COMPLEXITY OF LINEAR IDEMPOTENT MAL'CEV CONDITIONS

RESULTS ON THE COMPUTATIONAL COMPLEXITY OF LINEAR IDEMPOTENT MAL'CEV CONDITIONS

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Abstract

In this thesis we examine the computational complexity of determining the satisfaction of various Mal'cev conditions. First we present a novel classification of linear idempotent Mal'cev conditions based on the form of the equations with which they are represented. Using this classification we present a class of conditions which can be detected in polynomial time when examining idempotent algebras. Next we generalize an existing result of Freese and Valeriote by presenting another class of conditions whose satisfaction is exponential time hard to detect in the general case, and en route we prove that it is equally hard to detect local constant terms. The final new contribution is an extension of a recent result of Maróti to a subclass class of weak Mal'cev conditions, proving that their detection is decidable and providing a rough upperbound for the complexity of the provided algorithm for said detection. We close the thesis by reviewing the current state of knowledge with respect to determining satisfaction of linear idempotent Mal'cev conditions.

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1 Introduction

Since the development and proliferation of the modern computer it has become increasingly important to understand what can and cannot be computed. Often of a more practical concern is what can be computed with limited resources. The study of algorithmic complexity rose to fulfill this need using tools from numerous disparate mathematical disciplines. One such tool, recently risen to prominence, is the study of Constraint Satisfaction Problems (CSPs) within the framework of universal algebra.

We will begin this introduction with the basics of CSPs and its relationship to universal algebra, and continue with a summary of the results to be presented in the remainder of this document.

1.1 Constraint Satisfaction Problems

The content of this section is derived largely from [9], and that paper should be consulted should the reader desire more information about CSPs. Section 2 covers the basics of universal algebra, and should be consulted if the reader is unfamiliar.

Definition 1.1.1. A *Constraint Satisfaction Problem* is a decision problem where the input to the problem consists of a set A called the domain (or universe), a set V of variables and a set of constraints $(s_i, R_i)_{i\in I}$ where $s_i \subseteq V$ is a subset of the set of variables and $R_i \subseteq A^{s_i}$ is a relation on A of arity $|s_i|$ and the question to be decided is whether or not there is an assignment $f : V \to A$ of variables to elements of the domain such that for every i, $f(s_i) \in R_i$.

Example 1.1.2. We can encode the boolean formula $(\mathbf{x} \vee \neg\mathbf{y}) \wedge (\neg\mathbf{x} \vee \mathbf{z})$ as the following constraint satisfaction problem.

- The domain is the set $\{0, 1\}$.
- The set of variables is $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.
- There are two constraints:
  - A constraint with scope $(\mathbf{x}, \mathbf{y})$ and relation $\{(0, 0), (1, 0), (1, 1)\}$, and
  - A constraint with scope $(\mathbf{x}, \mathbf{z})$ and relation $\{(0, 0), (0, 1), (1, 1)\}$.

Definition 1.1.3. A *relational structure* is a set Γ of relations on A. We will often also refer to a relational structure as a *constraint language* when discussing CSPs. We say that the CSP $\mathcal{C}$ is in $CSP(\Gamma)$ if every relation which is part of a constraint of $\mathcal{C}$ is an element of Γ.

Example. Continuing Example 1.1.2, it is obvious that this CSP is in $CSP(\{\{(0, 0), (1, 0), (1, 1)\}, \{(0, 0), (0, 1), (1, 1)\}\})$, but it is also a CSP of a more interesting relational structure - the set whose elements are all 3-element subsets of $\{0, 1\}^2$. The reason that this relational structure is more interesting is that its associated collection of CSPs represents the problem of determining whether a particular kind of boolean formula is satisfiable. The boolean formulas in question are arbitrary (finite) conjunctions of disjunctions of pairs of propositional variables and negations of propositional variables, and the problem here represented is known as $2-SAT$.

In order to see how CSPs relate to universal algebra we will need one further definition.

Definition 1.1.4. Given a constraint language Γ on A and an (n-ary) operation f on A, say that f is a *polymorphism* of Γ if for every $R \in \Gamma$ and every $a_0, \ldots, a_{n-1} \in R$ we have that $f(a_0, \ldots, a_{n-1}) \in R$ as well, where f is applied coordinatewise.

Clearly (by Definition 2.2.1) whenever we have a constraint language Γ we can see that the underlying set A together with the set of polymorphisms of Γ forms an algebra $\mathbf{A}_\Gamma$. Similarly whenever we have an algebra $\mathbf{A}$ we can construct the constraint language $\Gamma_\mathbf{A}$ whose members are exactly the subalgebras of finite powers of $\mathbf{A}$, in which case the clone of $\mathbf{A}$ is exactly the set of polymorphisms of Γ. As it turns out, the complexity of solving $CSP(\Gamma)$ depends only on the algebra of polymorphisms of Γ when Γ and A are finite (see [7] and [17]). This connection allows us to begin applying the tools of universal algebra to Constraint Satisfaction Problems.

In general, CSPs are NP-complete (see Deinition 2.4.6), but by restricting to specific constraint languages we discover that there are many interesting subclasses of CSPs, many of which are also NP-complete, but others of which are in P (e.g. $2-SAT$ is in P). The central focus in the modern study of constraint satisfaction is a conjecture by Feder and Vardi in [13] which states that for every finite Γ, $CSP(\Gamma)$ is either NP-complete or in P. It is already known that if $\mathbf{A}_\Gamma$ generates a variety which admits type 1 (see Theorem 2.3.9) then $CSP(\Gamma)$ is NP-complete ([8]) and it has been conjectured (also in [8]) that the converse is true as well.

1.2 Results

From the preceding subsection we can see that knowing which Mal'cev conditions (see Definition 2.2.12) are satisfied by the variety an algebra generates can yield information about the tractability of its associated CSP. As such it becomes worthwhile to know for which Mal'cev conditions we can reasonably determine satisfaction by the variety generated by a given algebra. Now let us introduce the main results of this document, each of which provides some information about determining satisfaction of a Mal'cev condition from some restricted class.

The first main result of this document details a kind of condition whose satisfaction can be tractably determined in idempotent algebras. To introduce it we will begin by presenting the result that inspired it.

Theorem 1.2.1. *[15] Let $\mathbf{A}$ be a finite idempotent algebra. $\mathbf{A}$ supports a majority term (a term t such that $t(\mathbf{x},\mathbf{x},\mathbf{y}) = t(\mathbf{x},\mathbf{y},\mathbf{x}) = t(\mathbf{y},\mathbf{x},\mathbf{x}) = \mathbf{x}$) if and only if every triple of elements of $\mathbf{A}^3$ is a majority triple.*

For our purposes we do not need to present the definition of “majority triple”, but we need only know that it is equivalent to satisfying a local version of the equations defining a “majority term”. Whereas the preceding theorem was proven using an intricate knowledge of algebraic semantics, it turns out that it can also be proven through iterated composition of terms satisfying the aforementioned local versions of the majority equations (local terms). Generalizing this concept of local terms (Definition 3.1.1) provides us with a useful framework in which to prove similar equivalences for other conditions.

Example 1.2.2. A Mal’cev operation (not to be confused with a Mal’cev condition) is any ternary operation p satisfying the following two equations.

$$p(\mathbf{x},\mathbf{y},\mathbf{y}) = \mathbf{x} \tag{1}$$
$$p(\mathbf{y},\mathbf{y},\mathbf{x}) = \mathbf{x} \tag{2}$$

To demonstrate the kind of compositions used in the first major result, suppose that p_1 is an operation satisfying equation 2, but failing to satisfy equation 1 for some but not all $\mathbf{x}$’s and $\mathbf{y}$’s and suppose that p_2 is an operation which likewise satisfies equation 1 but not equation 2. Then both p' and p'', defined as follows, are Mal’cev operations (left as an excercise to the reader).

$$p'(\mathbf{x},\mathbf{y},\mathbf{z}) = p_2(p_1(\mathbf{x},\mathbf{y},\mathbf{z}),p_1(\mathbf{y},\mathbf{y},\mathbf{z}),\mathbf{z})$$
$$p''(\mathbf{x},\mathbf{y},\mathbf{z}) = p_1(\mathbf{x},p_2(\mathbf{x},\mathbf{y},\mathbf{y}),p_2(\mathbf{x},\mathbf{y},\mathbf{z}))$$

Both majority and Mal’cev operations are examples of strong term conditions (see Definition 2.5.5). The first major result of this document is to specify a condition on a strong term condition’s defining equations which, if satisfied, lets us know that determining satisfaction of those equations on an idempotent algebra is tractable.

Theorem. *(see Theorem 3.1.5) Fix a strong term condition whose defining equations have the property that for each column with more than one* $\mathbf{y}$*, we may change any single* $\mathbf{y}$ *in that column into an* $\mathbf{x}$ *and the result is still a column of the equations. Satisfaction of this strong term condition in an idempotent algebra can be determined in polynomial time.*

Section 4 bridges the gap between Section 3 and Section 5 by examining local constant terms (terms which are constant on a specific set). Local constant terms are interesting because determining their presence seems to be characteristic of their base set’s status as a subuniverse. In particular, determining whether a term exists which is constant on a specified subuniverse is easy (Lemma 4.1.2) whereas determining whether a term exists which is constant on a specified non-subuniverse set is hard (Theorem 4.2.6).

The next major result of this document generalizes another result of [15], in which it is proven that determining the satisfaction of certain conditions is intractable.

Theorem 1.2.3. *[15] The following problems are all* EXPTIME*-complete:*
Given a finite algebra $\mathbf{A}$*,*

- *does* $\mathbf{A}$ *have a semilattice term operation?*

- *does* $\mathbf{A}$ *generate a variety which omits all of the types in the set* T*, where* T *is one of:*
  - $\{1\}$,
  - $\{1, 2\}$,
  - $\{1, 5\}$,
  - $\{1, 2, 5\}$.
- *does* $\mathbf{A}$ *generate a congruence modular variety?*
- *does* $\mathbf{A}$ *generate a congruence distributive variety?*
- *for a fixed* $n > 3$*, does* $\mathbf{A}$ *have Jónsson level* n*?*

Originally this result was proven by embedding an arbitrary instance of $\mathsf{Gen\text{-}Clo}'$ (which is known from [4] to be $\mathsf{EXPTIME}$-complete) into an algebra $\mathbf{A}$ such that $\mathbf{A}$ has either no nontrivial idempotent operations or $\mathbf{A}$ has $t(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{z})$ as a term operation (treating $\mathbf{A}$ as a flat semilattice). It turns out that only the following two properties of t were used in the proof.

- $t(\mathbf{x}, \mathbf{x}, \mathbf{x}) = \mathbf{x}$
- There is a fixed $0 \in A$ such that for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$ we have that $t(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \{\mathbf{x}, 0\}$.

We call operations satisfying these two properties constant-projection blends, and extend the result as follows.

Theorem. *(see Theorem 5.1.5)Any term condition which is satisfiable by constant-projection blends is* $\mathsf{EXPTIME}$*-hard to detect.*

In particular there are several other useful conditions whose satisfaction we now know is difficult to detect.

Corollary 1.2.4. *The following problems are all* $\mathsf{EXPTIME}$*-complete:*
Given a finite algebra $\mathbf{A}$*,*

- *for a fixed* $n > 2$*, does* $\mathbf{A}$ *generate a variety which is congruence* n*-permutable?*
- *does* $\mathbf{A}$ *generate a variety which omits all types in the set* T*, where* T *is one of:*
  - $\{1, 4, 5\}$,
  - $\{1, 2, 4, 5\}$.
- *for a fixed* $n > 2$*, does* $\mathbf{A}$ *have a weak near unanimity term of arity* n*?*

Lastly we turn our attention to seemingly more difficult conditions.

Definition 1.2.5. A *near unanimity operation* is any operation (of arity at least 3) which, if all but one of its inputs are identical, produces the nearly unanimous input as its output.

It is worth noticing that a majority operation (as in Theorem 1.2.1) is a 3-ary near unanimity operation. As near unanimity operations do not have a specific required arity, it is not obvious that their detection should be decidable, and some earlier results ([25]) seemed to suggest that the opposite is true. The final major result of this document generalizes the following result.

Theorem 1.2.6. *[26] It is decidable whether or not a finite algebra supports a near unanimity operation.*

The proof of this theorem relies on use of "characteristic functions", among which near unanimity operations are easily recognizable. In Section 6 we show that the definition of characteristic functions can be broadened to encompass other kinds of operations, and that the corresponding result demonstrates that presence of one of these other kinds of operations is also decidable. The most relevant of this kind of operation to the study of CSPs is the edge operation (see Definition 2.3.18), as presence of an edge operation enables us to solve the corresponding CSP relatively quickly (see [6]).

2 Universal Algebra

2.1 Notation and Conventions

To begin, we will detail a few pieces of notation and terminology which will be used throughout this document. The first list of definitions concerns sets, functions and the natural numbers, while the second concerns matrices and tuples with variables as entries.

Definition 2.1.1.

- For any $n \in \mathbb{N}$, let $\mathbf{n} = \{0, \ldots, n-1\}$.
- Let $\overline{x} = x_0, \ldots, x_n$ (n will always be clear from context).
- Whenever $\mathbf{x}$ and $\mathbf{y}$ occur, they are treated as syntactic variables, suitable to be replaced by other objects.
- Given a finite set A and $n \in \mathbb{N}$, let $A^{[n]}$ be the set of n-element subsets of A.
- Let $\mathcal{O}_A$ be the set of finite-arity operations on the set A, and for any $\mathcal{F} \subseteq \mathcal{O}_A$ let $\mathcal{F}^{(n)}$ be the set of n-ary operations in $\mathcal{F}$.
- Given a finite set A and a finite set of operations $\mathcal{F}$ on A, let $\langle \mathcal{F} \rangle$ be the smallest subset of $\mathcal{O}_A$ which contains all the projection operations, is closed under composition of functions, and such that $\mathcal{F} \subseteq \langle \mathcal{F} \rangle$.

Definition 2.1.2.

- An $\mathbf{xy}$-matrix is a matrix with entries taken from $\{\mathbf{x}, \mathbf{y}\}$. Let $M_{n\times m}(\{\mathbf{x}, \mathbf{y}\})$ be the set of all $n \times m$ $\mathbf{xy}$-matrices, and $M_n(\{\mathbf{x}, \mathbf{y}\}) = M_{n\times n}(\{\mathbf{x}, \mathbf{y}\})$. Similarly, an $\mathbf{xy}$-tuple is a tuple with entries taken from $\{\mathbf{x}, \mathbf{y}\}$, and we will identify an $\mathbf{xy}$-tuple with the corresponding $\{\mathbf{x}, \mathbf{y}\}$-string, an appropriate row of an $\mathbf{xy}$-matrix, and the transpose of an appropriate column of an $\mathbf{xy}$-matrix.
- Let $E \in M_{n\times m}(\{\mathbf{x}, \mathbf{y}\})$, $0 \leq i < n$ and $0 \leq j < m$. We will denote the ith row of E with E_i, the jth column of E with E^j and the i, j-entry of E with E_i^j, $E_{i,j}$, or $E(i, j)$ interchangeably. Similarly, define $E_* = \{E_i : i < n\}$ and $E^* = \{E^j : j < m\}$.
- If w is an $\mathbf{xy}$-tuple, define $w(a, b)$ to be the tuple obtained by replacing each $\mathbf{x}$ in w by a and each $\mathbf{y}$ by b. Similarly, if f is an n-ary function and $w(\mathbf{x}, \mathbf{y})$ is an n-length $\mathbf{xy}$-tuple, then $f(w(\mathbf{x}, \mathbf{y}))$ is the binary function obtained by composition in the obvious fashion.
- If w is an $\mathbf{xy}$-tuple or the transpose of one, let $w(i)$ denote the ith entry in w.

2.2 Algebras

The utility exhibited by groups, rings, fields, vector spaces, and other such structures naturally leads to the idea that there may be utility in the common generalization of these structures. From this, we develop the idea of an algebra, and begin to investigate its properties. Unless otherwise stated, material in this section is adapted from the work of D. Hobby and R. McKenzie in [20], as well as that of R. McKenzie, G. McNulty and W. Taylor in [31].

Definition 2.2.1. An *algebra* $\mathbf{A}$ is a set A (called the *universe of* $\mathbf{A}$) with a set of finite-arity operations $\mathcal{F}$ on A (called the *basic operations of* $\mathbf{A}$). We say that $\mathbf{A}$ is finite if its universe and set of basic operations are both finite.

Example 2.2.2. Let A be the set of 4-tuples of all natural numbers less than 5, and let $\mathcal{F} = \{\overline{0}, +, m_0, \ldots, m_4\}$ where $\overline{0}$ is the constant operation whose value is $(0,0,0,0)$, $+$ is coordinate-wise addition modulo 5, and $m_c : A \to A$ is coordinatewise multiplication by c modulo 5. Clearly then $\mathbf{A}$ is $(\mathbb{F}_5)^4$ viewed as a vector space over the field $\mathbb{F}_5$.

Definition 2.2.3. Define the *clone of* $\mathbf{A}$ to be the smallest set of operations on A which is closed under composition and which contains the projection functions and the basic operations of $\mathbf{A}$.

We can easily see that not all collections of algebras will easily allow for Cartesian products, as there is no natural way of deciding which functions correspond to each other. For example if $\mathbf{A}$ is an algebra whose only basic operation is ternary and $\mathbf{B}$ is an algebra with two basic operations both of which are binary, there is no clear method of defining basic operations of the product algebra. In order to allow for constructions of this kind, we will define the *signature* of an algebra.

Definition 2.2.4.

- A *signature* is a set of *operation symbols*, each with an associated arity.
- Fix a signature $\mathcal{S}$. By *an algebra with signature* $\mathcal{S}$ we mean an algebra $\langle A, \mathcal{F}\rangle$ such that
  - for every operation symbol f in $\mathcal{S}$ with associated arity n there is a unique basic operation $g \in \mathcal{F}$ with arity n, called the *interpretation of* f *in* $\mathbf{A}$ and written $f^{\mathbf{A}}$, and
  - for every basic operation g of $\mathbf{A}$ there is a symbol $f \in \mathcal{S}$ such that g is the interpretation of f in $\mathbf{A}$.
- A *term* on signature $\mathcal{S}$ is a composition of the operation symbols of $\mathcal{S}$, and we define the interpretation of a term in an algebra by extension of the interpretation of its basic operations.

