex. 1.17]). For these singularities, the graph is a tree with weight δ on the central node, and weights on the branches given by a continued fraction decomposition $a_i/b_i = [t_{i1}, t_{i2}, ..., t_{im_i}]$ of the Seifert invariants. These weights are uniquely determined by the condition $t_{ij} \leq -2 - a_i < b_i < 0$ and

$$a_1 a_2 a_3 b_i / a_i \equiv 1 \pmod{a_i}, \tag{4.1}$$

where δ satisfies

$$\delta = \frac{-1}{a_1 a_2 a_3} + \sum_{i=1}^{3} \frac{b_i}{a_i} \leq -1. \tag{4.2}$$

Fintushel and Stern defined the R-invariant for Brieskorn homology spheres, which is an odd integer $R(a_1, a_2, a_3) \geq -1$. Moveover, if $\Sigma(a_1, a_2, a_3)$ bounds a smooth contractible manifold, then $R(a_1, a_2, a_3) = -1$ (see [15, Theorem 1.1]). Neumann and Zagier [27] gave the calculation

$$R(a_1, a_2, a_3) = -2\delta - 3.$$

This implies that the central node in the resolution graph of $\Sigma(a_1, a_2, a_3)$ has weight $\delta = -1$.

Equivariant plumbing on the defining graph Γ gives the minimal negative definite resolution $M(\Gamma)$, where each node in the graph is represented by an embedded 2-sphere with self-intersection number given by its weight. The circle action on $\Sigma(a, b, c)$ which arises from its Seifert fibering structure extends over the plumbing (see [29], [26]). By construction, the central node sphere is fixed under the circle action.

By restricting this circle action to $\pi = \mathbb{Z}/p$, for any integer p relatively prime to a, b, c, we obtain a simply connected, smooth 4-manifold $M(\Gamma)$ with a smooth, homologically trivial π-action, whose boundary is $\Sigma = \Sigma(a, b, c)$ with the standard free $\mathbb{Z}/p$-action.

Suppose that the standard free action on $\Sigma(a, b, c)$ also extends smoothly over another compact smooth 4-manifold W with $\partial W = \Sigma$. Then we obtain a smooth, closed 4-manifold

$$X = M(\Gamma) \cup_\Sigma (-W) \tag{4.3}$$

together with smooth, homologically trivial π-action. If W is *acyclic*, meaning that W has the integral homology of a point, and $\pi_1(W)$ is the normal closure of the image of $\pi_1(\Sigma)$, then X will be closed, *simply connected*, smooth 4-manifold with *odd* negative definite intersection form. In other words, $X \simeq \#_1^n \overline{\mathbb{C}P}^2$ where $n = b_2(M(\Gamma))$. To prove Theorem A, it is enough to consider actions of $\pi = \mathbb{Z}/p$ with p prime.

Theorem 4.4. *Suppose $\Sigma(a, b, c)$ bounds a smooth acyclic 4-manifold W, such that $\pi_1(W)$ is the normal closure of the image of $\pi_1(\Sigma(a, b, c))$. If p is a prime with $p \nmid abc$, then a free action of $\pi = \mathbb{Z}/p$ on $\Sigma(a, b, c)$ does not extend to a smooth action on W.*

Proof. We form the manifold $X = M(\Gamma) \cup_\Sigma (-W)$ from the given acyclic manifold W and the plumbed manifold $M(\Gamma)$ as described in (4.3). We have $X \simeq \#_1^n \overline{\mathbb{C}P}^2$ where $n = b_2(M(\Gamma))$. There is a basis for $H_2(X; \mathbb{Z})$ represented by the nodal 2-spheres in the plumbing construction. Since the plumbing is done equivariantly, we obtain a configuration of smoothly embedded π-fixed 2-spheres and π-invariant 2-spheres in X, with at least one π-fixed 2-sphere F_1 of self-intersection -1 (namely the central node in the graph Γ). We fix an ordering on the other nodes so that F_2 and F_3 are adjacent to F_1.

We give each of these 2-spheres the complex orientation and let

$$Q_X \colon H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \to \mathbb{Z}$$

denote the intersection form of X, expressed as a matrix with respect to the basis

$$\mathcal{F} = \{[F_1], [F_2], \dots, [F_n]\}.$$

In other words, Q_X is the plumbing matrix defined by the graph Γ, in which $[F_i]\cdot[F_j] = 1$, for $i \neq j$, whenever this intersection is non-zero.