Notice that Cartesian products of algebras with the same signature can now naturally be constructed through coordinatewise application of corresponding functions, and any such product will possess the same signature as each of its constituent parts. Additionally, the concept of a homomorphism now also applies to functions between algebras of the same signature. Next, we will examine the natural common generalization of subgroups, subrings, etc to this context.

Definition 2.2.5. Let $\mathbf{A} = \langle A, \mathcal{F} \rangle$ be an algebra. Say that $B \subseteq A$ is *a subuniverse of* $\mathbf{A}$ if B is closed under all operations in $\mathcal{F}$. Say that $\mathbf{B} = \langle B, \mathcal{G} \rangle$ is a subalgebra of $\mathbf{A}$ if B is a nonempty subuniverse of $\mathbf{A}$ and if $\mathcal{G} = \{f|_B : f \in \mathcal{F}\}$, the set whose members are the functions in $\mathcal{F}$ restricted to B. For any $S \subseteq A$, let $Sg_{\mathbf{A}}(S)$ denote the smallest subalgebra of $\mathbf{A}$ which contains S, called the *subalgebra of* $\mathbf{A}$ *generated by* S.

Notice that, if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, then they have the same signature.

Next we will examine the natural common generalization of normal subgroups, ideals, and so on.

Definition 2.2.6. Let $\mathbf{A} = \langle A, \mathcal{F} \rangle$ be an algebra. Say that $\alpha \subseteq A^2$ is a *congruence on* $\mathbf{A}$ and write $\alpha \in Con(\mathbf{A})$ if α is an equivalence relation which is closed under coordinatewise application of the basic operations of $\mathbf{A}$. Also define $\mathbf{0}_{\mathbf{A}} = \{(a, a) : a \in A\}$ to be the unique minimal congruence on $\mathbf{A}$ and $\mathbf{1}_{\mathbf{A}} = A^2$ to be the unique maximal congruence on $\mathbf{A}$ with respect to the partial ordering of $\subseteq$.

Example 2.2.7. Let G be any group. Every congruence on G is of the form $\{(a, b) \in G : a, b \in cN, \text{ for some } c \in G\}$ where N is a normal subgroup of G, and every normal subgroup of G gives rise to a distinct congruence of this form. In other words, if $N \trianglelefteq G$ is a congruence class (equivalence class) of $\alpha \in Con(\mathbf{A})$, then the congruence classes of α are precisely the cosets of N in G.

We can see from the preceding example that in many of our available examples a congruence is not actually a subalgebra, rather it is the equivalence relation which partitions the original algebra into cosets of a substructure. This leads us to define the quotient of an algebra by a congruence in the following way.

Definition 2.2.8. Let $\mathbf{A}$ be an algebra and $\alpha \in Con(\mathbf{A})$. Define the *quotient of* $\mathbf{A}$ *by* α, written $\mathbf{A}/\alpha$, to be the algebra with the same signature as $\mathbf{A}$ whose universe is the set of equivalence classes of α and whose basic operations are the application of the basic operations of $\mathbf{A}$ to those classes. Specifically if f is a basic operation of $\mathbf{A}$, then the application of f to the equivalence classes of α proceeds by choosing representatives of each class, applying f to those representatives, and concludes with the equivalence class of that application.

That quotients are well-defined can be easily checked.

In each of the structures from which we are generalizing, there are equations which the basic operations must satisfy, and it is to these we turn next.

Definition 2.2.9. A class of algebras $\mathcal{V}$ of the same signature is called a *variety* if it is the class of all algebras with that signature satisfying a given set of equations on the terms of that signature. Say that a variety is *finitely axiomatizable* if there is a finite set of equations which can be used to define $\mathcal{V}$.

Example 2.2.10. The variety of lattices is the class of algebras whose basic operations consist of two binary functions $\vee$ and $\wedge$ such that $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the identities $\mathbf{x} \vee (\mathbf{x} \wedge \mathbf{y}) = \mathbf{x} = \mathbf{x} \wedge (\mathbf{x} \vee \mathbf{y})$.

A variety can equivalently be defined as any class of algebras of the same signature which is closed under products, subalgebras and homomorphic images (See the HSP Theorem by G. Birkhoff in [10]). Since this document will largely be concerned with varieties generated by a single algebra, we will provide that definition next.

Definition 2.2.11. Given an algebra $\mathbf{A}$ define the *variety generated by* $\mathbf{A}$, written $V(\mathbf{A})$, to be the collection of all homomorphic images of subalgebras of products of $\mathbf{A}$.

There is a relationship between varieties akin to a partial ordering which it is relevant to mention, that of *interpretability*, as it is a step along the path to begin classification of varieties in terms of *Mal'cev conditions*.

Definition 2.2.12. ([16])

- Let $\mathcal{U}$ and $\mathcal{V}$ be varieties and let $\{f_i : i \in I\}$ be the operation symbols of $\mathcal{U}$. Say that $\mathcal{U}$ *is interpretable in* $\mathcal{V}$, written $\mathcal{U} \leq \mathcal{V}$, if for every $i \in I$ there is a $\mathcal{V}$-term t_i of the same arity as f_i, such that for all $\mathbf{A} \in \mathcal{V}$, the algebra $\langle A, t_i^{\mathbf{A}}(i \in I)\rangle$ is a member of $\mathcal{U}$.

- If $\mathcal{U}$ is a variety, say that $\mathcal{U}$ *is finitely presented* if it has finitely many operation symbols and is finitely axiomatizable.

- If $\mathcal{U}$ is a finitely presented variety then the class of all varieties $\mathcal{V}$ with $\mathcal{U} \leq \mathcal{V}$ is called the *strong Mal'cev class defined by* $\mathcal{U}$, and the condition $\mathcal{U} \leq \mathcal{V}$ on $\mathcal{V}$ is called the *strong Mal'cev condition defined by* $\mathcal{U}$.

- If $\{\mathcal{U}_i\}_{i\geq 0}$ is a decreasing sequence of finitely presented varieties (relative to interpretability), then the class $\{\mathcal{V} : \mathcal{U}_i \leq \mathcal{V} \text{ for some } i\}$ is called the *Mal'cev class defined by* $\{\mathcal{U}_i\}_{i\geq 0}$, and the condition of membership in this class is called the *Mal'cev condition defined by* $\{\mathcal{U}_i\}_{i\geq 0}$.

- Say that an operation $t : A^n \to A$ is *idempotent* if $t(a, a, \ldots, a) = a$ for every $a \in A$. Say that an algebra is idempotent if all its basic operations are idempotent. Say that a variety is idempotent if all its elements are idempotent.

- Say that a Mal'cev condition is *idempotent* if the variety (-ies) used to define it is (are) idempotent. Say that a Mal'cev condition is *proper* if it is not equivalent to a strong Mal'cev condition.

- Say that the strong Mal'cev condition defined by $\mathcal{U}$ is *linear* if the equations used to define $\mathcal{U}$ are linear (i.e. do not involve composition of operation symbols). Say that a Mal'cev condition is *linear* if each variety in the sequence which defines it, when considered separately from that sequence, defines a strong linear Mal'cev condition.

Example 2.2.13. Let $\mathcal{U}$ be the variety with one ternary operation symbol f, and whose defining equations are as follows.

$$f(\mathbf{x},\mathbf{x},\mathbf{x}) = \mathbf{x}$$

$$f(\mathbf{y},\mathbf{x},\mathbf{x}) = f(\mathbf{x},\mathbf{y},\mathbf{x}) = f(\mathbf{x},\mathbf{x},\mathbf{y})$$

Then the strong Mal'cev condition defined by $\mathcal{U}$ (later referred to as that of having a weak majority term) is linear as the above equations do not involve composition of operation symbols. On the other hand if we were to include the following additional defining equation, then the strong Mal'cev condition defined by $\mathcal{U}$ would no longer be linear.

$$f(f(\mathbf{y},\mathbf{x},\mathbf{x}),\mathbf{x},\mathbf{x}) = f(\mathbf{y},\mathbf{x},\mathbf{x})$$

Another way to view the concept of a strong Mal'cev condition, is to consider instead the finite set of equations defining a finitely presented variety. A variety $\mathcal{V}$ will satisfy a strong Mal'cev condition exactly if it has terms which satisfy the associated equations. Under this view, a Mal'cev condition is the countable disjunction of a decreasing sequence of such strong Mal'cev conditions. A Mal'cev condition is idempotent, then, if its defining equations imply idempotence of its operation symbols; sometimes it will simply be stated that a Mal'cev condition is idempotent, in which case it is implied that the equations specifying this idempotence are invisibly included among those defining the condition.

In this document we are primarily concerned with linear idempotent Mal'cev conditions, and will be introducing a novel classification of them in section 2.5. Examples of linear idempotent Mal'cev conditions permeate the next section and should aid in clarifying the view of Mal'cev conditions underlying the remainder of this document.

2.3 Congruence Lattices and Omitting Types

Recall that a *lattice* can also be viewed as a partially ordered set with the property that every pair of elements has a unique least upper bound (called their join and written $a \vee b$) and a unique greatest lower bound (called their meet and written $a \wedge b$). Notice that $Con(\mathbf{A})$ naturally forms a lattice for any algebra $\mathbf{A}$, when equipped with the partial ordering $\subseteq$. With this structure, we can examine properties of a congruence lattice such as distributivity and modularity.

Recall 2.3.1. Let L be a lattice.

- Say that L is *modular* if $(a \wedge c) \vee (b \wedge c) = ((a \wedge c) \vee b) \wedge c$ for all $a, b, c \in L$.

- Say that L is *meet-semidistribuive* if for every $a, b, c \in L$ if $a \wedge b = a \wedge c$ then $a \wedge b = a \wedge (b \vee c)$.
- Say that L is *join-semidistributive* if for every $a, b, c \in L$ if $a \vee b = a \vee c$ then $a \vee b = a \vee (b \wedge c)$.
- Say that L is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

Theorem 2.3.2. *[21] Given an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$, the following are equivalent.*

- *For every $\mathbf{B} \in V(\mathbf{A})$, $Con(\mathbf{B})$ is distributive (say that $\mathbf{B}$ is congruence distributive), and*
- *For some $n \geq 2$ there are ternary operations $d_0, \ldots, d_n \in \langle \mathcal{F} \rangle$ such that for all $x, y, z \in A$,*
  - $d_0(x, y, z) = x$,
  - $d_n(x, y, z) = z$,
  - $d_i(x, y, x) = x$ *for all* $i \leq n$,
  - $d_i(x, y, y) = d_{i+1}(x, y, y)$ *for all odd* $i < n$, *and*
  - $d_i(x, x, y) = d_{i+1}(x, x, y)$ *for all even* $i < n$.

Say that any sequence of terms satisfying the conditions in the second part of the preceding theorem is a *sequence of Jónsson terms*. If the equivalent conditions of the theorem are satisfied for some n, we can say that the variety is $CD(n)$. Clearly $CD(n)$ comprises a strong Mal'cev condition, and congruence distributivity as a whole is a Mal'cev condition.

Theorem 2.3.3. *[18] Given an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$, the following are equivalent.*

- *For every $\mathbf{B} \in V(\mathbf{A})$, $Con(\mathbf{B})$ is modular (say that $\mathbf{B}$ is congruence modular), and*
- *There are ternary operations $d_0, \ldots, d_n, q \in \langle \mathcal{F} \rangle$ for some even $n \geq 2$ (call them Gumm terms) such that for all $x, y, z \in A$,*
  - $d_0(x, y, z) = x$,
  - $d_i(x, y, x) = x$ *for all* $i \leq n$,
  - $d_i(x, x, y) = d_{i+1}(x, x, y)$ *for all even* $i < n$,
  - $d_i(x, y, y) = d_{i+1}(x, y, y)$ *for all odd* $i < n$,
  - $d_n(x, y, y) = q(x, y, y)$, *and*
  - $q(x, x, y) = y$.
- *[12] There are 4-ary operations $d_0, \ldots, d_n \in \langle \mathcal{F} \rangle$ (call them Day terms) such that for all $x, y, z, w \in A$,*

– $d_0(x, y, z, w) = x$,
– $d_n(x, y, z, w) = w$,
– $d_i(a, b, b, a) = a$ *for all* $i \leq n$,
– $d_i(a, a, d, d) = d_{i+1}(a, a, d, d)$ *for all even* $i < n$, *and*
– $d_i(a, b, b, d) = d_{i+1}(a, b, b, d)$ *for all odd* $i < n$.

Definition 2.3.4. Given an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ and $n \geq 2$, say that $\mathbf{A}$ is *congruence n-permutable* if for all $\alpha, \beta \in Con(\mathbf{A})$, $\alpha \circ_n \beta = \beta \circ_n \alpha = \alpha \vee \beta$, where $\circ_n$ is defined inductively by:

$$\alpha \circ_n \beta = \begin{cases} (\alpha \circ \beta) & \text{if } n = 2 \\ (\alpha \circ_{n-1} \beta) \circ \beta & \text{if } n > 2 \text{ and } n \text{ is even} \\ (\alpha \circ_{n-1} \beta) \circ \alpha & \text{if } n \text{ is odd} \end{cases}.$$

Note: $\circ$ in the preceding definition denotes relational composition, namely

$$\alpha \circ \beta = \{(x, z) \in A^2 : \exists y \in A, (x, y) \in \alpha, (y, z) \in \beta\}.$$

We may abbreviate congruence 2-permutability by instead referring to congruence permutability.

Theorem 2.3.5. *[19] Given an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ and $n \geq 2$, the following are equivalent.*

- *For every $\mathbf{B} \in V(\mathbf{A})$, $\mathbf{B}$ is congruence n-permutable, and*
- *There are ternary terms $d_0, \ldots, d_n \in \langle \mathcal{F} \rangle$ such that for all $x, y, z \in A$,*
  – $d_0(x, y, z) = x$,
  – $d_n(x, y, z) = z$, *and*
  – $d_i(x, x, y) = d_{i+1}(x, y, y)$ *for all* $i < n$.

Say that a sequence of terms satisfying the conditions in the second part of the preceding theorem is *an n-length sequence of Hagemann-Mitschke terms*. Also, say that a term satisfying the conditions in the second part of the preceding theorem when $n = 2$ is a *Mal'cev term*. Specifically, a Mal'cev operation is a ternary operation t on A such that for all $x, y \in A$,

$$t(x, y, y) = t(y, y, x) = x.$$

Theorem 2.3.6. *[32] Given an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$, the following are equivalent.*

- *$V(\mathbf{A})$ is both congruence permutable and congruence distributive, and*
- *There is a ternary term $p \in \langle \mathcal{F} \rangle$ such that for all $x, y \in A$,*

$$p(x, y, x) = p(x, y, y) = p(y, y, x) = x.$$

Say that a term satisfying the second condition in the preceding theorem is a *Pixley term*.

There are many other term conditions which arise naturally from the study of tame congruence theory, which is the study of how local properties of algebras affect its global structure. In order to understand how these term conditions arise we need to define polynomials, neighbourhoods and traces (see chapter 2 of [20].

Definition 2.3.7. Given an algebra $\mathbf{A}$, say that $p(\overline{x}) \in \mathcal{O}_A^{(n)}$ is a *polynomial of* $\mathbf{A}$ if there is a term operation f on $\mathbf{A}$ of arity $n + k$ for some $k \geq 0$ and elements $\overline{a} \in A^k$ such that

$$p(\overline{x}) = f(\overline{x}, \overline{a}).$$

Say that two algebras on the same underlying set are *polynomially equivalent* if their sets of polynomials are equal.

Definition 2.3.8. Let $\mathbf{A}$ be a finite algebra and $\alpha \in Con(\mathbf{A})$ be minimal among those congruences not equal to $\mathbf{0_A}$.

- Call congruences satisfying the above condition the *minimal congruences of* $\mathbf{A}$.
- Say that $U \subseteq A$ is *an α-minimal set of* $\mathbf{A}$ if U is minimal with respect to containment such that there is a unary polynomial $p(x)$ with $p(A) = U$ and p is not constant on at least one α-class.
- Say that $N \subseteq A$ is *an α-trace of* $\mathbf{A}$ if $|N| > 1$ and there is an α-minimal set U and an α-class C with $N = U \cap C$.
- Let $B \subseteq A$, then the *algebra induced by* $\mathbf{A}$ *on* B, denoted $\mathbf{A}|_B$, is the algebra with universe B whose basic operations consist of the restriction to B of all polynomials of $\mathbf{A}$ under which B is closed.

An α-minimal set of $\mathbf{A}$ or an α-trace of $\mathbf{A}$ may be referred to as an α-minimal set or an α-trace respectively, when $\mathbf{A}$ is clear from the context.

Theorem 2.3.9. *Let $\mathbf{A}$ be a finite algebra and α a minimal congruence of $\mathbf{A}$.*

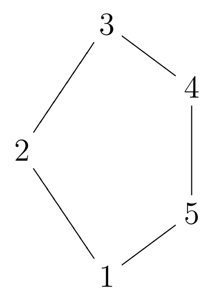
- *If N and M are α-traces, then $\mathbf{A}|_N$ and $\mathbf{A}|_M$ are isomorphic via the restriction of some polynomial of $\mathbf{A}$.*
- *If N is an α-trace then $\mathbf{A}|_N$ is polynomially equivalent to one of:*

  1. *A unary algebra whose basic operations are all permutations (unary type),*
  2. *A one-dimensional vector space over a finite field (affine type),*
  3. *A 2-element boolean algebra (boolean type),*
  4. *A 2-element lattice (lattice type), or*

5. *A* 2*-element semilattice (semilattice type).*

The preceding theorem allows us to assign a type to each minimal congruence, according to the behaviour of its associated traces (the enumeration of the types in the theorem is standard). We can extend this idea by saying that if β covers α in $Con(\mathbf{A})$ (that is, $\alpha < \beta$ and there is no congruence strictly between the two), then the type of β over α is the type of β/α in the algebra $\mathbf{A}/\alpha$. From this we can define the type set of an algebra (the set of types of covering pairs of congruences of the algebra) and the type set of a variety (the union of the type sets of all algebras in the variety). Say that a variety *omits type* i if i does not appear in the type set of the variety.

We can obtain a useful ordering on types by considering the ordering induced by containment on their associated clones on 2-element sets.



The following theorems (2.3.11 through 2.3.16) present characterizations of varieties which omit certain sets of types. In particular there is such a theorem for each order ideal in the natural ordering on types, while the question of whether or not a variety omits a type set which does not form an order ideal is undecidable. It is worth noticing that omission of type 1 and omission of types $\{1, 2\}$ are both strong Mal'cev conditions (when restricted to finitely generated varieties), while omission of any other order ideal is a proper Mal'cev condition ([1]).

Before presenting the theorems which characterize omission of certain type sets, it will be useful to consider one more definition.

Definition 2.3.10. Given $B, C \in M_{m\times n}(\{\mathbf{x}, \mathbf{y}\})$ and n-ary operation f on set A, write $f(B) = f(C)$ if for every $i < m$ and every $a, b \in A$

$$f(B_i(a, b)) = f(C_i(a, b)).$$

Theorem 2.3.11. *Given a finite algebra* $\mathbf{A}$*, the following are equivalent.*

1. $V(\mathbf{A})$ *omits type* 1*,*

2. *There is an* $n \geq 3$ *and an* n*-ary idempotent term* f *on* $\mathbf{A}$ *such that for all* $x, y \in A$

$$f(y, x, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, x, x, \dots, y)$$

(call this a weak near unanimity operation),

3. *There is a* 4*-ary idempotent term* f *on* $\mathbf{A}$ *such that for all* $x, y \in A$

$$f(y, y, x, x) = f(y, x, y, x) = f(x, x, x, y)$$

(call this a Siggers operation), and

4. *For some* $n > 0$ *there are matrices* $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ *and an* n*-ary idempotent term* t *(called a Taylor term) such that*
   - B *has* $\mathbf{x}$*'s on the diagonal,*
   - C *has* $\mathbf{y}$*'s on the diagonal, and*
   - $t(B) = t(C)$ *holds in* $\mathbf{A}$.

Proof. $1 \Leftrightarrow 2$: [27].
$1 \Leftrightarrow 3$: [34].
$1 \Leftrightarrow 4$: Theorem 9.6 in [20]. □

Theorem 2.3.12. *Given a finite algebra* $\mathbf{A}$*, the following are equivalent.*

1. $V(\mathbf{A})$ *omits types* 1 *and* 5*,*

2. *For some* $n \geq 0$ *there are ternary terms* $d_0, \ldots, d_n, p, e_0, \ldots, e_n$ *on* $\mathbf{A}$ *such that for all* $x, y, z \in A$
   - $d_0(x, y, z) = x$*,*
   - $d_i(x, y, y) = d_{i+1}(x, y, y)$*,* $e_i(x, y, y) = e_{i+1}(x, y, y)$ *and* $e_i(x, y, x) = e_{i+1}(x, y, x)$ *for all even* $i < n$*,*
   - $d_i(x, x, y) = d_{i+1}(x, x, y)$*,* $e_i(x, x, y) = e_{i+1}(x, x, y)$ *and* $d_i(x, y, x) = d_{i+1}(x, y, x)$ *for all odd* $i < n$*,*
   - $d_n(x, y, y) = p(x, y, y)$*,*
   - $p(x, x, y) = e_0(x, x, y)$*, and*
   - $e_n(x, y, z) = z$*, and*

3. *For some* $n > 0$ *there are matrices* $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ *and an* n*-ary idempotent term* t *such that*
   - B *has* $\mathbf{x}$*'s on and below the diagonal,*
   - C *has* $\mathbf{y}$*'s on the diagonal, and*
   - $t(B) = t(C)$ *holds in* $\mathbf{A}$.

Proof. Theorem 9.8 in [20]. □

Theorem 2.3.13. *Given a finite algebra* $\mathbf{A}$*, the following are equivalent.*

1. *$V(\mathbf{A})$ omits types 1 and 2,*
2. *$V(\mathbf{A})$ is congruence meet-semidistributive,*
3. *There is a ternary weak near unanimity term p and a 4-ary weak near unanimity term q such that $p(baa) = q(baaa)$ for every $a, b \in A$, and*
4. *For some $n > 0$ there are matrices $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ and an n-ary idempotent term t such that*
   - *B has $\mathbf{x}$'s on the diagonal,*
   - *C has $\mathbf{y}$'s on the diagonal,*
   - *$B_{i,j} = C_{i,j}$ for all $i, j < n$ with $i \neq j$, and*
   - *$t(B) = t(C)$ holds in $\mathbf{A}$.*

Proof. $1 \Leftrightarrow 2$: Theorem 9.10 in [20].
$2 \Leftrightarrow 3$: [1].
$2 \Leftrightarrow 4$: [3]. □

Theorem 2.3.14. *Given a finite algebra $\mathbf{A}$, the following are equivalent.*

1. *$V(\mathbf{A})$ omits types 1, 2 and 5,*
2. *$V(\mathbf{A})$ is congruence join-semidistributive,*
3. *For some $n \geq 0$ there are ternary terms $d_0, \dots, d_n$ on $\mathbf{A}$ (call them Hobby-McKenzie terms) such that for all $x, y, z \in A$*
   - *$d_0(x, y, z) = x$,*
   - *$d_n(x, y, z) = z$,*
   - *$d_i(x, y, y) = d_{i+1}(x, y, y)$ and $d_i(x, y, x) = d_{i+1}(x, y, x)$ for all even $i < n$, and*
   - *$d_i(x, x, y) = d_{i+1}(x, x, y)$ for all odd $i < n$, and*
4. *For some $n > 0$ there are matrices $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ and an n-ary idempotent term t such that*
   - *B has $\mathbf{x}$'s on and below the diagonal,*
   - *C has $\mathbf{y}$'s on the diagonal and $\mathbf{x}$'s below the diagonal, and*
   - *$t(B) = t(C)$ holds in $\mathbf{A}$.*

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3$: Theorem 9.11 in [20].
$2 \Leftrightarrow 4$: [1]. □

Theorem 2.3.15. *Given a finite algebra* $\mathbf{A}$*, the following are equivalent.*

1. $V(\mathbf{A})$ *omits types* 1*,* 4 *and* 5*,*
2. *For some* $n \geq 2$*,* $V(\mathbf{A})$ *is congruence* n*-permutable, and*
3. *For some* $n > 0$ *there are matrices* $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ *and an* n*-ary idempotent term* t *such that*
   - B *has* $\mathbf{x}$*'s on and below the diagonal,*
   - C *has* $\mathbf{y}$*'s on and above the diagonal, and*
   - $t(B) = t(C)$ *holds in* $\mathbf{A}$*.*

Proof. $1 \Leftrightarrow 2$: Theorems 9.13 and 9.14 in [20].
$2 \Leftrightarrow 3$: [1] □

Theorem 2.3.16. *Given a finite algebra* $\mathbf{A}$*, the following are equivalent.*

1. $V(\mathbf{A})$ *omits types* 1*,* 2*,* 4 *and* 5*,*
2. $V(\mathbf{A})$ *is congruence* n*-permutable for some* n *and* $V(\mathbf{A})$ *is congruence meet-semidistributive, and*
3. *For some* $n \geq 0$ *there are* 4*-ary terms* $d_0, \ldots, d_n$ *on* $\mathbf{A}$ *such that for all* $x, y, z \in A$
   - $d_0(x, y, y, z) = x$*,*
   - $d_n(x, x, y, z) = z$*,*
   - $d_i(x, x, y, x) = d_{i+1}(x, y, y, x)$ *and* $d_i(x, x, y, y) = d_{i+1}(x, y, y, y)$ *for all* $i < n$*.*

Proof. Theorem 9.15 in [20]. □

M. Valeriote has also conjectured that the equivalent conditions in theorem 2.3.16 are also equivalent to the following: for some $n > 0$ there are matrices $B, C \in M_n(\{\mathbf{x}, \mathbf{y}\})$ and an n-ary idempotent term t such that

- B is the unique $n \times n$ matrix with $\mathbf{x}$'s on and below the diagonal and $\mathbf{y}$'s above the diagonal,
- C has $\mathbf{y}$'s on and above the diagonal, and
- $t(B) = t(C)$ holds in $\mathbf{A}$.

Clearly this condition implies the final conditions of Theorems 2.3.15 and 2.3.13, though the validity of the reverse implication remains an open question.

There are two more kinds of term conditions that are worth discussing here, as they occur in both the literature and in this document: near unanimity terms and edge terms.

Definition 2.3.17. Say that $t : A^n \to A$ is an n-ary *near unanimity operation* if the following identities hold.

$$t(\mathbf{x}^i \mathbf{y} \mathbf{x}^{n-i-1}) = \mathbf{x} \quad \text{for each } 0 \leq i < n.$$

Near unanimity operations arise naturally in the study of algebras, and any algebra possessing one has many other useful properties. For example if $\mathbf{A}$ has a near unanimity term of arity n then subalgebras of powers of $\mathbf{A}$ are completely determined by their projections onto sets of $n-1$ variables ([5]).

Definition 2.3.18. Say that $t : A^{k+1} \to A$ is a k*-edge operation* if the following identities hold.

$$\begin{aligned} t(\mathbf{y}\mathbf{y}\mathbf{x}^{k-1}) &= \mathbf{x} \\ t(\mathbf{y}\mathbf{x}\mathbf{y}\mathbf{x}^{k-2}) &= \mathbf{x} \\ t(\mathbf{x}^i \mathbf{y} \mathbf{x}^{k-i}) &= \mathbf{x} \quad \text{for each } 2 \leq i \leq k. \end{aligned}$$

Edge operations are one generalization of near unanimity operations and Mal'cev operations (see 2.3.5) and they also bestow interesting properties on algebras. For example $\mathbf{A}$ has an edge term if and only if $\mathbf{A}$ has few subpowers, i.e. there are relatively few subalgebras of finite powers of $\mathbf{A}$ (see [6] for the definition and result). In addition, the following result demonstrates the connection between edge terms and the constraint satisfaction problem.

Theorem 2.3.19. *([6]) If $\mathbf{A} = \langle A, \mathcal{F} \rangle$ supports an edge operation and A is finite then there is a finite set R of relations on A such that the set of all term operations on $\mathbf{A}$ is equal to the set of all operations on A which preserve the relations in R ($\mathbf{A}$ is finitely related).*

2.4 Complexity and Algorithms

The original results in this document focus primarily on the computational complexity of solving various problems about algebras, so it will be necessary to have a basic familiarity with a model of computation before proceeding further. In this section we will outline Turing machines as a precise model of computation and use them to describe a few of the standard computational complexity classes. We will then present the way in which algorithms will be treated throughout the rest of this document.

Except where otherwise noted, the contents of this section are based on [35].

Definition 2.4.1. A *Turing machine* is a structure $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r \rangle$ where

- Q, Σ, and Γ are finite sets,
- Q is the set of *internal states* (or simply states) of M,
- Σ is the *input alphabet* of M, not containing the *blank symbol* $\sqcup$.

- Γ is the *tape alphabet* of M, with $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the *transition function*,
- $q_0 \in Q$ is the *start state*,
- $q_a \in Q$ is the *accept state*, and
- $q_r \in Q$ is the *reject state*, with $q_a \neq q_r$.

Typically a Turing machine is imagined to be a machine which has a read/write head and an infinite-length tape (the tape is considered infinite to the right, but not necessarily to the left). When the Turing machine is started, there is some finite sequence of symbols from Σ on the tape (called the input), and if the Turing machine halts it is said to accept or reject the input if it halts in the accept or reject state respectively. We will only consider inputs which are finite, i.e. inputs which consist of finitely many symbols from Σ followed only by blanks.

At any given time the Turing machine is in some state q with read/write head at position i, and tape contents $w_0w_1 \dots w_{n-1} \in \Gamma^*$ followed by blanks. The Turing machine's action at this stage is to evaluate $\delta(q, w_i) = (q', \gamma, D)$ and then to write symbol γ to position i, move in direction D and change its internal state to q'. For brevity we can refer to each of these applications of δ as a "step".

If the read/write head attempts to move off the left side of the tape, it instead remains at position 0. If the Turing machine enters state q_a or q_r it halts immediately.

To proceed further, we will need the concept of order notation for functions.

Definition 2.4.2. Given $f, g : \mathbb{N} \to \mathbb{N}$, say that $f(n) = O(g(n))$ if there are some constants $c, n_0 \in \mathbb{N}$ such that for all $n > n_0$, $f(n) \leq cg(n)$.

We will evaluate the complexity of the algorithm a Turing machine enacts using two main criteria, namely time and space.

Definition 2.4.3.

- For $w \in \Sigma^n$ say that w has *length or size* n and write $\|w\| = n$.
- The *runtime* or *time complexity* of Turing machine M is the function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ where $f(n)$ is the maximum number of steps that M takes on any input of length n. In this case we may also say that M runs *in time* $f(n)$.
- The *space complexity* of Turing machine M is the function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ where $f(n)$ is the maximum number of non-blank symbols on the tape at any step of M's computation beginning with any input of length n. In this case we may also say that M runs *in space* $f(n)$.

Notice that for any n there are only finitely many inputs of length n, so the only way that ∞ will be in the range of the time or space complexity function for Turing machine M will be if M fails to halt on some input.

It should be fairly clear that for any particular decision problem (problem whose answer is ”yes” or ”no”) there may be numerous (often infinitely many) Turing machines capable of solving it. In order to discuss the complexity of a problem rather than the complexity of a particular machine then, we need to introduce the concept of a language.

Definition 2.4.4.

- For any finite set Σ, the set of *words on* Σ written $\Sigma^* = \bigcup_{k \geq 0} \Sigma^k$ is the set of all finite sequences whose elements are members of Σ.
- For any set Σ, say that L is a Σ-*language* (or simply a *language* when Σ is clear from context) if $L \subseteq \Sigma^*$.
- Say that Turing machine M *decides language* L if M accepts every member of L and rejects every word not in L.
- For language L and word w, write $w \in_? L$ to denote the question of whether or not w is a member of L.

Every decision problem can be reformulated as a problem of deciding whether or not a word is a member of a language, though in practice we usually conceive of decision problems as testing for set membership, as we shall see. First, we will define some well-known classes of languages.

Definition 2.4.5.

- P (polynomial time) is the class of languages L such that $w \in_? L$ can be decided by a Turing machine which runs in time $O(\|w\|^k)$ for some $k \geq 0$.
- PSPACE (polynomial space) is the class of languages L such that $w \in_? L$ can be decided by a Turing machine which runs in space $O(\|w\|^k)$ for some $k \geq 0$
- $\mathsf{EXPTIME}$ (exponential time) is the class of languages L such that $w \in_? L$ can be decided by a Turing machine which runs in time $f(\|w\|)$ where $\log_2 f(\mathbf{x}) = O(\mathbf{x}^k)$ for some $k \geq 0$.
- Say that a language A is *decidable* if there a Turing machine which decides A. Otherwise say that it is *undecidable*.

Typically P is regarded as a reasonable categorization of those problems which are in principle tractable, which is why these particular complexity classes have been the subject of much investigation. One additional complexity class which is often seen is NP (nondeterministic polynomial time), whose definition we must precede with the introduction of one additional concept.

Definition 2.4.6.

- Given a language L say that Turing machine M is a *verifier* for L if:
  - For every $w \in L$ there is a word w' (not necessarily in L) such that M accepts input ww', and
  - For every $w \notin L$ and every word w', M rejects input ww'.
- NP (nondeterministic polynomial time) is the class of languages L such that there is a verifier M which will run in time $O(\|w\|^k)$ for some $k \geq 0$ on every input ww'.
- co-NP is the class of languages whose complement is in NP. In other words $L \subseteq \Sigma^*$ is in co-NP if and only if $\Sigma^* \setminus L$ is in NP.

One classic example of an NP problem is that of Boolean satisfiability (SAT), which consists of all boolean formulas which are satisfiable. To see that SAT is in NP, one need only note that it is easy to check whether a particular assignment of free variables to truth values satisfies a given boolean formula; if w is a satisfiable boolean formula then a satisfying assignment can be supplied as w' in order to verify this fact. All problems in NP share a form similar to that of SAT, in that it is easy to check whether or not a potential solution (like a satisfying assignment of variables) is correct, but finding such a correct solution appears difficult.

The complexity class P (along with the others mentioned above) are, as it turns out, independent of the model of (deterministic) computation in use. In other words if a problem can be solved in polynomial time on a Turing machine, then it can likewise be solved in polynomial time on a many-tape Turing machine, or on a desktop computer (with the assumption of unlimited space). This fact allows us to ignore the specifics of Turing machines when considering the time and space complexity of algorithms, provided we require no finer detail than polynomial. As such, the remainder of this section and the rest of this document will present algorithms as though they are being written for a modern programming language, albeit in pseudo-code (i.e. we will not worry about the syntax of any specific programming language).

It should be fairly clear that $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXPTIME}$. The time hierarchy theorem proves that $\mathsf{P} \neq \mathsf{EXPTIME}$, but no more is known about the relationships between these four classes at the time of writing. This kind of uncertainty regarding the distinctness of various complexity classes has led to the idea of a problem being reducible to another.

Definition 2.4.7.

- Say that a function $f : \Sigma^* \to \Sigma^*$ is *polynomial-time computable* if there is a polynomial-time Turing machine that halts with exactly $f(w)$ on its tape when started on any input w.

- Given two Σ-languages A and B, say that A *is (polynomial-time) reducible to* B, written $A \leq_p B$, if there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that $w \in A$ if and only if $f(w) \in B$.
- Given a complexity class Q and a language A, say that A *is $\leq_p$-hard for* Q (usually shortened to Q-hard) if $B \leq_p A$ for every $B \in \mathsf{Q}$.
- Given a complexity class Q and a language A, say that A *is $\leq_p$-complete for* Q (or Q-complete) if A is $\leq_p$-hard for Q and $A \in \mathsf{Q}$.

In practice we can demonstrate polynomial-time reducibility by describing a polynomial-time algorithm for deciding membership in A which makes use of a B-oracle (a presumed function which decides membership in B in time 1). That this is equivalent to the existence of a relevant polynomial-time computable function is left as an exercise to the reader.

We will end this section with one further definition, and a result which demonstrates the concepts of computational complexity in the setting of universal algebra.

Definition 2.4.8. [15] If $\mathbf{A}$ is a finite algebra, let $|\mathbf{A}| = |A|$ be the cardinality of the universe of $\mathbf{A}$ and let $\|\mathbf{A}\|$ be the input size of $\mathbf{A}$, namely

$$\|\mathbf{A}\| = \sum_{i=0}^{r} k_i |A|^i$$

where k_i is the number of basic operations of arity i and r is the largest arity of a basic operation of $\mathbf{A}$ (in future, use $arity(\mathbf{A})$ to denote this).