Let $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ denote a standard diagonal basis given by an (orientation-preserving) homotopy equivalence $X \simeq \#_1^n \overline{\mathbb{C}P}^2$, and the orientation convention given in Definition 3.5. Let C denote the change of basis matrix (with respect to $\mathcal{E}$ and $\mathcal{F}$), so that $C^t Q_X C = -I$ is in diagonal form with respect to the basis $\mathcal{E}$. Then the columns of C give the components of each e_i in terms of the basis $\mathcal{F}$, and similarly the columns of C^{-1} give the expressions for each F_i in terms of the standard diagonal basis $\mathcal{E}$.

Since F_1 is a fixed 2-sphere with $[F_1]\cdot[F_1] = -1$, we may assume that $e_1 = \pm[F_1]$ in the diagonal basis $\mathcal{E}$. Suppose first that $e_1 = [F_1]$. The inverse C^{-1} then has the form

$$C^{-1} = \begin{pmatrix} 1 & -1 & -1 & * & \cdots & * \\ 0 & a_2 & b_2 & * & \cdots & * \\ 0 & a_3 & b_3 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_n & b_n & * & \cdots & * \end{pmatrix} \tag{4.5}$$

where we have labelled the base node F_1 and two adjacent nodes F_2 and F_3, such that $[F_2]\cdot[F_1] = [F_3]\cdot[F_1] = 1$, but $[F_2]\cdot[F_3] = 0$. By construction, F_2 and F_3 are π-invariant (but not fixed) embedded 2-spheres. This configuration always occurs in the plumbing graph for $M(\Gamma)$. By Remark 3.11, the complex orientation for F_2 and F_3 in the plumbing is opposite to the standard orientation.

It follows that $[F_2] = -e_1 + a_2e_2 \dots$, and similarly that $[F_3] = -e_1 + b_2e_2 \dots$. By Theorem 3.9 we can conclude that all the non-zero entries in the second and third column are actually *negative*. On the other hand, since

$$0 = [F_2]\cdot[F_3] = -1 - \sum_{i=2}^{n} a_i b_i$$

and each term $a_i b_i \geq 0$, we have a contradiction. If $e_1 = -[F_1]$, then F_2 and F_3 have the standard orientation and all the non-zero entries in the second and third columns of C^{-1} must be positive (by Theorem 3.9). We obtain a contradiction as before. □

5. Locally Linear Extensions

In this section we briefly survey some results of Edmonds [10] and Kwasik-Lawson [22]. First it should be noted, by the work of Freedman [18], that every integral homology 3-sphere Σ bounds a topological contractible 4-manifold W. That every free action on Σ can be extended to a topological action on a topological contractible 4-manifold was first noted by Ruberman and Kwasik-Vogel [23]. The question of extending a free action of $\pi = \mathbb{Z}/p$ on Σ to a locally linear action on a contractible 4-manifold was studied by Edmonds [10] for p a given prime, including the case of an involution $p = 2$. This work was generalized by Kwasik-Lawson [22] to cover actions of any finite cyclic group.

Conventions. In this section, the notation π denotes a finite cyclic group, not necessarily of prime order. We also write $\pi = \mathbb{Z}/p$ to specify the order. All π-actions are orientation-preserving.

The result for locally linear actions will involve additional spectral and torsion invariants. The equivariant eta invariant is the g-signature defect term for manifolds with boundary. Let $\partial W = \Sigma$ and $Q = \Sigma/\pi$, then the relation between the rho invariants $\rho(Q,\gamma)$ of the orbit space and the equivariant eta invariant is given by

$$\eta_t(\Sigma) = \sum_\gamma \rho(Q,\gamma)\overline{\chi}_\gamma(t), \quad \text{for } t \in \pi,\ t \neq 1, \tag{5.1}$$

where the sum contains values $\chi_\gamma(t)$ of the characters of the irreducible representations γ of $\pi = \mathbb{Z}/p$. There is also a Fourier transform formula [2, 2.8] expressing rho invariants in terms of the equivariant eta invariant η_t:

$$\rho(Q,\gamma) = \frac{1}{p}\sum_{t\neq 1} \eta_t(\Sigma)(\chi_\gamma(t) - \dim(\gamma)). \tag{5.2}$$