Lemma 2.4.9. *[15] Given any $S \subseteq A$ we can compute $Sg_{\mathbf{A}}(S)$ in time $O(arity(\mathbf{A})\|\mathbf{A}\|)$.*

Noticing that $\|\mathbf{A}^k\| \leq \|\mathbf{A}\|^k$ for all k, this tells us that for any $S \subseteq A^k$ we can compute $Sg_{\mathbf{A}^k}(S)$ in time $O(arity(\mathbf{A})\|\mathbf{A}\|^k)$.

2.5 Term Conditions

Here we present a novel classification of idempotent term conditions to be used throughout this document. We will classify term conditions based on properties of the equations whose satisfaction they require by first identifying these term conditions with $\mathbf{xy}$-matrices.

Definition 2.5.1. Given $\mathbf{xy}$-matrices ${}_0E, \ldots, {}_{k-1}E$ with ${}_iE \in M_{m_i \times n}(\{\mathbf{x}, \mathbf{y}\})$, a set A, and an idempotent operation $t : A^n \to A$, say that t *is a* $\{{}_0E, \ldots, {}_{k-1}E\}$*-term (on* A*)* if for all $i < k$ and all rows v and w in ${}_iE$,

$$t(v(a, b)) = t(w(a, b)) \text{ for all } a, b \in A$$

Notice that any linear strong Mal'cev condition involving finitely many idempotent terms on the same algebra can be restated as a condition involving a single term, and hence can be rephrased as such a term condition.

Example 2.5.2. An n-length sequence of Jónsson terms (see Theorem 2.3.2) can be restated as a single term. For example from a 3-length sequence of Jónsson terms $d_0, \ldots, d_3$ we can define a 9-ary term t as follows:

$$t(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = d_1(d_2(x_0, x_1, x_2), d_2(x_3, x_4, x_5), d_2(x_6, x_7, x_8))$$

Then we find that $d_0, \ldots, d_3$ is a 3-length sequence of Jónsson terms if and only if t is an E-term where

$$E = \left\{ \begin{pmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{y} & \mathbf{x} & \mathbf{x} & \mathbf{y} & \mathbf{x} & \mathbf{x} & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \end{pmatrix}, \begin{pmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{y} \end{pmatrix} \right\}.$$

Definition 2.5.3. If $E \in M_{m \times n}(\{\mathbf{x}, \mathbf{y}\})$, A is a set, and $t : A^n \to A$ is an idempotent operation, say that t *is a weak* E*-term (on* A*)* if for all rows v and w in E_*,

$$t(v(a, b)) = t(w(a, b)) \text{ for all } a, b \in A.$$

Notice that t is a weak E-term if and only if t is an $\{E\}$-term.

Example 2.5.4. A Siggers term (also in Theorem 2.3.11) is an idempotent term t such that

$$t(\mathbf{y}, \mathbf{y}, \mathbf{x}, \mathbf{x}) = t(\mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{x}) = t(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{y}).$$

Clearly a Siggers term is a weak $\begin{pmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} \end{pmatrix}$-term.

Definition 2.5.5. If $E \in M_{m\times n}(\{\mathbf{x},\mathbf{y}\})$, A is a set, and $t : A^n \to A$ is an idempotent operation, say that t *is a strong E-term (on A)* if for every $v \in E_*$,

$$t(v(a,b)) = a \text{ for all } a, b \in A.$$

Facts.

- Notice that for any $\mathbf{xy}$-matrix E, t is a strong E-term if and only if t is a weak F-term, where F is the matrix obtained by adjoining a row of $\mathbf{x}$'s to E.
- If E^i is a column of $\mathbf{x}$'s then every clone possesses a strong E-term, namely the projection on the ith coordinate. Since this is a trivial case, it will be ignored after this.
- If E_i is a row of $\mathbf{y}$'s then the only clones possessing a strong E-term are clones on a one-element set. Since this is also a trivial case, it will be ignored after this.

Example 2.5.6. A Mal'cev term (from Definition 4.5 of [20]) is a term t such that

$$t(\mathbf{x},\mathbf{y},\mathbf{y}) = \mathbf{x} = t(\mathbf{y},\mathbf{y},\mathbf{x})$$

Clearly a Mal'cev term is a strong $\begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{x} \end{pmatrix}$-term.

Definition 2.5.7. Given a finite set A, integers $0 < n < m$, $0 < k$ and operations $t : A^n \to A$ and $t' : A^m \to A$, say that t' is a *k-extension of t* if there is an injection $\sigma : \mathbf{n} \to \mathbf{m}$ such that

$$t'(x_0, x_1, \ldots, x_{m-1}) = t(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)})$$

for all $x_0, \ldots, x_{m-1} \in A$, and such that $\sigma(i) = i$ for every $i < k$. Say that t' is an *extension of t* if t' is a k-extension of t for some $k \geq 0$

Essentially, k-extensions of an operation t are those functions which can be obtained from t through the addition of dummy variables and a permutation of the input variables, so long as the first k variables remain fixed.

Definition 2.5.8. Let $\{S_i\}_{i\in\omega}$ be a sequence of finite sets of $\mathbf{xy}$-matrices (or a sequence of $\mathbf{xy}$-matrices), where all $\mathbf{xy}$-matrices in S_i have width n_i, such that for all finite sets A, all $i < j < \omega$, and all idempotent operations $t : A^{n_i} \to A$ where t is an S_i-term (or weak or strong S_i-term) on A, there is an idempotent operation $t' : A^{n_j} \to A$ which extends t and where t' is an S_j-term (or weak or strongS_i-term) on A. (Notice that this implies that if $i < j$ then $n_i \leq n_j$.) Given a finite set A and an idempotent operation $t : A^n \to A$, say that t *is a sequential (weak) $\{S_i\}$-term (on A)* if there is an i such that $n = n_i$ and t is a (weak/strong) S_i-term on A.

Example 2.5.9. As in Example 2.5.2, we can rephrase each n-length sequence of Jónsson terms as an E-term for some E and so we can see that a sequence of Jónsson terms (of unspecified length) can be rephrased as a sequential term condition.

Example 2.5.10. Recall from Definition 2.3.18 that a k-edge term is a $k+1$-ary term t such that

$$\begin{array}{rl} t(\mathbf{y}\mathbf{y}\mathbf{x}^{k-1}) = \mathbf{x} & \\ t(\mathbf{y}\mathbf{x}\mathbf{y}\mathbf{x}^{k-2}) = \mathbf{x} & \text{and} \\ t(\mathbf{x}^{i}\mathbf{y}\mathbf{x}^{k-i}) = \mathbf{x} & \text{for each } 2 < i < k+1 \end{array}.$$

Clearly then t is a k-edge term if and only if t is an E-term, where

$$E = \begin{pmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} & & \mathbf{x} & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{x} & \dots & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} & & \mathbf{x} & \mathbf{x} \\ & \vdots & & & \ddots & & \vdots \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \dots & \mathbf{x} & \mathbf{y} \end{pmatrix}$$

So possession of an edge term (of unspecified arity) can be rephrased as a sequential strong term condition.

It may be worth noticing that the classification of term conditions here presented does not rely on its defining equations containing only two variables. The only point of confusion when increasing the number of available variables regards strong term conditions, but since any equation of the form $t(w) = \sigma$ (where t is a term symbol, Σ is a set of variables with $\sigma, \mathbf{x} \in \Sigma$ and $w \in \Sigma^*$) is equivalent to one of the form $t(w') = \mathbf{x}$ (where $w' \in \Sigma^*$ with $\|w\| = \|w'\|$), this does not pose a barrier to extending the definition of strong term conditions.

3 Local Term Conditions

3.1 The Local-Global Property

When considering whether or not an algebra supports a term operation satisfying particular equations, it is often helpful to instead find terms which behave similarly on small subsets of the algebra, as these may be easier to detect. In this section we examine a class of idempotent strong term conditions (see Definition 2.5.5) and the corresponding local terms. From this examination we conclude that the presence of a sufficient set of local terms guarantees us that a term satisfying the appropriate equations is present, and so we derive that we can quickly detect the presence of such a global term in an idempotent algebra.

Definition 3.1.1. Given a finite set A, matrix $E \in M_{m\times n}(\{\mathbf{x}, \mathbf{y}\})$, operation $t \in \mathcal{O}_A^{(n)}$, and $S \subseteq A^2 \times \mathbf{m}$, say that t is a local strong E-operation on S if t is idempotent and if for each $(a, b, i) \in S$ it is true that $t(E_i(a, b)) = a$.

Clearly t is a strong E-term if and only if t is a local strong E-term on $A^2 \times \mathbf{m}$.

Definition 3.1.2. Say that $m \times n$ $\mathbf{xy}$-matrix E has the *local-global property of size* k where $k > 0$ if for all finite algebras $\mathbf{A}$, $\mathbf{A}$ has a strong E-term if and only if $\mathbf{A}$ has local strong E-terms on S for every $S \in (A^2 \times \mathbf{m})^{[k]}$.

Say that E has the *local-global property* if it has the local-global property for some k.

Notice that one direction of the local-global property is trivial. Namely, if an algebra has a strong E-term then it has a local strong E-term on every subset.

Lemma 3.1.3. *For any $m \times n$ $\mathbf{xy}$-matrix E which has the local-global property there is a polynomial-time algorithm to determine whether or not an idempotent algebra has a strong E-term.*

Proof. By the definition of the local-global property there is a $k > 0$ such that an algebra $\mathbf{A}$ will have a strong E-term if and only if it has local strong E-terms on S for all $S \in (A^2 \times \mathbf{m})^{[k]}$.

For each $S \in (A^2 \times \mathbf{m})^{[k]}$ construct matrices B and V as follows. For each $(a, b, j) \in S$ attach row $E_j(a, b)$ to the bottom of B and attach a to the bottom of matrix V. This will result in a $k \times n$ matrix B and a $k \times 1$ matrix V such that $\mathbf{A}$ will have a local strong E-term on S if and only if $V \in Sg_{\mathbf{A}^k}(B^*)$. Therefore we can search for a strong E-term by testing this subalgebra membership for each $S \in (A^2 \times \mathbf{m})^{[k]}$. □

Algorithm.
Fixed: $m \times n$ $\mathbf{xy}$-matrix E with the local-global property of size k
Input: Finite idempotent algebra $\mathbf{A}$
Output: Whether or not $\mathbf{A}$ has a strong E-term.
Runtime: $O(arity(\mathbf{A})nm^k|A|^{2k}\|\mathbf{A}\|^k)$

Step	Runtime	Repetitions
(1) For each $S \in (A^2 \times \mathbf{m})^{[k]}$ do:		$O(m^k\|A\|^{2k})$
(1.1) Construct matrices B and V	$O(kn)$	
(1.2) Test if $V \in Sg_{\mathbf{A}^k}(B^*)$	$O(arity(\mathbf{A})\\|\mathbf{A}\\|^k)$	

Note that the runtime of step 1.2 in the preceding algorithm is obtained from Lemma 2.4.9.

Now that the utility of the local-global property is clear, we find ourselves in need of a characterization of $\mathbf{xy}$-matrices which possess it. A partial characterization follows, along with a demonstration that this characterization is not complete.

Definition 3.1.4. Say that $E \in M_{m\times n}(\{\mathbf{x},\mathbf{y}\})$ satisfies the *downward column condition* (DCC) if $E^* \cup \{(\mathbf{x}^m)^T\}$ is an order ideal (a downward closed set) in $\{\mathbf{x},\mathbf{y}\}^m$ where $\{\mathbf{x},\mathbf{y}\}$ has partial order $\mathbf{x} \leq \mathbf{y}$.

Equivalently, E satisfies the DCC if for every $w \in E^*$, if w has more than one $\mathbf{y}$ then for every $i < m$ with $w(i) = \mathbf{y}$ there is a $v \in E^*$ such that $v(i) = \mathbf{x}$ and $w(j) = v(j)$ for each $j \neq i$.

Theorem 3.1.5. *If $E \in M_{m\times n}(\{\mathbf{x},\mathbf{y}\})$ satisfies the DCC then it has the local-global property of size m.*

Proof. Let $\mathcal{F}$ be a clone on finite set A such that for each $S \in (A^2\times\mathbf{m})^{[m]}$ there is a $t_S \in \mathcal{F}$ such that t_S is a local strong E-term on S. It suffices to prove that for each $S \subseteq A^2 \times \mathbf{m}$ there is $t_S \in \mathcal{F}$ which is a local strong E-term on S, and we will prove this by induction on $|S|$.

We are given that our claim is true when $|S| = m$, so let us assume that our claim is proven for all $S \subseteq A^2 \times \mathbf{m}$ with $|S| < k$ and choose any $S \in (A^2 \times \mathbf{m})^{[k]}$. Since $k > m$, we can choose an $(a, b, j) \in S$ such that $|S \cap (A^2 \times \{j\})| > 1$ and define

$$T = S \setminus \{(a, b, j)\} \text{ and}$$

$$R = (S \setminus (A^2 \times \{j\})) \cup \{(a, t_T(E_j(a, b)), j)\}.$$

Clearly $|T| = |S| - 1$ and $|R| < |S|$, so we have appropriate terms t_T and t_R.

For each $i < m$ define

$$z_i(\overline{x}) = \begin{cases} x_i & \text{if } E(j,i) = \mathbf{x} \\ t_T(\overline{x}) & \text{if } E^i \text{ has exactly one } \mathbf{y}\text{, at } E^i_j \\ t_T(E_j(x_q, x_i)) & \text{else} \end{cases}$$

where, in the third case, $q < n$ is such that E^i and E^q differ only at position j, with $E(j, i) = \mathbf{y}$ and $E(j, q) = \mathbf{x}$ (The DCC gives us this and this is the only step in the proof where the DCC is used). Clearly, each of these $z_i(\overline{x})$ is in $\langle\mathcal{F}\rangle$. Then let us define $t_S = t_R(z_0(\overline{x}), z_1(\overline{x}), \ldots, z_{n-1}(\overline{x}))$. Clearly $t_S \in \langle\mathcal{F}\rangle$ and so we must now prove that t_S is a local strong E-term on S.

Case 1: Consider the triple (a, b, j). We know that

$$t_S(E_j(a, b)) = t_R(z_0(E_j(a, b)), z_1(E_j(a, b)), \ldots, z_{n-1}(E_j(a, b))).$$

For each $i < n$:

- If $E(j, i) = \mathbf{x}$ then $z_i(E_j(a, b)) = a$,
- If $E(j, i) = \mathbf{y}$ and E^i has no other $\mathbf{y}$'s then $z_i(E_j(a, b)) = t_T(E_j(a, b))$,
- Otherwise, $z_i(E_j(a, b)) = t_T(E_j(a, b))$ since $E^i_j = \mathbf{y}$ and $E^q_j = \mathbf{x}$.

Therefore, $t_S(E_j(a, b)) = t_R(E_j(a, t_T(E_j(a, b)))) = a$ since $(a, t_T(E_j(a, b)), j) \in R$.

Case 2: For each $(c, d, j) \in T$, for each $i < n$:

- If $E(j, i) = \mathbf{x}$ then $z_i(E_j(c, d)) = c$,
- If $E(j, i) = \mathbf{y}$ and E^i has no other $\mathbf{y}$'s then $z_i(E_j(c, d)) = t_T(E_j(c, d)) = c$ since $(c, d, j) \in T$,
- Otherwise, $z_i(E_j(c, d)) = t_T(E_j(c, d)) = c$ since $(c, d, j) \in T$.

Therefore, $t_S(E_j(c, d)) = t_R(E_j(c, c)) = c$ since t_R is idempotent.

Case 3: For each $(c, d, j') \in T$ with $j' \neq j$, for each $i < n$:

- If $E(j, i) = \mathbf{x}$ then $z_i(E_{j'}(c, d)) = E^i_{j'}(c, d)$,
- If $E(j, i) = \mathbf{y}$ and E^i has no other $\mathbf{y}$'s then $z_i(E_{j'}(c, d)) = t_T(E_{j'}(c, d)) = c = E^i_{j'}(c, d)$ since $(c, d, j') \in T$ and $E^i_{j'} = \mathbf{x}$,
- Otherwise, $z_i(E_{j'}(c, d)) = t_T(E_j(E^q_{j'}(c, d), E^i_{j'}(c, d))) = E^i_{j'}(c, d)$ since t_T is idempotent and $E^i_{j'} = E^q_{j'}$.

Therefore, $t_S(E_{j'}(c, d)) = t_R(E_{j'}(c, d)) = c$ since $(c, d, j') \in R$.

These three cases show that t_S is a local strong E-term on S, and since S was arbitrary this proves that this applies to any $S \in (A^2 \times \mathbf{m})^{[k]}$, completing the induction. □

A similar result to Theorem 3.1.5 was independently arrived at by McKenzie, regarding fixed-arity near unanimity terms and Mal'cev terms ([29]). Using quite different methods, polynomial-time algorithms were already known for majority terms (3-ary near unanimity terms) and Mal'cev terms in [15]. The corollaries after the following note replicate and expand slightly on these existing results.

It is worth noticing that Theorem 3.1.5 can be generalized to matrices whose entries belong to any finite set of variables, provided that the DCC is also generalized to refer to those matrices whose column set together with $\{\mathbf{x}^m\}$ forms an order ideal in the appropriate power of the flat semilattice with $\mathbf{x}$ at the root ($\mathbf{x}$ is the unique minimum element and no two other elements are comparable). No useful strong term conditions requiring more than two variables to be expressed are known to the author at the time of writing, however some useful examples of the listed result using two variables follow.

Corollary 3.1.6. *For fixed $n > 2$, there is a polynomial-time algorithm to decide whether or not an idempotent algebra supports an n-ary near unanimity term.*

Proof. An n-ary near unanimity term is a strong E_n-term where

$$E_n = \overbrace{\begin{pmatrix} \mathbf{y} & \mathbf{x} & \mathbf{x} & & \mathbf{x} \\ \mathbf{x} & \mathbf{y} & \mathbf{x} & \dots & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{y} & & \mathbf{x} \\ & \vdots & & \ddots & \vdots \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \dots & \mathbf{y} \end{pmatrix}}^{n}$$

□

Corollary 3.1.7. *For fixed $k \geq 2$, there is a polynomial-time algorithm to decide whether or not an idempotent algebra supports a k-edge term.*

Proof. A k-edge term is a strong E_k-term where

$$E_k = \overbrace{\begin{pmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} & & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} & \mathbf{x} & \dots & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} & & \mathbf{x} \\ & & \vdots & & \ddots & \vdots \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \dots & \mathbf{y} \end{pmatrix}}^{k+1}$$

□

While the downward column condition is sufficient to prove that enough local E-terms necessitate a global E-term, the DCC is not a necessary condition, as the following lemma demonstrates. Notice that the defining matrix for a Pixley term does not satisfy the DCC.

Lemma 3.1.8. *A Pixley term (see Theorem 2.3.6) on $\mathbf{A}$ is a ternary term t such that for each $a, b \in A$, $t(a, b, a) = a$, $t(a, b, b) = a$ and $t(b, b, a) = a$. If $\mathbf{A}$ has a local Pixley term on S for each $(A^2 \times \mathbf{3})^{[3]}$ then it has a (global) Pixley term.*

Equivalently: $\begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{x} \end{pmatrix}$ *has the local-global property of size* 3.

Proof. Clearly a Pixley term is a strong $\begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{x} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{x} \end{pmatrix}$-term, this proof will proceed largely along the same lines as the proof of Theorem 3.1.5.

Assume that for each $i < k$ and each $S \in (A^2 \times \mathbf{3})^{[i]}$ there is a term t_S on $\mathbf{A}$ which is a local Pixley term on S. Given any $S \in (A^2 \times \mathbf{3})^{[3]}$, we will construct a local Pixley term on S. Choose $(a, b, j) \in S$ such that $|S \cap (A^2 \times \{j\})| > 1$.

Case 1: If $j = 0$, define

$$T = S \setminus \{(a, b, 0)\},$$
$$Q = \{(d, c, 1) : (c, d, 2) \in T\} \cup \{(d, c, 2) : (c, d, 1) \in T\} \cup \{(a, t_T(a, b, a), 0)\},$$
$$R = S \setminus (A^2 \times \{0\}) \text{ and}$$
$$t_S(x_0, x_1, x_2) = t_R(x_0, t_Q(x_0, t_T(x_0, x_1, x_2), x_2), x_2).$$

(1.1) Examining $(a, b, 0)$,

$$\begin{array}{rl|l} t_S(a,b,a) = & t_R(a, t_Q(a, t_T(a,b,a), a), a) & \\ = & t_R(a,a,a) & \text{since } (a, t_T(a,b,a), 0) \in Q \\ = & a & \text{since } t_R \text{ is idempotent} \end{array}$$

(1.2) For any $(c, d, 0) \in T$,

$$\begin{array}{rl|l} t_S(c,d,c) = & t_R(c, t_Q(c, t_T(c,d,c), c), c) & \\ = & t_R(c, t_Q(c,c,c), c) & \text{since } (c,d,0) \in T \\ = & t_R(c,c,c) & \text{since } t_Q \text{ is idempotent} \\ = & c & \text{since } t_R \text{ is idempotent} \end{array}$$

(1.3) For any $(c, d, 1) \in S$,

$$\begin{array}{rl|l} t_S(c,d,d) = & t_R(c, t_Q(c, t_T(c,d,d), d), d) & \\ = & t_R(c, t_Q(c,c,d), d) & \text{since } (c,d,1) \in T \\ = & t_R(c,d,d) & \text{since } (d,c,2) \in Q \\ = & c & \text{since } (c,d,1) \in R \end{array}$$

(1.4) For any $(c, d, 2) \in S$,

$$\begin{array}{rl|l} t_S(d,d,c) = & t_R(d, t_Q(d, t_T(d,d,c), c), c) & \\ = & t_R(d, t_Q(d,c,c), c) & \text{since } (c,d,2) \in T \\ = & t_R(d,d,c) & \text{since } (d,c,1) \in Q \\ = & c & \text{since } (c,d,2) \in R \end{array}$$

So t_S is a local Pixley term on S.

Case 2: If $j = 1$, define

$$T = S \setminus \{(a, b, 1)\},$$
$$R = (S \cap (A^2 \times \{2\})) \cup \{(c, t_T(c, d, d), 0) : (c, d, 0) \in T\} \cup \{(a, t_T(a, b, b), 1)\} \text{ and}$$
$$t_S(x_0, x_1, x_2) = t_R(x_0, t_T(x_0, x_1, x_1), t_T(x_0, x_1, x_2)).$$

(2.1) Examining $(a, b, 1)$,

$$\begin{aligned} t_S(a,b,b) &= t_R(a, t_T(a,b,b), t_T(a,b,b)) && \\ &= a && \text{since } (a, t_T(a,b,b), 1) \in R \end{aligned}$$

(2.2) For any $(c, d, 0) \in S$,

$$\begin{aligned} t_S(c,d,c) &= t_R(c, t_T(c,d,d), t_T(c,d,c)) && \\ &= t_R(c, t_T(c,d,d), c) && \text{since } (c,d,0) \in T \\ &= c && \text{since } (c, t_T(c,d,d), 0) \in R \end{aligned}$$

(2.3) For any $(c, d, 1) \in T$,

$$\begin{aligned} t_S(c,d,d) &= t_R(c, t_T(c,d,d), t_T(c,d,d)) && \\ &= t_R(c,c,c) && \text{since } (c,d,1) \in T \\ &= c && \text{since } t_R \text{ is idempotent} \end{aligned}$$

(2.4) For any $(c, d, 2) \in S$,

$$\begin{aligned} t_S(d,d,c) &= t_R(d, t_T(d,d,d), t_T(d,d,c)) && \\ &= t_R(d, d, t_T(d,d,c)) && \text{since } t_T \text{ is idempotent} \\ &= t_R(d,d,c) && \text{since } (c,d,2) \in T \\ &= c && \text{since } (c,d,2) \in R \end{aligned}$$

So t_S is a local Pixley term on S.

Case 3: If $j = 2$, define

$$T = S \setminus \{(a,b,2)\},$$

$$R = (S \cap (A^2 \times \{1\})) \cup \{(c, t_T(d,d,c), 0) : (c,d,0) \in T\} \cup \{(a, t_T(b,b,a), 2)\} \text{ and}$$

$$t_S(x_0, x_1, x_2) = t_R(t_T(x_0,x_1,x_2), t_T(x_1,x_1,x_2), x_2).$$

(3.1) Examining $(a, b, 2)$,

$$\begin{aligned} t_S(b,b,a) &= t_R(t_T(b,b,a), t_T(b,b,a), a) && \\ &= a && \text{since } (a, t_T(b,b,a), 2) \in R \end{aligned}$$

(3.2) For any $(c, d, 0) \in S$,

$$\begin{aligned} t_S(c,d,c) &= t_R(t_T(c,d,c), t_T(d,d,c), c) && \\ &= t_R(c, t_T(d,d,c), c) && \text{since } (c,d,0) \in T \\ &= c && \text{since } (c, t_T(d,d,c), 0) \in R \end{aligned}$$

(3.3) For any $(c, d, 1) \in S$,

$$\begin{aligned} t_S(c,d,d) &= t_R(t_T(c,d,d), t_T(d,d,d), d) && \\ &= t_R(t_T(c,d,d), d, d) && \text{since } t_T \text{ is idempotent} \\ &= t_R(c,d,d) && \text{since } (c,d,1) \in T \\ &= c && \text{since } (c,d,1) \in R \end{aligned}$$

(3.4) For any $(c, d, 2) \in T$,

$$\begin{aligned} t_S(d,d,c) &= t_R(t_T(d,d,c), t_T(d,d,c), c) && \\ &= t_R(c,c,c) && \text{since } (c,d,2) \in T \\ &= c && \text{since } t_R \text{ is idempotent} \end{aligned}$$

So t_S is a local Pixley term on S.

This completes the inductive hypothesis, proving that for any $S \in (A^2 \times \mathbf{3})^{[k]}$ there is a local Pixley term on S. By induction therefore, $\mathbf{A}$ supports a (global) Pixley term. $\square$

Corollary 3.1.9. *There is a polynomial-time algorithm which will decide whether or not an idempotent algebra has a Pixley term.*

Proof. Since the defining matrix of a Pixley term has the local-global property, Lemma 3.1.3 gives us a polynomial-time algorithm to test for its presence.

Alternatively, one can argue as follows. A Pixley term is present in an algebra if and only if that algebra has a Mal'cev term and a majority term (see Theorem 12.5 of [10]), so we can use Corollaries 3.1.6 and 3.1.7 to search for those two terms instead. $\square$

It is not clear what generalizations of the downward column condition might include a Pixley term, and there is not yet any example of a strong term condition of the kind examined here for which it is possible to have the local terms but not the appropriate global term.

3.2 Other Local Terms

The ability to discover properties of an algebra based on properties of its parts is often quite useful, and as such there are many existing results which delve into different concepts of locality in an algebraic context. Future extensions of the main result of the preceding section should aim to provide a common generalization of as many such results as possible. This section collects several of these results.

First we will present two results which already are special cases of Theorem 3.1.5, and which initially suggested such a theorem was feasible.

Theorem 3.2.1. *[15] Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ generates a congruence permutable variety if and only if for every $a, b, c, d \in A$,*

$$\left(\begin{smallmatrix} b & a \\ d, & c \end{smallmatrix}\right) \in Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & a \\ c, & d \end{smallmatrix}\right) \circ Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & b \\ d, & d \end{smallmatrix}\right)$$

where $\mathbf{B} = Sg^{\mathbf{A}^2}(\{\begin{smallmatrix} a & a & b \\ c, & d, & d \end{smallmatrix}\})$.

A new proof of this result from the main result of the preceding section follows.

Proof. $\Rightarrow$: Suppose that $\mathbf{A}$ generates a congruence permutable variety. Since $\left(\begin{smallmatrix} a & b \\ c, & d \end{smallmatrix}\right) \in Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & a \\ c, & d \end{smallmatrix}\right) \circ Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & b \\ d, & d \end{smallmatrix}\right)$ the fact that these congruences permute tells us that $\left(\begin{smallmatrix} b & a \\ d, & c \end{smallmatrix}\right)$ is also a member of that set.

$\Leftarrow$: Suppose that for every $a, b, c, d \in A$ we have that

$$\left(\begin{smallmatrix} b & a \\ d, & c \end{smallmatrix}\right) \in Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & a \\ c, & d \end{smallmatrix}\right) \circ Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & b \\ d, & d \end{smallmatrix}\right)$$

where $\mathbf{B} = Sg^{\mathbf{A}^2}(\{\begin{smallmatrix} a & a & b \\ c, & d, & d \end{smallmatrix}\})$. Therefore for every $a, b, c, d \in A$ there are 4-ary term operations t_1 and t_2 on $\mathbf{B}$ such that

- $t_1(\genfrac{}{}{0pt}{}{a}{c},\genfrac{}{}{0pt}{}{a}{c},\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{b}{d}) = \genfrac{}{}{0pt}{}{b}{d}$,
- $t_1(\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{a}{c},\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{b}{d}) = \genfrac{}{}{0pt}{}{b}{c}$,
- $t_2(\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{a}{c},\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{b}{d}) = \genfrac{}{}{0pt}{}{b}{c}$, and
- $t_2(\genfrac{}{}{0pt}{}{b}{d},\genfrac{}{}{0pt}{}{a}{c},\genfrac{}{}{0pt}{}{a}{d},\genfrac{}{}{0pt}{}{b}{d}) = \genfrac{}{}{0pt}{}{a}{c}$.

In other words,

- $t_1(a, a, a, b) = b$,
- $t_1(c, c, d, d) = d$,
- $t_1(d, c, d, d) = c$,
- $t_2(a, a, a, b) = b$,
- $t_2(b, a, a, b) = a$, and
- $t_2(d, c, d, d) = c$.

Therefore $t_1(\mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is a local $(\genfrac{}{}{0pt}{}{\mathbf{y}}{\mathbf{x}},\genfrac{}{}{0pt}{}{\mathbf{y}}{\mathbf{y}},\genfrac{}{}{0pt}{}{\mathbf{x}}{\mathbf{y}})$-term (i.e. a local Mal'cev term) on $\{(b, a, 0), (c, d, 1)\}$. Since we have such terms for every $a, b, c, d \in A$ we know by Theorem 3.1.5 that $V(\mathbf{A})$ supports a Mal'cev term and so $\mathbf{A}$ generates a congruence permutable variety. □

Theorem 3.2.2. *[15] A finite idempotent algebra $\mathbf{A}$ has a majority term if and only if for all $0, 1, 2, 3, 4, 5 \in A$ there are $6, 7, 8 \in A$ such that*

$$(a, d) \in (Cg^{\mathbf{B}}(a, b) \wedge Cg^{\mathbf{B}}(a, c)), \textit{ and}$$
$$(d, c) \in (Cg^{\mathbf{B}}(b, c) \wedge Cg^{\mathbf{B}}(a, c))$$

where $a = (0, 1, 2)$, $b = (3, 1, 4)$, $c = (0, 5, 4)$, $d = (6, 7, 8)$ and $\mathbf{B} = Sg^{\mathbf{A}^3}(\{a, b, c\})$.

Next, we present a result regarding edge terms which contains a significantly different concept of locality to that presented in the preceding section.

Theorem 3.2.3. *[28] A finite idempotent algebra $\mathbf{A}$ has an edge term if and only if for every $a, b \in A$ there is an $E \in M_{m\times n}(\{\mathbf{x}, \mathbf{y}\})$ for some $m, n > 0$ and an n-ary term operation t on $\mathbf{A}$ such that*

- *E has a $\mathbf{y}$ in every column, and*
- *$t(E_i) = a$ for each $i < m$.*

Next we will examine local conditions associated with various properties of congruence lattices which appear similar to the preceding local conditions, but which do not have a clear translation into a syntactic context. As such it seems likely that some kind of local term condition can be derived from these theorems, though what the definitions may be are unclear.

Theorem 3.2.4. *[15] Let* $\mathbf{A}$ *be a finite idempotent algebra such that*

$$(a,c) \in (Cg^{\mathbf{B}}(a,b) \wedge Cg^{\mathbf{B}}(a,c)) \vee (Cg^{\mathbf{B}}(b,c) \wedge Cg^{\mathbf{B}}(a,c))$$

where $\mathbf{B} = Sg^{\mathbf{A}^3}(\{a,b,c\})$*, for all elements* $a,b,c \in A^3$ *of the form*

$$a = (0,1,1), b = (3,1,2), \text{ and } c = (0,2,2)$$

for some $0,1,2,3 \in A$*. Then* $V(\mathbf{A})$ *is congruence distributive.*

Theorem 3.2.5. *[15] Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $V(\mathbf{A})$ *is congruence modular if and only if for every* $a,b,c,d \in A$ *we have that*

$$\left(\begin{smallmatrix} a & a \\ b, & d \end{smallmatrix}\right) \in Cg^{\mathbf{B}}\left(\begin{smallmatrix} c & c \\ b, & d \end{smallmatrix}\right) \vee \left(Cg^{\mathbf{B}}\left(\left(\begin{smallmatrix} a & a \\ b, & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & c \\ b, & d \end{smallmatrix}\right)\right) \wedge Cg^{\mathbf{B}}\left(\left(\begin{smallmatrix} a & c \\ b, & b \end{smallmatrix}\right), \left(\begin{smallmatrix} a & c \\ d, & d \end{smallmatrix}\right)\right)\right).$$

Theorem 3.2.6. *[15] Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $V(\mathbf{A})$ *is congruence semidistributive if and only if*

$$\left(\begin{smallmatrix} a & a \\ a, & b \end{smallmatrix}\right) \in Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & b \\ b, & a \end{smallmatrix}\right) \vee \left(Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & b \\ a, & a \end{smallmatrix}\right) \wedge Cg^{\mathbf{B}}\left(\begin{smallmatrix} a & a \\ a, & b \end{smallmatrix}\right)\right)$$

for every $a,b \in A$ *where* $\mathbf{B} = Sg^{\mathbf{A}^2}(\left\{\begin{smallmatrix} a & a & b \\ a, & b, & a \end{smallmatrix}\right\})$*.*

Local results for congruence n-permutability (for fixed n) have been elusive, though there are some partial results.

Definition 3.2.7. [22] If $\mathbf{A}$ is a finite algebra say that $\mathbf{A}$ satisfies the condition HM_n if for every $\alpha \prec \beta$ in $Con(\mathbf{A})$ we have that $\beta = \rho \circ_{n-1} \rho$ where

$$\rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta}) \text{ and}$$

$$T_{\alpha,\beta} = \{(x,x) : x \in A\} \cup \{N^2 : N\},$$

Where N ranges over complete preimages (under the quotient map) of the β/α-traces of $\mathbf{A}/\alpha$.

Recall that $\circ_n$ was defined in Definition 2.3.4.

Theorem 3.2.8. *[22] Let* $\mathbf{A}$ *be a finite algebra and let* $n = 2$ *or* $n = 3$*.* $V(\mathbf{A})$ *is congruence* n*-permutable if and only if for every finite* $\mathbf{B} \in V(\mathbf{A})$*,*

- $1 \notin typ(\mathbf{B})$ *and*
- $\mathbf{B}$ *satisfies the condition* HM_n*.*

Unfortunately no similar results are known for $n \geq 4$ at the time of writing. While HM_n is obviously a local condition, its definition is so different from the preceding local conditions that it is doubtful a corresponding local term condition exists.

In closing this section, we will present a version of local term conditions for the type omission theorems (2.3.11 through 2.3.16) derived from a single result of Freese and Valeriote presented after the following definition.

Definition 3.2.9.

- Say that an algebra is *strictly simple* if it has no nontrivial congruences and no nontrivial subalgebras.
- Say that two algebras are *term equivalent* if they have the same associated clone.
- Say that an algebra is *functionally complete* if its clone contains all operations on its underlying set.
- Given a clone $\mathcal{C}$, let the *idempotent reduct of* $\mathcal{C}$ be the set of all operations in $\mathcal{C}$ which are idempotent.

Theorem 3.2.10. *[15] If* $\mathbf{A}$ *is a finite idempotent algebra and* $i \in typ(V(\mathbf{A}))$ *then there is a finite strictly simple algebra* $\mathbf{S}$ *of type* j *for some* $j \leq i$ *(under the type ordering) such that* $\mathbf{S}$ *is the quotient of a subalgebra of* $\mathbf{A}$*. If*

- $j = 1$ *then* $\mathbf{S}$ *is term equivalent to a* 2*-element set;*
- $j = 2$ *then* $\mathbf{S}$ *is term equivalent to the idempotent reduct of a module;*
- $j = 3$ *then* $\mathbf{S}$ *is functionally complete;*
- $j = 4$ *then* $\mathbf{S}$ *is polynomially equivalent to a* 2*-element lattice;*
- $j = 5$ *then* $\mathbf{S}$ *is term equivalent to a* 2*-element semilattice.*

Let $\mathbf{A}$ be a finite idempotent algebra.