As an example that we will use later, the rho invariants of classical lens spaces are given in terms of the represention $\gamma_\ell(t) = e^{2\ell\pi i/p}$:

$$\rho(L(p;r,s),\gamma_\ell) = \frac{4}{p}\sum_{k=1}^{p-1} \cot(\frac{\pi k r}{p})\cot(\frac{\pi k s}{p})\sin^2(\frac{\pi k \ell}{p}) \tag{5.3}$$

which can be easily computed from the above formula using the equivariant eta invariant $\eta_t(S^3)$ of the 3-sphere with the action extending to a disk with rotation number (r,s).

$$\eta_t(S^3) = \frac{(t^r+1)(t^s+1)}{(t^r-1)(t^s-1)}, \quad \text{for } t \in \pi,\ t \neq 1. \tag{5.4}$$

We will also need the notion of Reidemeister torsions before we state the main result about locally linear extensions. This torsion invariant arises from an acyclic chain complex as follows. Give Q a cell structure and let Σ be given the induced cell structure from the regular covering. Then $C_*(\Sigma)$ is a chain complex of free $\mathbb{Z}[\pi]$ modules. Using the natural homomorphisms

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\zeta] \to \mathbb{Q}[\zeta]$$

where $t \mapsto \zeta = e^{2\pi i/p}$, we see that the twisted homology of $C_*(\Sigma)\otimes\mathbb{Q}[\zeta]$ is acyclic with torsion $\Delta(Q)$ in $\mathbb{Q}[\zeta]^\times$. The Reidemeister torsion of the lens space $L(p;r,s)$ is $\Delta(L(p;r,s)) \sim (\zeta^r-1)(\zeta^s-1)$.

Theorem 5.5 (Edmonds [10], Kwasik-Lawson [22, p. 32])**.**

(i) *A free action of $\pi = \mathbb{Z}/p$ on an integral homology 3-sphere Σ extends to a locally linear action on a contractible 4-manifold W with one fixed point if and only if the quotient rational homology sphere $Q = \Sigma/\pi$ is $\mathbb{Z}[\pi]$ h-cobordant to a classical lens space L.*

(ii) *A rational homology sphere $Q = \Sigma/\pi$ is $\mathbb{Z}[\pi]$ h-cobordant to classical lens space L if and only if there is a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L$ under which their rho invariants are equal and the Reidemeister torsions satisfy $\Delta(Q) \sim u^2\Delta(L)$ where u is the image of a unit in $\mathbb{Z}[\pi]$.*

Recall that a $\mathbb{Z}[\pi]$ h-cobordism V between Q and L is one where $H_*(V,Q;\mathbb{Z}[\pi]) = 0$ with local coefficients; equivalently, the $\mathbb{Z}/p$-cover is an integral h-cobordism. To find a locally linear extension, one needs to find a lens space $L(p,q)$ for some integer $q \pmod p$ and a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L(p,q)$ satisfying the conditions above. To do this, start with the classifying map of the cover $Q = \Sigma/\pi$, so a map $f\colon Q \to B\pi$. By general position arguments we can take the image to be a 3-dimensional lens space $L(p;r,s)$ and arrange so that the map is of degree one [10], thus giving a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L(p;r,s)$. When Σ is a Seifert fibered space the following theorem gives the constraint on the lens space:

Theorem 5.6 (Kwasik-Lawson [22, p. 35])**.** *Let Q be a Seifert fibered space with Seifert invariants $\{(a_i,b_i)\}$ with $\alpha\sum b_i/a_i = p$ where α is the product of the a_i. Then there is a degree one map $f\colon Q \to L(p;r,s)$ which is a $\mathbb{Z}[\pi]$-homology equivalence if and only if $\alpha \equiv rs \pmod p$.*

In the case when we have a simple homology equivalence the Reidemeister torsion condition is fulfilled.

Theorem 5.7 (Kwasik-Lawson [22, p. 37])**.** *There is a simple $\mathbb{Z}[\pi]$-homology equivalence between the rational homology sphere $Q = \Sigma(a,b,c)/\pi$ and a lens space $L(p;r,s)$, respecting the orientation and the preferred generators of $H_1(Q)$ and $H_1(L)$, if and only if $\{a,b,c\}$ are congruent to $\{r,s,1\} \pmod p$ up to sign and $abc \equiv rs \pmod p$.*

We now use the above results to show an infinite family in the list of Stern [31] admits locally linear pseudo free extensions to a contractible 4-manifold. First we will need the following

Lemma 5.8. *For each positive integer k, each of the Brieskorn homology 3-spheres $\Sigma(r, rs\pm 1, 2r(rs\pm 1)+rs\pm 2)$ bounds an indefinite smooth 4-manifold X_0 with signature equal to -2.*

Proof. We can realize $\Sigma(r, rs\pm 1, 2r(rs\pm 1)+rs\pm 2)$ as the boundary of the following plumbed indefinite 4-manifold X_0 (see Fickle [13]): The signature is determined via an

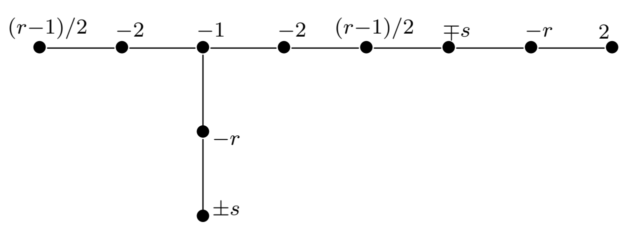


Figure 1. The boundary of this plumbed 4-manifold is the homology 3-sphere $\Sigma(r, rs\pm 1, 2r(rs\pm 1)+rs\pm 2)$.

algorithm which amounts to a graph version of the Gaussian diagonalization process over the rationals (see [12, p. 153]). ☐

Theorem 5.9. *Let r be odd, and let p be an integer relatively prime to $2r(r+1)$. Then for each positive integer k, the standard free action of $\pi = \mathbb{Z}/p$ on $\Sigma(r, rkp\pm 1, 2r(rkp\pm 1)+rkp\pm 2)$ extends to locally linear action on a smooth contractible 4-manifold W with a single fixed point of rotation number $(r, 2r+2)$.*

Proof. There is a simple $\mathbb{Z}[\pi]$-homology equivalence from the quotient

$$Q = \Sigma(r, rkp \pm 1, 2r(rkp \pm 1) + rkp \pm 2)/\pi$$

to the classical lens space $L(p; r, 2r + 2)$. We need to show that these have equivalent rho invariants and we do this by showing that their equivariant eta invariants are equal. Equivariant plumbing (see Fintushel [14, §4]) on the graph in Figure 1 simplifies the computation: we will see that it produces cancelling pairs of rotation numbers .

For each integer a, let $D^2(a)$ denote the unit disk in $\mathbb{C}$ with S^1-action given by $z \mapsto z^a$. Given relatively prime integers a and b, we have a circle action on $D^2(a) \times D^2(b)$ given by the formula

$$z \cdot (re^{i\theta}, se^{i\tau}) = (re^{i(\theta+at)}, se^{i(\tau+bt)}), \tag{5.10}$$

where $z = e^{it} \in S^1$. Write $S^2 = D^2_+ \cup D^2_-$ as the upper and lower hemispheres and consider the trivial D^2-bundle over each hemisphere. The formula in (5.10) defines an S^1-action on the trivial bundle $D^2_+ \times D^2$, and similarly for the lower hemisphere with a and b replaced with c and d. We glue these trivial equivariant bundles together using the map

$$F\colon \partial D^2_+ \times D^2 \to \partial D^2_- \times D^2$$

defined by $F(e^{i\theta}, se^{i\tau}) = (e^{-i\theta}, se^{i(-k\theta+\tau)})$. We obtain an S^1-equivariant D^2-bundle E_k over $S^2 = D^2_+(a) \cup D^2_-(-a)$ with Euler number k, provided that

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{5.11}$$

To equivariantly plumb with another such D^2 bundle we identify over a trivialized hemisphere by interchanging base and fibre coordinates.

The extended $\mathbb{Z}/p$-action is part of the circle action and is therefore isotopic to the identity (hence homologically trivial). We can thus identify the equivariant signature of the manifold X_0 with its usual signature: $\operatorname{sign}(X_0) = -2$ (see Lemma 5.8). The rotation numbers arising from equivariant plumbing on the graph in Figure 1 are

$$(2, r), (2, r), (-1, 2), (-1, 2), (r, -2), (r, -2), (-1, r), (1, r), (r, 2r + 2)$$

and one fixed 2-sphere with self-intersection -1 with rotation number 1 acting on the normal fiber. The Euler characteristic of the fixed set $\chi(\mathrm{Fix}(X_0)) = 11$ and signature equal to -2. After removing the cancelling pairs the G-signature theorem for manifolds with boundary simplifies to

$$\eta_t(\Sigma) = -2\Big(\frac{t+1}{t-1}\Big)\Big(\frac{t^2+1}{t^2-1}\Big) + \frac{4t}{(t-1)^2} - \operatorname{sign}(X_0) + \Big(\frac{t^r+1}{t^r-1}\Big)\Big(\frac{t^{2r+2}+1}{t^{2r+2}-1}\Big). \tag{5.12}$$

It is easy to check that the first three terms above cancel leaving the equivariant eta invariant of the classical lens space $L(p; r, 2r + 2)$ as was to be shown. □

6. An Infinite Family of Examples

In this section we give an infinite family of examples of non-smoothable locally linear extensions.