Corollary 3.2.11. *The following are equivalent.*

1. $V(\mathbf{A})$ *omits type* 1*,*
2. *for every* $a, b \in A$ *there is an* n*-ary term operation* t *on* $\mathbf{A}$ *for some* $n \geq 3$ *(a local weak near unanimity operation) such that*

$$t(b, a, a, \ldots, a) = t(a, b, a, \ldots, a) = \ldots = t(a, a, a, \ldots, a, b), \text{ and}$$

$$t(a, b, b, \ldots, b) = t(b, a, b, \ldots, b) = \ldots = t(b, b, b, \ldots, b, a)$$

3. *for every* $a, b \in A$ *there is a* 4*-ary term operation* t *on* $\mathbf{A}$ *(a local Siggers operation) such that*

$$t(b, b, a, a) = t(b, a, b, a) = t(a, a, a, b), \text{ and}$$

$$t(a, a, b, b) = t(a, b, a, b) = t(b, b, b, a).$$

Proof. $\mathbf{1} \Rightarrow \mathbf{2}$ and $\mathbf{1} \Rightarrow \mathbf{3}$: If $V(\mathbf{A})$ omits type 1 then Theorem 2.3.11 says that $\mathbf{A}$ supports a weak near unanimity term and a Siggers term.

$\mathbf{2} \Rightarrow \mathbf{1}$: Suppose that $V(\mathbf{A})$ admits type 1, then Theorem 3.2.10 says there is an algebra $\mathbf{B}$ which is the quotient of a subalgebra of $\mathbf{A}$ such that $\mathbf{B}$ is term equivalent to a 2-element set. Choose $a, b \in A$ such that a and b are representatives of the elements of $\mathbf{B}$. If $\mathbf{A}$ supports a local weak near unanimity operation on a and b, then $\mathbf{B}$ supports a weak near unanimity operation, contradicting the fact that it is term-equivalent to a 2-element set.

$\mathbf{3} \Rightarrow \mathbf{1}$: Identical to the preceding argument, replacing "weak near unanimity" by "Siggers". □

Corollary 3.2.12. *$V(\mathbf{A})$ omits types 1 and 2 if and only if for every $a, b \in A$ there is a ternary term operation p on $\mathbf{A}$ and a 4-ary term operation q on $\mathbf{A}$ such that*

- *p and q are local weak near unanimity operations on $\{a, b\}$ (see the preceding corollary),*
- *$p(b, a, a) = q(b, a, a, a)$, and*
- *$p(a, b, b) = q(a, b, b, b)$.*

Proof. If $V(\mathbf{A})$ omits types 1 and 2 then Theorem 2.3.13 gives us the associated global term operations.

If $V(\mathbf{A})$ admits type 1 or 2 then let $\mathbf{B}$ be the algebra given to us by Theorem 3.2.10. If $\mathbf{B}$ is of type 1, then it cannot admit a weak near unanimity operation at all. If $\mathbf{B}$ is of type 2 then it is strictly simple and term equivalent to the idempotent reduct of a module. Let 0 be a representative of the additive identity of $\mathbf{B}$ and let a be a representative of any other element of $\mathbf{B}$. Suppose that p and q are the operations specified in the statement of this corollary for 0 and a; we will examine their interpretations in $\mathbf{B}$ ($p^{\mathbf{B}}$ and $q^{\mathbf{B}}$) to find a contradiction.

Since $\mathbf{B}$ is term-equivalent to the idempotent reduct of a module, there are ring elements $b_0, \ldots, b_2$ and $c_0, \ldots, c_3$ such that

$$p^{\mathbf{B}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = b_0\mathbf{x} + b_1\mathbf{y} + b_2\mathbf{z} \text{ and}$$

$$q^{\mathbf{B}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = c_0\mathbf{x} + c_1\mathbf{y} + c_2\mathbf{z} + c_3\mathbf{w}$$

on $\mathbf{B}$. Examining how these operations behave on $[0]$ and $[a]$, we can see that

$$b_0[a] = b_1[a] = b_2[a] = c_0[a] = c_1[a] = c_2[a] = c_3[a] = r[a]$$

for some r in the ring. We can also see that

$$p^{\mathbf{B}}([0], [a], [a]) = 2r[a] = 3r[a] = q^{\mathbf{B}}([0], [a], [a], [a])$$

and so $r[a] = [0]$ which means that

$$[0] = 3r[a] = p^{\mathbf{B}}([a], [a], [a]) = [a]$$

which contradicts our initial choice of a, completing the proof. □

Corollary 3.2.13. *$V(\mathbf{A})$ omits types 1, 2 and 5 if and only if for every $a, b \in \mathbf{A}$ there are ternary term operations $d_0, \ldots, d_n$ for some $n \geq 2$ such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{a, b\}$*

- $d_0(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}$,
- $d_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}$,
- $d_i(\mathbf{x}, \mathbf{y}, \mathbf{y}) = d_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{y})$ *and* $d_i(\mathbf{x}, \mathbf{y}, \mathbf{x}) = d_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{x})$ *for all even* $i < n$*, and*
- $d_i(\mathbf{x}, \mathbf{x}, \mathbf{y}) = d_{i+1}(\mathbf{x}, \mathbf{x}, \mathbf{y})$ *for all odd* $i < n$ *(call these local Hobby-McKenzie terms on* $\{a, b\}$*).*

Proof. If $V(\mathbf{A})$ omits types 1, 2 and 5 then Theorem 2.3.14 says that we have a sequence of terms which satisfy these equations for all a and b, so there is nothing left to prove.

Suppose that $V(\mathbf{A})$ admits type 1, 2 or 5 and has the specified sequences of terms. Let $\mathbf{B}$ be the algebra given to us by Theorem 3.2.10, let $a, b \in \mathbf{A}$ be elements of the preimage of $\mathbf{B}$ in $\mathbf{A}$ such that $[a] \neq [b]$ in $\mathbf{B}$, and let $d_0, \ldots, d_n$ be a sequence of terms satisfying the above equations for $[a]$ and $[b]$ with minimal value of n.

If $\mathbf{B}$ is term equivalent to a set or a semilattice we have a contradiction, as $|B| = 2$ in either case and neither algebra can support terms globally satisfying these equations.

If $\mathbf{B}$ is term equivalent to the idempotent reduct of a module then without loss of generality we may choose $[b] = 0$. For every $i \leq n$ write $d_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = r_i\mathbf{x} + s_i\mathbf{y} + t_i\mathbf{z}$, and now let us examine d_{n-1}.

Suppose n is odd, then we can derive the following facts.

- $d_{n-1}(0, [a], 0) = 0$ so $s_{n-1}[a] = 0$,
- $d_{n-1}([a], 0, 0) = 0$ so $r_{n-1}[a] = 0$, and
- $d_{n-1}([a], 0, [a]) = [a]$ so $t_{n-1}[a] = [a]$.

Therefore d_{n-1} and d_n are identical when restricted to $\{0, [a]\}$ and so the sequence $d_0, \ldots, d_{n-1}$ is a shorter sequence of local Hobby-McKenzie terms on $\{[a], 0\}$, contradicting the minimality of n.

Suppose $n > 2$ is even, then we can derive the following facts.

- $d_{n-1}([a], [a], 0) = 0$, so $(r_{n-1} + s_{n-1})[a] = 0$,
- $d_{n-1}(0, 0, [a]) = [a]$ so $t_{n-1}[a] = [a]$,
- $d_{n-2}([a], 0, 0) = d_{n-1}([a], 0, 0)$ so $r_{n-2}[a] = r_{n-1}[a]$,
- $d_{n-2}(0, [a], 0) = d_{n-1}(0, [a], 0)$ so $s_{n-2}[a] = s_{n-1}[a]$, and
- $d_{n-2}(0, [a], [a]) = d_{n-1}(0, [a], [a])$ so $t_{n-2}[a] = t_{n-1}[a]$.

Therefore d_{n-2} and d_{n-1} are identical when restricted to $\{0, [a]\}$ and so the sequence $d_0, \ldots, d_{n-2}, d_n$ is a shorter sequence of local Hobby-McKenzie terms on $\{[a], 0\}$, contradicting the minimality of n.

Suppose $n = 2$, then $d_1([a], 0, 0) = d_1(0, 0, [a]) = [a]$ and $d_1(0, [a], 0) = 0$ and so $[a] = d_1([a], [a], [a]) = [a] + [a]$, therefore $[a] = 0$ which contradicts our choice of a. This completes the proof. □

Corollary 3.2.14. *$V(\mathbf{A})$ omits types 1, 2, 4 and 5 if and only if for every $a, b \in \mathbf{A}$ there are 4-ary term operations $d_0, \ldots, d_n$ for some $n \geq 2$ such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{a, b\}$*

- $d_0(\mathbf{x}, \mathbf{y}, \mathbf{y}, \mathbf{z}) = \mathbf{x}$,
- $d_n(\mathbf{x}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}$,
- $d_i(\mathbf{x}, \mathbf{x}, \mathbf{y}, \mathbf{x}) = d_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{y}, \mathbf{x})$ *for all $i < n$, and*
- $d_i(\mathbf{x}, \mathbf{x}, \mathbf{y}, \mathbf{y}) = d_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{y}, \mathbf{y})$ *for all $i < n$.*

Proof. If $V(\mathbf{A})$ omits types 1, 2, 4 and 5 then Theorem 2.3.16 says $\mathbf{A}$ supports terms globally satisfying the above equations.

If $V(\mathbf{A})$ admits type 1, 2, 4 or 5 then Theorem 3.2.10 gives us a strictly simple algebra $\mathbf{B}$ which is the homomophic image of a subalgebra of $\mathbf{A}$ and which has one of the listed types. Let a and b be elements of the preimage of $\mathbf{B}$ in $\mathbf{A}$ with $[a] \neq [b]$ in $\mathbf{B}$.

If $\mathbf{B}$ is term equivalent to a two-element semilattice or set, or if $\mathbf{B}$ is polynomially equivalent to a two-element lattice, then it cannot support the required term operations on $\{[a], [b]\}$ as that is its entire universe.

If $\mathbf{B}$ is term equivalent to the idempotent reduct of a module, then without loss of generality we may assume that $[b] = 0$ and that for all $i \leq n$ we have that $d_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = r_i\mathbf{x} + s_i\mathbf{y} + t_i\mathbf{z} + u_i\mathbf{w}$ are the appropriate term operations for $\{a, b\}$ in $\mathbf{A}$. Then we can derive the following facts.

- $d_0([a], 0, 0, 0) = [a]$ so $r_0[a] = [a]$,
- $d_0([a], 0, 0, [a]) = [a]$ so $u_0[a] = 0$, and
- for every $i < n$ if $u_i[a] = 0$ then

  $$d_{i+1}(0, [a], [a], 0) = d_i(0, 0, [a], 0) = t_i[a] = d_i(0, 0, [a], [a]) = d_{i+1}(0, [a], [a], [a])$$

  and so $u_{i+1}[a] = 0$.

Therefore $u_n[a] = 0$ and so $[a] = d_n(0, 0, 0, [a]) = 0$ which contradicts our initial choice of elements, completing the proof. □

The two remaining local versions of type omission theorems are all similar to Corollary 3.2.11 and are left as an exercise for the reader.

Notice that each of these corollaries naturally leads to a polynomial time algorithm for determining which idempotent algebras generate a variety omitting the associated types as we can detect these forms of type omission by searching for the presence of associated local terms. The exception to this is the case of weak near unanimity terms, as there is currently no known algorithm which can detect the presence of a local weak near unanimity term of unspecified arity (though we can still detect omission of type 1 through a search for local Siggers terms).

4 Local Constant Terms

4.1 The Easy Case

In this section we will step aside from the study of idempotent term conditions and examine what can be learned about algebras with constant terms. We will define an idea of local constant terms and demonstrate that there is a stark contrast between the ease of finding a local constant term on a subuniverse and the difficulty of finding a local constant term on an arbitrary subset.

Definition 4.1.1.
- Given a finite set A and $S \subseteq A$, say that $f : A \to A$ is a local constant on S if $f(a) = f(b)$ for all $a, b \in S$. If $f(a) = d$ for each $a \in S$, we may also say that f is a local constant on S with value d.
- For each $k > 0$ let $\mathsf{Loc\text{-}Const}'_k$ denote the class of tuples $(A, \mathcal{F}, S, d)$ where A is a finite set, $S \subseteq A$, with $|S| \leq k$, $d \in A$ and $\mathcal{F}$ is a finite set of operations on A such that $\langle \mathcal{F} \rangle$ contains a local constant on S with value d.
- For each $k > 0$ let $\mathsf{Loc\text{-}Const}_k$ denote the class of tuples $(A, \mathcal{F}, S)$ where A, $S \subseteq A$ with $|S| \leq k$ and $\mathcal{F} \subseteq \mathcal{O}_A$ are all finite sets such that $\langle \mathcal{F} \rangle$ contains a local constant on S.
- Let $\mathsf{Loc\text{-}Const}' = \bigcup\limits_{k>0} \mathsf{Loc\text{-}Const}'_k$ and let $\mathsf{Loc\text{-}Const} = \bigcup\limits_{k>0} \mathsf{Loc\text{-}Const}_k$.
- Let $\mathsf{Loc\text{-}Const\text{-}Sub}'$ be the subclass of $\mathsf{Loc\text{-}Const}'$ where S is a subalgebra of $\mathbf{A}$ and similarly let $\mathsf{Loc\text{-}Const\text{-}Sub}$ be the subclass of $\mathsf{Loc\text{-}Const}$ where S is a subalgebra of $\mathbf{A}$.

Lemma 4.1.2. *There is a polynomial-time algorithm which, for fixed $k > 0$, can determine membership in $\mathsf{Loc\text{-}Const}_k$ and $\mathsf{Loc\text{-}Const}'_k$.*

Proof. Given finite sets A and $\mathcal{F} \subseteq \mathcal{O}_A$ and $S \subseteq A$ with $|S| \leq k$. Without loss of generality say that $S = \{a_0, \ldots, a_{k-1}\}$. Clearly $\overline{a} \in A^k$. It is easy to see that $\langle \mathcal{F} \rangle$ will contain a local constant on S with value d (for some $d \in A$) if and only if $d^k \in Sg_{\mathbf{A}^k}(\overline{a})$. Generating this subalgebra will take $O(arity(\mathbf{A})\|\mathbf{A}\|^k)$ time (see Lemma 2.4.9).

For fixed $k > 0$ then, we know that membership in $\mathsf{Loc\text{-}Const}'_k$ can be determined in polynomial time, and since there are only $|A|$ possible values any such local constant could take we know that membership in $\mathsf{Loc\text{-}Const}_k$ can be determined in $O(|A| arity(\mathbf{A})\|\mathbf{A}\|^k)$ time, which is also polynomial in the size of the input. □

Now let us consider what might be needed in order to determine whether or not an algebra has a global constant term.

Lemma 4.1.3. *Let $\mathbf{A}$ be a finite algebra, and c an element in A. If for every $a \in A$ there is a unary term t such that $t(a) = t(c) = c$, then there is a unary term t' such that for every $b \in A$, $t'(b) = c$.*

Proof. Let us begin by setting $S_0 = A$, and $t_0(\mathbf{x}) = \mathbf{x}$.

For each $i > 0$, if $|S_{i-1}| = 1$, let $S_i = S_{i-1}$ and $t_i = t_{i-1}$, otherwise pick any $a \in S_{i-1}$ with $a \neq c$, and let t be the unary term given by our premise for the element a. Then let S_i be the image of S_{i-1} under t (i.e. $S_i = t(S_{i-1})$) and let t_i be $t \circ t_{i-1}$ (i.e. $t_i(x) = t(t_{i-1}(x))$). Since $t(a) = t(c) = c$, it is clear that $|S_i| < |S_{i-1}|$ and $c \in S_i$.

At each step this sequence of sets gets smaller (if it is not already a singleton) and $t_i(A) = S_i$, therefore $|S_{|A|}| = 1$ and $t_{|A|}(A) = S_{|A|}$. Since $c \in S_i$ for all i, this tells us that $t_{|A|}(A) = \{c\}$, i.e. $t_{|A|}$ is a constant function on $\mathbf{A}$. □

Corollary 4.1.4. *There is a polynomial-time algorithm to determine whether or not an algebra supports a unary constant term.*

Algorithm. Specified Constant
Notation: $SC(\mathbf{A}, c)$
Input: A finite algebra $\mathbf{A}$, and an element $c \in A$.
Output: Whether or not $\mathbf{A}$ supports a constant term with value c.
Runtime: $O(|A| arity(\mathbf{A}) \|\mathbf{A}\|^2)$

Step	Runtime	Repetitions
(1) For each $a \in A$ do:		$O(\|A\|)$
(1.1) Construct $B = Sg_{\mathbf{A}^2}(\{\binom{c}{a}\})$	$O(arity(\mathbf{A})\|\mathbf{A}\|^2)$	
(1.2) If $\binom{c}{c} \notin B$, return False	$O(1)$	
(2) Otherwise, return True		

Algorithm. Constant
Input: A finite algebra $\mathbf{A}$
Output: Whether or not $\mathbf{A}$ supports a constant term.
Runtime: $O(|A|^2 arity(\mathbf{A}) \|\mathbf{A}\|^2)$

Step	Runtime	Repetitions
(1) For each $c \in A$ do:		$O(\|A\|)$
(1.1) If $SC(\mathbf{A}, c)$ is True, return True	$O(\|A\| arity(\mathbf{A})\|\mathbf{A}\|^2)$	
(2) Otherwise, return False		

Notice that this also tells us that membership in Loc-Const-Sub and Loc-Const-Sub′ can be determined in polynomial time, as we can simply restrict the given basic operations to our subalgebra and run the above algorithm there.

4.2 The Hard Case

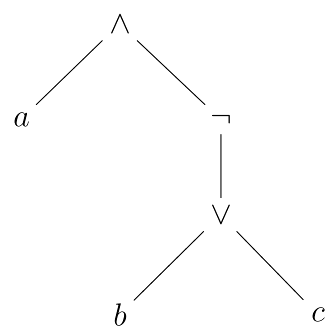
Now we will turn our attention to the strikingly different case of determining membership in Loc-Const and Loc-Const′. It turns out that testing for membership in either class is EXPTIME-complete. In order to prove this we will first introduce bottom-up tree automata, as they are presented in [4].