Example 6.1. The Brieskorn homology 3-sphere $\Sigma = \Sigma(3, 16, 113)$ bounds a smooth contractible 4-manifold W, and admits free $\pi = \mathbb{Z}/5$-action. It is part of the infinite family of the form $\Sigma(r, rs+1, 2r(rs+1)+rs+2)$ given by Stern's examples with $r = 3$ and $s = 5$. It follows from Theorem B that the standard $\mathbb{Z}/5$-action on $\Sigma(3, 16, 113)$ extends to a locally linear action on W with one fixed point whose rotation data is $(3, 3)$. However, Theorem A shows that there is no such smooth action. It follows that $\Sigma(3, 16, 113)$ admits a $\mathbb{Z}/5$-equivariant embedding into a homotopy 4-sphere with a locally linear $\mathbb{Z}/5$-action.

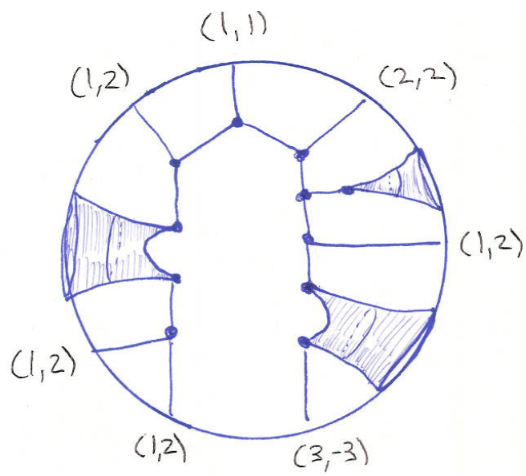


Figure 2. The fixed set pattern in the moduli space $(\mathcal{M}(X), \pi)$ for $\Sigma(3, 16, 113)$. Each vertex in the interior is a reducible connection whose link is a complex projective space with a linear π-action. The isotropy representations then resemble that of an equivariant connected sum of linear actions on complex projective spaces.

The associated negative definite smooth 4-manifold $M(\Gamma)$ has signature -11. Equivariant

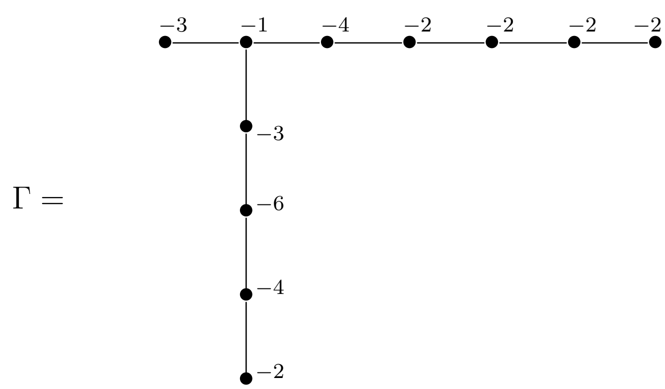


Figure 3. The canonical negative definite plumbing diagram for $\Sigma(3, 16, 113)$.

plumbing beginning with the central vertex produces 6 fixed points with rotation data $\{(1,1), (1,2), (1,2), (1,2), (1,2), (2,2)\}$ and 3 fixed 2-spheres F_1, F_2, F_3, two of which represent homology classes of self-intersection -2 with normal rotation number $c_F = 3$ and one representing a homology class (center vertex) of self-intersection -1 with normal rotation $c_F = 1$. For the locally linear action on $X = M(\Gamma) \cup_{\Sigma(3,16,113)} -W$, we have one additional fixed point with rotation data $(3, -3)$ coming from $-W$.