Definition 4.2.1. [4] A *ranked alphabet* Σ is a disjoint union of a finite family $\{\Sigma_k : 0 \leq k \leq r\}$ of finite sets with $\Sigma_0 \neq \emptyset$. We say that r is the rank of Σ if $|\Sigma_r| > 0$.

If Σ is a ranked alphabet, the set of Σ-trees, $\mathcal{T}_\Sigma$, is the smallest set X of finite sequences such that

- $\Sigma_0 \subseteq X$
- If $k \geq 1$, $\sigma \in \Sigma_k$ and $t_0, t_1, \ldots, t_{k-1} \in X$, then $\sigma t_0 t_1 \ldots t_{k-1} \in X$.

We can reliably think of members of $\mathcal{T}_\Sigma$ graphically (as a tree), for example if $a, b, c \in \Sigma_0$, $\neg \in \Sigma_1$, and $\wedge, \vee \in \Sigma_2$ then the tree $\wedge a \neg \vee bc$ (which we would usually write $a \wedge \neg(b \vee c)$ in the language of boolean logic) can be seen as the following tree.



Definition 4.2.2. [4]

- A *bottom-up tree automaton* (BTA) is a structure $\langle \Sigma, Q, Q^*, R \rangle$ in which Q is a finite set (the set of states), Σ is a ranked alphabet, $Q^* \subseteq Q$ (the set of accepting states) and $R : \bigcup_{k=0}^{r} \Sigma_k \times Q^k \to Q$ (the transition function). The rank of the automaton is the same as the rank of the alphabet. Say that $\langle \Sigma, Q, Q^*, R \rangle$ is a *restricted bottom-up tree automaton* if $|Q^*| = 1$.
- For any tree $t \in \mathcal{T}_\Sigma$ and any BTA $M = \langle \Sigma, Q, Q^*, R \rangle$ define the *action of M on t*, written $M(t)$, inductively such that for each $\sigma \in \Sigma_n$, $M(\sigma t_0 t_1 \ldots t_{n-1}) = R(\sigma, M(t_0), M(t_1), \ldots, M(t_{n-1}))$. Clearly M behaves as a function from $\mathcal{T}_\Sigma$ to Q and it can effectively be thought of as repeated application of its transition function.
- Say that M *accepts* t if $M(t) \in Q^*$, and say that the *tree language recognized by* M is the set $L(M) = \{t \in \mathcal{T}_\Sigma : M(t) \in Q^*\}$.

If M is a BTA, $\sigma \in \Sigma_k$ and $a_i \in Q$ for each $i < k$, for the sake of simplicity we will write $R(\sigma, a_0, a_1, \ldots, a_{k-1})$ instead of $R(\sigma, (a_0, a_1, \ldots, a_{k-1}))$. Since Σ_0 is nonempty and $|Q^0| = 1$, there is a unique state of M associated with each element of Σ_0 through R. Specifically, for each $\alpha \in \Sigma_0$ let $q_\alpha = R(\alpha)$ and call q_α the initial state of M associated with α.

Definition 4.2.3. [4] For each $r > 0$, $\mathsf{Int\text{-}BTL}^1_r$ denotes the set of all finite sequences $\langle M_0, \ldots, M_{\ell-1}\rangle$ of restricted BTAs sharing a common ranked alphabet of rank at most r with $\bigcap_{k<\ell} L(M_k) \neq \emptyset$.

Theorem 4.2.4. $\mathsf{Int\text{-}BTL}^1_2$ *is* $\mathsf{EXPTIME}$*-hard.*

Proof. Theorem 2.5 of [4]. □

From Theorem 4.2.4 we know that if we can demonstrate a polynomial-time reduction of $\mathsf{Int\text{-}BTL}^1_2$ to $\mathsf{Loc\text{-}Const}$, this will demonstrate that $\mathsf{Loc\text{-}Const}$ is $\mathsf{EXPTIME}$-hard as well. This reduction will take place in two stages, the first of which is reduction to a special case of $\mathsf{Int\text{-}BTL}^1_2$.

Lemma 4.2.5. *For every rank* 2 *alphabet* Σ *there is a rank* 2 *alphabet* Σ' *with* $|\Sigma'_0| = 1$ *with the property that for every sequence of restricted BTAs* $\langle M_0, \ldots, M_{\ell-1}\rangle$ *over* Σ *there is a sequence of restricted BTAs* $\langle M'_0, \ldots, M'_{\ell-1}\rangle$ *over* Σ' *such that the* M'_i*'s accept a common tree if and only if the* M_i*'s accept a common tree. Furthermore this construction can be done in an amount of time which is polynomial in the size of the input.*

Proof. Let Σ be a rank 2 alphabet. Fix $\mathbf{x} \in \Sigma_0$ and define Σ' as follows:

- $\Sigma'_0 = \{\mathbf{x}\}$
- $\Sigma'_1 = \Sigma_1 \cup \Sigma_0 \setminus \{\mathbf{x}\}$
- $\Sigma'_2 = \Sigma_2$

Given any sequence of restricted BTAs $M_i = \langle \Sigma, Q_i, \{q^*_i\}, R_i\rangle$ for $i < \ell$ let $M'_i = \langle \Sigma', Q_i, \{q^*_i\}, R'_i\rangle$ where R'_i is defined as follows for each $i < \ell$:

- Let $R'_i(\mathbf{x}) = R_i(\mathbf{x})$,
- For each $\sigma \in \Sigma_0 \setminus \{\mathbf{x}\}$ and each $q \in Q_i$, let $R'_i(\sigma, q) = R_i(\sigma)$,
- For each $\sigma \in \Sigma_1$ and each $q \in Q_i$, let $R'_i(\sigma, q) = R_i(\sigma, q)$, and
- For each $\sigma \in \Sigma_2$ and each $q, q' \in Q_i$, let $R'_i(\sigma, q, q') = R_i(\sigma, q, q')$.

Now let us recursively define a function τ which transforms Σ-trees into Σ'-trees as follows:

- Let $\tau(\mathbf{x}) = \mathbf{x}$,
- For every $\sigma \in \Sigma_0 \setminus \{\mathbf{x}\}$, let $\tau(\sigma) = \sigma\mathbf{x}$,
- For every $\sigma \in \Sigma_1$ and Σ-tree T, let $\tau(\sigma T) = \sigma\tau(T)$, and
- For every $\sigma \in \Sigma_2$ and Σ-trees T, T', let $\tau(\sigma T T') = \sigma\tau(T)\tau(T')$

It is clear that M_i accepts tree T if and only if M_i' accepts tree $\tau(T)$, and so the M_i's accepts a common tree if and only if the M_i''s accept a common tree (it is left as an exercise to the reader to show that every Σ'-tree is in the image of τ). $\square$

Theorem 4.2.6. *Testing for membership in* $\mathsf{Loc\text{-}Const}$ *or in* $\mathsf{Loc\text{-}Const}'$ *is* $\mathsf{EXPTIME}$-*complete.*

Proof. We can clearly determine membership in $\mathsf{Loc\text{-}Const}$ and $\mathsf{Loc\text{-}Const}'$ in $\mathsf{EXPTIME}$ by constructing all unary functions in the appropriate clone and checking them individually, so we now need to prove that $\mathsf{Loc\text{-}Const}$ and $\mathsf{Loc\text{-}Const}'$ are $\mathsf{EXPTIME}$-hard. In order to do this, we will take an arbitrary sequence of BTAs and construct from them a set of operations $\mathcal{F}$ and a distinguished subset S of their universe such that the sequence of BTAs will accept a common tree if and only if $\langle \mathcal{F} \rangle$ supports a local constant on S, demonstrating the reduction of $\mathsf{Int\text{-}BTL}_2^1$ to $\mathsf{Loc\text{-}Const}$. We will see during this proof that this same construction also demonstrates the reduction of $\mathsf{Int\text{-}BTL}_2^1$ to $\mathsf{Loc\text{-}Const}'$.

Let Σ be an alphabet of rank 2, without loss of generality $\Sigma_0 = \{\mathbf{x}\}$ (by Lemma 4.2.5), and let $M_i = \langle \Sigma, Q_i, \{q_i^*\}, R_i \rangle$ for $i < \ell$. Without loss of generality, we can assume that the Q_i's are pairwise disjoint. We will now construct another sequence of BTAs M_i' which will accept a common tree if and only if the M_i's do, and from this we will construct a set of operations and a distinguished subset of their universe.

Choose a set of distinct elements $\{\hat{q}_i : i \leq \ell\} \cup \{p\}$ not yet in use and define $Q_i' = Q_i \cup \{\hat{q}_i\}$ for each $i < \ell$. Define $Q = \bigcup\limits_{i<\ell} Q_i' \cup \{\hat{q}_\ell, p\}$ and choose a new rank 1 symbol r for our ranked alphabet and let $\Sigma' = \Sigma \cup \{r\}$. We will now define new transition functions $R_i' : \Sigma' \times \bigcup\limits_{k=0}^{2} Q^k \to Q$ for each $i < \ell$ as follows:

1. Let $R_i'(\mathbf{x}) = R_i(\mathbf{x})$

2. For each $\sigma \in \Sigma_1$ and each $q \in Q$ let

$$R_i'(\sigma, q) = \begin{cases} R_j(\sigma, q) & \text{if } q \in Q_j \text{ for some } j < \ell \\ \hat{q}_j & \text{if } q = \hat{q}_j \text{ for some } j \leq \ell \\ \hat{q}_\ell & \text{if } q = p \end{cases}$$

3. For each $q \in Q$ let

$$R_i'(r, q) = \begin{cases} p & \text{if } q = q_j^* \text{ for some } j < \ell \\ \hat{q}_j & \text{if } q \in Q_j' \setminus \{q_j^*\} \text{ for some } j < \ell \\ \hat{q}_\ell & \text{if } q \in \{p, \hat{q}_\ell\} \end{cases}$$

4. For each $\sigma \in \Sigma_2$ and each $q, q' \in Q$ let

$$R_i'(\sigma, q, q') = \begin{cases} R_j(\sigma, q, q') & \text{if } q, q' \in Q_j \text{ for some } j < \ell \\ \hat{q}_j & \text{if } q \in Q_j \text{ and } q' \notin Q_j \text{ for some } j < \ell \\ \hat{q}_j & \text{if } q = \hat{q}_j \text{ for some } j \leq \ell \\ \hat{q}_\ell & \text{if } q = p \end{cases}$$

Clearly part 1 of the above definition is the only part which depends on i, and so $R'_i = R'_j$ for all i, j when restricted to $\Sigma' \setminus \Sigma'_0$ in the first input.

Letting $M'_i = \langle \Sigma', Q, \{p\}, R'_i \rangle$, we can easily see that each M'_i will only accept trees of the form rT where T is a Σ-tree, and that M'_i will accept tree rT if and only if M_i accepts tree T.

Next we will construct from the M'_i's a set of operations on Q and a distinguished subset $S \subseteq Q$.

For each $\sigma \in \Sigma_1 \cup \{r\}$ define $f_\sigma : Q \to Q$ as

$$f_\sigma(q) = R'_i(\sigma, q)$$

and for each $\sigma \in \Sigma_2$ define $f_\sigma : Q^2 \to Q$ as

$$f_\sigma(q, q') = R'_i(\sigma, q, q')$$

Notice that these functions are well-defined since the same function is obtained regardless of which $i < \ell$ is chosen.

Let $\mathcal{F} = \{f_\sigma : \sigma \in \Sigma' \setminus \Sigma'_0\}$ and let $S = \{R_i(\mathbf{x}) : i < \ell\}$.

Since p is the only accepting state of the M'_i's, it is obvious that the M'_i's accept a common tree if and only if there is a function $f \in \langle \mathcal{F} \rangle$ (whose formation from the elements of $\mathcal{F}$ is essentially identical to a Σ'-tree T) such that $f(a) = p$ for every $a \in S$. This completes the reduction of $\textsf{Int-BTL}_2^1$ to $\textsf{Loc-Const}'$ as we have now shown that the M_i's accept a common tree if and only if $(Q, \mathcal{F}, S, p) \in \textsf{Loc-Const}'$. Next we will prove that this construction also provides us with a reduction of $\textsf{Int-BTL}_2^1$ to $\textsf{Loc-Const}$.

Claim: For all unary $f \in \langle \mathcal{F} \rangle$, if f is a local constant on S then $f(S) = \{p\}$ or $f(S) = \{\hat{q}_\ell\}$.

Proof. For each $q \in Q'_j$ we have that $f_\sigma(q) \in Q'_j$ for each $\sigma \in \Sigma_1$ and that $f_\sigma(q, q') \in Q'_j$ for each $\sigma \in \Sigma_2$ and each $q' \in Q$. Therefore for any unary $f \in \langle \mathcal{F} \rangle$ and any $q \in Q'_i$, $f(q) \in Q'_i \cup \{\hat{q}_\ell, p\}$. Since S has nonempty intersection with every Q'_i and the Q'_i's are pairwise disjoint we can conclude that if $f \in \langle \mathcal{F} \rangle^{(1)}$ is a local constant on S then $f(S) \in \{\{\hat{q}_\ell\}, \{p\}\}$. $\square$

From this claim we can conclude that it is sufficient to show that if $\langle \mathcal{F} \rangle$ contains a local constant on S with value $\hat{q}_\ell$ then it also contains a local constant on S with value p.

In order to prove this let $\mathcal{F}_0 = \{f_\sigma : \sigma \in \Sigma \setminus \{\mathbf{x}\}\}$, then the proof of the preceding claim also tells us that $\mathcal{F}_0$ fixes (not pointwise) the set $Q \setminus \{\hat{q}_\ell, p\}$. Notice also that if $f \in \langle \mathcal{F} \rangle^{(k)}$ and $\overline{q} \in Q^k$ with some $q_i \in \{\hat{q}_j : j \leq \ell\} \cup \{p\} = f_r(Q)$, then $f(\overline{q}) \in f_r(Q)$ as well. Another way of stating this is that if $g \in \langle \mathcal{F} \rangle^{(k)}$ with $g \notin \langle \mathcal{F}_0 \rangle$ and any $\overline{a} \in Q^k$, $g(\overline{a}) \in f_r(Q)$.

Suppose that $f \in \langle \mathcal{F} \rangle^{(1)}$ with $f(S) = \{\hat{q}_\ell\}$, and suppose that f has minimal compositional depth with this property.

Case 1: If $f(\mathbf{x}) = f_\sigma(f'(\mathbf{x}))$ for some $\sigma \in \Sigma_1 \cup \{r\}$ and $f' \in \langle \mathcal{F} \rangle^{(1)}$, then $f'(S) \subseteq \{\hat{q}_\ell, p\}$.

Case 2: If $f(\mathbf{x}) = f_\sigma(f'(\mathbf{x}), g'(\mathbf{x}))$ for some $\sigma \in \Sigma_2$ and $f', g' \in \langle \mathcal{F} \rangle^{(1)}$, then $f'(S) \subseteq \{\hat{q}_\ell, p\}$.

Either way, we have $f' \in \langle \mathcal{F} \rangle^{(1)}$ with $f'(S) \subseteq \{\hat{q}_\ell, p\}$.

Case a: If $f'(S) = \{\hat{q}_\ell\}$ then this contradicts the minimality of the compositional depth of f.

Case b: If $f'(S) = \{p\}$ then f' is the function we are looking for and we are finished.

Case c: Otherwise, $f'(S) = \{\hat{q}_\ell, p\}$. Since f_r is the only function in $\mathcal{F}$ whose image includes p, we know that $f'(x) = f_r(f''(x))$ for some $f'' \in \langle \mathcal{F} \rangle^{(1)}$.

1. By the definition of f_r we know that $f''(S) \subseteq \{q_i^* : i < \ell\} \cup \{\hat{q}_\ell, p\}$.
2. Since $f'(S) \neq \{p\}$ we know that $f''(S) \cap \{\hat{q}_\ell, p\} \neq \emptyset$.
3. Since $f'(S) \neq \{\hat{q}_\ell\}$ we know that $f''(S) \cap \{q_i^* : i < \ell\} \neq \emptyset$.
4. Since $f_r(Q) \cap \{q_i^* : i < \ell\} = \emptyset$ and f_r is the only function in $\mathcal{F}$ whose image contains p, we know that $p \notin f''(S)$. Together with point 2, this proves that $\hat{q}_\ell \in f''(S)$.
5. Since $\hat{q}_\ell \in f''(S)$ and $\hat{q}_\ell, p \notin S$, we know that $f'' \notin \langle \mathcal{F}_0 \rangle$, so $f''(S) \subseteq f_r(Q)$, therefore by point 1 $f''(S) \subseteq \{\hat{q}_\ell, p\}$ which contradicts point 3.

To summarize: cases 1 and 2 are exhaustive and each give us a function of simpler composition to examine. This function is either the function we are looking for (i.e. $f'(S) = \{p\}$) or we can break it down to an even simpler function to continue searching. Since every function in $\langle \mathcal{F} \rangle$ is produced by finitely many compositions, this process must terminate and the only way it can do so is to find the function we are searching for, completing the proof. $\square$

Having proven that detecting a local constant function in a clone which contains a non-unary function is EXPTIME-complete, it is worth asking what the computational complexity is of detecting a local constant term in a clone which consists only of unary functions. To this end, we will provide a slight modification of Theorem 4.2.6, proving that we can construct a similar set of operations and distinguished subset from Deterministic Finite Automata.

Definition 4.2.7. A *Deterministic Finite Automaton (DFA)* is a structure $\langle Q, \Gamma, \delta, q^*, F \rangle$ where Q is a finite set of states, Γ is a finite set of symbols, $\Delta : \Sigma \times Q \to Q$ is the transition function, q^* is the initial state and F is the set of accepting states. Say that a DFA is a *restricted DFA* if $|F| = 1$.

Let Int-DFA be the collection whose elements are finite sets of DFAs $\{F_0, \ldots, F_{k-1}\}$ over a common alphabet Σ such that all the F_i's accept a common word in Σ^*. (Let Int-DFA^1 be the corresponding problem on restricted DFAs).

Lemma 4.2.8. Int-DFA *is* PSPACE*-complete.*

Proof. Lemma 3.2.3 in [23]. □

Lemma 4.2.9. Int-BTL$^1_1$ *is* PSPACE*-hard.*

Proof. Given any sequence of restricted DFAs $F_i = \langle Q_i, \Gamma, \delta_i, q_i^*, \{p_i\}\rangle$, $i < k$, over a common alphabet Γ, we can easily generate restricted BTAs $M_i = \langle \Sigma, Q_i, \{p_i\}, R_i\rangle$, $i < k$ where $\Sigma_1 = \Gamma$ and $\Sigma_0 = \{\mathbf{x}\}$ for some $\mathbf{x}$ not used anywhere else, and where $R_i(\sigma, q) = \delta_i(\sigma, q)$ for each $\sigma \in \Gamma$ and $q \in Q_i$ and $R_i(\mathbf{x}) = q_i^*$ for each $i < k$. Clearly the M_i's will accept a common tree if and only if the F_i's accept a common word.