The intersection form Q_X is given by

$$Q_X = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \tag{6.2}$$

and by Donaldson's Theorem A there exists an invertible integer matrix C such that $C^tQC = -I$, then the change of basis matrix C^{-1} taking the basis in the plumbing diagram to a diagonal basis $\{e_i\}$ can be computed to be

$$C^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{6.3}$$

The fixed 2-spheres are given in terms of a diagonal basis as the first, sixth and tenth columns:

$$\begin{aligned} F_1 &= e_1 \\ F_6 &= -e_6 - e_7 \\ F_{10} &= -e_5 - e_{11}. \end{aligned}$$

The rest of the columns give expressions for the invariant 2-spheres in terms of the diagonal basis. Now any other such matrix is obtained from C by either permutations of the standard basis $\{e_i\}$ or a change of sign ($e_i \mapsto -e_i$) since these are the automorphisms of the standard diagonal form. As we have seen, this information contradicts the existence of the general position equivariant moduli space $\overline{\mathcal{M}(X)}$. Note that Figure 2 shows that this action is not ruled out just by the rotation numbers and the singular set in the moduli space.

Example 6.4. Before finding the general argument presented above, we worked out a particular infinite family of examples. Here is a way to simultaneously diagonalize all their intersection forms. Let (M_k, π) denote the canonical negative definite resolution of

$$\Sigma_k = \Sigma(3, 3kp + 1, 6(3kp + 1) + 3kp + 2)$$

together with an action of $\pi = \mathbb{Z}/p$, for p relatively prime to 6, extending the standard free π-action on Σ_k via equivariant plumbing. If the action also extends to a smooth action

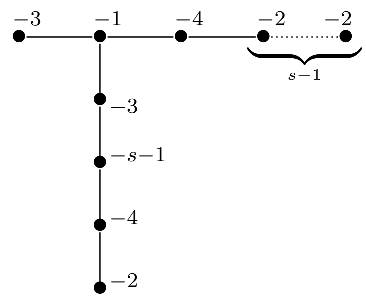
$\Gamma_k =$ 

Figure 4. The canonical negative definite resolution plumbing diagram for $\Sigma(3, 3s + 1, 6(3s + 1) + 3s + 2)$, where $k = sp$.

on a contractible 4-manifold W then $X_k = M_k \cup -W$ is a simply connected, negative definite 4-manifold with a smooth, homologically trivial π-action. The intersection form of X_k is given by the symmetric matrix indexed by k (of size depending on $s = kp$):

$$Q_{X_k} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We now claim that the matrix C^{-1} in $C^t Q_{X_k} C = -I$ is given in terms of a diagonal basis by the following expressions. Those which do not depend on the parameter are given by:

$$F_1 = e_1, \quad F_2 = -e_1 - e_2 + e_3, \quad F_3 = -e_1 + e_2 + e_4, \quad F_5 = -e_2 - e_3 + e_4 + e_6,$$
$$F_6 = -e_6 - e_7, \quad F_7 = -e_1 - e_3 - e_4 + e_8, \quad F_8 = -e_8 + e_9$$

and the rest of the basis is obtained inductively by:

$$F_4 = -e_4 + e_5 - e_8 - e_9 - e_{10} \cdots - e_n, \quad F_n = -e_5 - e_n \quad F_{n-1} = -e_{n-1} + e_n$$

where $n = 6 + s$, with $s \geq 3$. Once again, the point is that there is no consistent choice of sign in the expression of all the $[F_i]$, and moreover one cannot achieve such consistency by an automorphism of the standard form. Thus the actions in Theorem B for $r = 3$ and $s = 5$ do not extend smoothly.

The proof of Corollary C. If $\Sigma = \Sigma(a, b, c)$ is a Brieskorn homology 3-sphere which bounds a smooth contractible 4-manifold W, then the manifold $N = W \cup_\Sigma (-W)$ is a smooth homotopy 4-sphere in which $\Sigma(a, b, c)$ is a smoothly embedded submanifold. Now the examples of Theorem B provide a locally linear extension of the free $\pi = \mathbb{Z}/p$-actions to N with two isolated fixed points. Conversely, suppose that (N, π) is a smooth π-action on a homotopy 4-sphere. Then if (Σ, π) embeds smoothly and equivariantly into N, it follows that the action on N has two isolated fixed points, and that $N = W \cup_\Sigma W'$ is a smooth equivariant decomposition of N as the union of compact 4-manifolds W and W' with boundary Σ. By the van Kampen Theorem, the image of $\pi_1(\Sigma)$ normally generates $\pi_1(W)$, so we obtain a contradiction by Theorem 4.4. □

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