The proof of Lemma 4.2.8 (in [23]) converts an arbitrary PSPACE-bounded Turing machine with input x and converts it into a sequence of DFAs whose languages will have a nonempty intersection if and only if the Turing machine accepts x. It is worth noticing that all the DFAs generated in this proof have unique accepting states, and thus are restricted DFAs, proving that Int-DFA^1 is PSPACE-complete.

The construction with which we began this proof therefore proves that Int-BTL$^1_1$ is reducible to Int-DFA^1 and so both are PSPACE-hard. □

Corollary 4.2.10. *The local constant problem restricted to unary functions is* PSPACE*-complete.*

Proof. If we begin with a set of restricted BTAs on a common rank 1 alphabet, we can trace the proof of Theorem 4.2.6 and we will obtain $\langle A, \mathcal{F}, S\rangle$ such that $\mathcal{F}$ is a set of unary functions on A, $S \subseteq A$ and our initial BTAs will accept a common tree if and only if $\langle \mathcal{F}\rangle$ has a local constant function on S. Since Int-BTL$^1_1$ is PSPACE-hard (compared to Int-BTL$^1_2$ being EXPTIME-hard) this tells us that the local constant problem is PSPACE-hard when restricted to unary functions. In order to complete the proof, we need to prove that the local constant problem on unary functions is in PSPACE.

Given finite sets A, $\mathcal{F} \subseteq \mathcal{O}_A^{(1)}$ and $S \subseteq A$ and element $p \in A$, we wish to determine whether or not $\langle \mathcal{F}\rangle$ supports a local constant term on S. Without loss of generality say that $S = \{a_0, a_1, \ldots, a_{k-1}\}$. Let us define a sequence of DFAs as follows. For each $i < k$ let $D_i = \langle A, \mathcal{F}, \delta, a_i, \{p\}\rangle$ be a DFA where $\delta(f, a) = f(a)$ for each $f \in \mathcal{F}$ and each $a \in A$. Clearly this construction takes time at most polynomial in the size of the input. If $w = f_0 f_1 \ldots f_{n-1} \in \mathcal{F}^n$ is any word on $\mathcal{F}$, then the state D_i will be in after reading w will be exactly $f_0 \circ f_1 \circ \ldots \circ f_{n-1}(a_i)$, and so any word accepted by all D_i's will have a corresponding function $f \in \langle \mathcal{F}\rangle$ such that $f(S) = \{p\}$ and vice versa. This demonstrates the reduction of Loc-Const$'$ restricted to unary functions to Int-DFA, and since Int-DFA is in PSPACE we know that Loc-Const$'$ restricted to unary functions is as well.

Since Loc-Const can be solved with polynomially many applications of Loc-Const$'$ (one for each potential value of any local constant function) this also tells us that Loc-Const restricted to unary functions is in PSPACE, completing the proof. □

5 Difficulty of Deciding Term Conditions

5.1 Constant-Projection Blends

In this section, we show that a wide class of term conditions are difficult to detect in non-idempotent algebras. We accomplish this by reducing the clone membership problem to the problem of determining whether or not an algebra has particular types of nontrivial idempotent operations.

Definition 5.1.1. (See Section 9 of [15]) The clone membership problem ($\mathsf{Gen\text{-}Clo}'$) is the problem which takes as input a finite set of operations $\mathcal{F}$ on a finite set A and a single unary operation h on A, and determines whether or not $h \in \langle \mathcal{F} \rangle$.

In [4] it is proven through a similar reduction to that used in Lemma 4.2.6 that $\mathsf{Gen\text{-}Clo}'$ is $\mathsf{EXPTIME}$-complete. We will now proceed to generalize Corollary 9.3 of [15] demonstrating a class of idempotent Mal'cev conditions which are similarly difficult to detect.

Definition 5.1.2. Given a finite set B and n-ary operation g on B, say that g is a *Constant-Projection Blend* (CPB) if there is an element $0 \in B$ and $i < n$ such that $g(a_0, \ldots, a_{n-1}) \in \{0, a_i\}$ for all $a_0, \ldots, a_{n-1} \in B$. If this is the case, say that g is CPB_0 (on coordinate i).

Notice that the composition of CPB_0 operations is also CPB_0, and so the set of all idempotent CPB_0 operations on a set B is a clone.

We will define a procedure which will begin with a set of operations $\mathcal{F}$ on A and a unary operation h on A and construct an algebra such that this algebra will have nontrivial idempotent terms if and only if $h \in \langle \mathcal{F} \rangle$. This will be used to demonstrate that any idempotent Mal'cev condition satisfiable by CPB_0 operations will also be $\mathsf{EXPTIME}$-hard to detect in general.

Given a finite set of operations $\mathcal{F}$ on a finite set A and a unary operation h on A, choose elements $0, 1 \notin A$ and define $B = A \cup \{0, 1\}$. For any $f : A^n \to A$ define $f' : B^n \to B$ to be

$$f'(\overline{x}) = \begin{cases} 0 & \text{if any } x_i \notin A \\ f(\overline{x}) & \text{otherwise} \end{cases}.$$

Let $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$.

Let $\mathcal{U}$ be a finite set of idempotent operations on B which are CPB_0 on coordinate 0. For each n-ary $g \in \mathcal{U}$, define $t_g : B^{n+1} \to B$ to be

$$t_g(x_0, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{if } x_0 = h'(x_1) \\ 0 & \text{otherwise} \end{cases}$$

and define $\mathcal{G} = \mathcal{F}' \cup \{t_g : g \in \mathcal{U}\}$.

Lemma 5.1.3. *For any $q \in \langle \mathcal{G} \rangle$ with $q(A^n) \subseteq A$ there is a $p \in \langle \mathcal{F} \rangle$ such that $p(\overline{x}) = q(\overline{x})$ for all $\overline{x} \in A^n$.*

Proof. We will prove this by induction on the compositional depth of q. Clearly all projections in $\langle\mathcal{G}\rangle$ satisfy this claim. Notice that every $q \in \langle\mathcal{G}\rangle$ with $q(A^n) \subseteq A$ can be written in the form $q = g(q_0, \ldots, q_{k-1})$ for some $g \in \mathcal{G}^{(k)}$ and some $q_i \in \langle\mathcal{G}\rangle$ with $q_i(A^n) \subseteq A$ as well.

Let $q_0, \ldots, q_{k-1} \in \langle\mathcal{G}\rangle$ be n-ary operations such that $q_i(A^n) \subseteq A$ for all $i < k$, and let $p_0, \ldots, p_{k-1} \in \langle\mathcal{F}\rangle$ be operations such that $p_i(\overline{x}) = q_i(\overline{x})$ for all $\overline{x} \in A^n$ and all $i < k$.

Case 1: Suppose $q = f'(q_0, \ldots, q_{k-1})$ for some $f \in \mathcal{F}$, then let $p = f(p_0, \ldots, p_{k-1})$. Clearly $p \in \langle\mathcal{F}\rangle$, and for every $\overline{x} \in A^n$,

$$\begin{array}{rl|l} q(\overline{x}) = & f'(q_0(\overline{x}), \ldots, q_{k-1}(\overline{x})) & \\ = & f'(p_0(\overline{x}), \ldots, p_{k-1}(\overline{x})) & \text{since } p_i(\overline{x}) = q_i(\overline{x}) \\ = & f(p_0(\overline{x}), \ldots, p_{k-1}(\overline{x})) & \text{since } p_i(\overline{x}) \in A \\ = & p(\overline{x}) & \end{array}$$

so the inductive hypothesis also holds for q.

Case 2: Suppose $q = t_g(q_0(\overline{x}), q_1(\overline{x}), \ldots, q_{k-1}(\overline{x}))$ with $q(A^n) \subseteq A$ for some $g \in \mathcal{U}$. Let $p(\overline{x}) = p_1(\overline{x})$. Then for any $\overline{x} \in A^n$ it is clear that $q(\overline{x}) \in \{0, q_1(\overline{x})\}$. Since $q(A^n) \subseteq A$, this tells us that $q(\overline{x}) = q_1(\overline{x}) = p_1(\overline{x}) = p(\overline{x})$, showing that the inductive hypothesis also holds for q.

This completes the induction, and so the proof. □

Lemma 5.1.4. *If $h \in \langle\mathcal{F}\rangle$ then $\mathcal{U} \subseteq \langle\mathcal{G}\rangle$, and if $h \notin \langle\mathcal{F}\rangle$ then $\langle\mathcal{G}\rangle$ contains no nontrivial idempotent operations.*

Proof. Suppose that $h \in \langle\mathcal{F}\rangle$. Clearly $'$: $\langle\mathcal{F}\rangle \to \langle\mathcal{G}\rangle$ distributes over functional composition, and so $h' \in \langle\mathcal{G}\rangle$. Therefore $t_g(h'(x_0), x_0, x_1, \ldots, x_{k-1}) = g(x_0, x_1, \ldots, x_{k-1}) \in \langle\mathcal{G}\rangle$ for every $g \in \mathcal{U}$.

Now suppose that $p \in \langle\mathcal{G}\rangle$ is a nontrivial idempotent operation. Since $\mathcal{F}'$ contains no surjective functions, we must have $p(\overline{x}) = t_g(q_0(\overline{x}), \ldots, q_{k-1}(\overline{x}))$ for some $q_i \in \langle\mathcal{G}\rangle$ and some $g \in \mathcal{U}$. For each $i < k$ let $u_i(x) = q_i(x, x, \ldots, x)$, and so $p(x, x, \ldots, x) = t_g(u_0(x), u_1(x), \ldots, x_{k-1}(x)) = x$. Since $t_g(u_0(x), u_1(x), \ldots, u_{k-1}(x)) \in \{0, u_1(x)\}$, we know that $u_1(x) = x$ and $u_0(x) = h'(x)$ for all $x \in B \setminus \{0\}$. By Lemma 5.1.3, there is an $h'' \in \langle\mathcal{F}\rangle$ such that $h'(x) = h''(x)$ for all $x \in A$, i.e. $h'' = h \in \langle\mathcal{F}\rangle$. □

Theorem 5.1.5. *A term condition is* EXPTIME*-hard if there is a polynomial-time algorithm which takes as input a set B with $0 \in B$ and produces CPB_0 operations on B which satisfy the term condition.*

Proof. Fix a term condition which, on any finite set B with $0 \in B$, is satisfiable by CPB_0 operations such that said operations are computable in time polynomial in $|B|$. Let M be a Turing machine which, when given an algebra as input, halts in an accepting state if the algebra supports term operations which satisfy the term condition and halts in a rejecting state otherwise. To obtain a contradiction, suppose that M runs in time less than exponential in the size of the input. Then the following algorithm will decide Gen-Clo$'$ in time less than exponential in the size of the input.

Algorithm.
Input: Finite sets A and $\mathcal{F} \subseteq \mathcal{O}_A$ and unary operation $h \in \mathcal{O}_A^{(1)}$.
Output: Whether or not $h \in \langle \mathcal{F} \rangle$.

(1) Let $B = A \cup \{0, 1\}$.

(2) Let $\mathcal{U}$ be a finite set of idempotent CPB_0 operations on B which satisfy the fixed term condition (without loss of generality, they are all CPB_0 on coordinate 0).

(3) Let $\mathcal{G} = \{f' : f \in \mathcal{F}\} \cup \{t_g : g \in \mathcal{U}\}$.

(4) Run M on input $\langle B, \mathcal{G} \rangle$ and output its response.

Steps (1) and (3) are obviously polynomial-time in the input size, step (2) runs in polynomial-time by the premise of this theorem, and step (4) runs in less than exponential-time by our choice of M, hence this algorithm runs in less than exponential time. Lemma 5.1.4 guarantees the correctness of this algorithm, completing the proof. □

Now let us make use of Theorem 5.1.5 by applying it to several different term conditions.

Recall the definition of Jónsson terms from Theorem 2.3.2.

Corollary 5.1.6. *For any $n > 2$, testing whether or not an algebra has a Jónsson sequence of length n is* EXPTIME*-complete. Also, testing whether or not an algebra has a Jónsson sequence at all is* EXPTIME*-complete.*

Proof. Theorem 2.3.2 tells us also that the presence of Jónsson terms is detectable in EXPTIME, as is the presence of any fixed-length sequence thereof.

Given any finite set A with $0 \in A$ define

$$g_1(x, y, z) = \begin{cases} x & \text{if } x \in \{y, z\} \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(x, y, z) = \begin{cases} x & \text{if } z \in \{x, y\} \\ 0 & \text{otherwise} \end{cases}$$

Clearly g_1 and g_2 are CPB_0, and the following defines a Jónsson sequence of length 3 on A.

$$\begin{aligned} d_0(x, y, z) &= x \\ d_1(x, y, z) &= g_1(x, y, z) \\ d_2(x, y, z) &= g_2(z, y, x) \\ d_3(x, y, z) &= z \end{aligned}$$

□

The preceding corollary was also proven in Corollary 9.3 of [15].

Corollary 5.1.7. *For any $n > 2$, testing whether or not an algebra generates a variety which is congruence n-permutable is* EXPTIME*-complete. Also, testing whether or not an algebra is congruence n-permutable for unspecified n is* EXPTIME*-complete.*

Proof. From 2.3.5 we know that a variety has n-permutable congruences if and only if it has ternary terms (called Hagemann-Mitschke terms) $p_0, \ldots, p_n$ such that

$$\begin{aligned} p_0(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \mathbf{x} \\ p_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \mathbf{z} \\ p_i(\mathbf{x}, \mathbf{x}, \mathbf{y}) &= p_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{y}) \quad \text{for all } i < k \end{aligned}.$$

This tells us that n-permutability for fixed n can be determined in EXPTIME, as can n-permutability for unspecified n.

Given any finite set A with $0 \in A$ define

$$g(x, y, z) = \begin{cases} x & \text{if } y = z \\ 0 & \text{otherwise} \end{cases}.$$

Clearly g is CPB_0, and the following defines a 3-length sequence of Hagemann-Mitschke terms on A.

$$\begin{aligned} p_0(x, y, z) &= x \\ p_1(x, y, z) &= g(x, y, z) \\ p_2(x, y, z) &= g(z, y, x) \\ p_3(x, y, z) &= z \end{aligned}$$

□

Corollary 5.1.8. *Testing whether or not an algebra generates a variety which omits types 1, 2, 4 and 5 is* EXPTIME*-complete.*

Proof. From 2.3.16 we know that a variety omits types $\{1, 2, 4, 5\}$ if and only if it supports 4-ary idempotent terms $f_0, \ldots, f_n$ for some n such that

$$\begin{aligned} f_0(x, y, y, z) &= x \\ f_n(x, x, y, z) &= z \\ f_i(x, x, y, x) &= f_{i+1}(x, y, y, x) \quad \text{for all } i < n \\ f_i(x, x, y, y) &= f_{i+1}(x, y, y, y) \quad \text{for all } i < n \end{aligned}.$$

This equivalence tells us that omission of types $\{1, 2, 4, 5\}$ is in EXPTIME.

Given any finite set A with $0 \in A$ define

$$g(x, y, z) = \begin{cases} x & \text{if } y = z \\ 0 & \text{otherwise} \end{cases}.$$

Clearly g is CPB_0, and the following sequence of terms satisfies the equations necessary to prove that their presence ensures that a variety omits types $\{1, 2, 4, 5\}$.

$$f_0(x,y,z,u)=x$$
$$f_1(x,y,z,u)=g(x,y,z)$$
$$f_2(x,y,z,u)=g(u,x,y)$$

□

Notice that the above Corollary proves that it is EXPTIME-hard to determine whether or not an algebra generates a variety omitting type set T, where T is any subset of $\{1,2,4,5\}$. In particular it shows that these tests are EXPTIME-complete when T is any one of:

(a) $\{1\}$

(b) $\{1,2\}$

(c) $\{1,5\}$

(d) $\{1,2,5\}$

(e) $\{1,4,5\}$

(f) $\{1,2,4,5\}$.

Note that all of the above can be determined in EXPTIME, as seen in Theorems 2.3.11 through 2.3.16. Note also that the first four of the above are proven in Corollary 9.3 of [15], and that the omission of any other type set is undecidable (see [30]).

Recall the definition of weak near unanimity terms from Theorem 2.3.11.

Corollary 5.1.9. *Testing whether or not an algebra has a weak near unanimity term (or a cyclic term) of arity n is* EXPTIME*-complete.*

Proof. We can easily construct a CPB_0 term which satisfies these equations on a finite set A with $0 \in A$ as follows

$$f(x_0, x_1, \ldots, x_{n-1}) = \begin{cases} x_0 & \text{if } x_0 = x_1 = \ldots = x_{n-1} \\ 0 & \text{otherwise} \end{cases}.$$

That a weak near unanimity term of arity n can be detected in EXPTIME is obvious since we can simply examine every n-ary term operation on an algebra in EXPTIME. □

Notice that a strong E-term for any nontrivial E cannot be CPB for any element, so we have not learned anything about the complexity of detecting such terms. In particular, this explains why the condition $n > 2$ was necessary in Corollary 5.1.7, as a Mal'cev term cannot be CPB. We can, however, conclude that it is difficult to detect the corresponding local strong terms for any such strong term condition.

Recall the definition of local strong term conditions from Definition 3.1.1.

Corollary 5.1.10. *Let E be an $m \times n$ $\mathbf{xy}$-matrix with a column containing exactly one $\mathbf{y}$. Then determining whether or not an algebra has a local E-term on S is* EXPTIME*-complete, where $S = \{(a_i, b_i, i) : i < m\}$ for some a_i, b_i in the algebra.*

Proof. Let A be a finite set, assume without loss of generality that E^0 has exactly one $\mathbf{y}$, namely at E^0_0, and assume without loss of generality that $a_0 = 0$. Then define the following function on A.

$$g(x_0, \ldots, x_{n-1}) = \begin{cases} 0 & \text{if } \overline{x} \text{ is of the form } E_0(a,b) \text{ for some } a, b \in A \text{ with } a \neq b \\ x_0 & \text{otherwise} \end{cases}.$$

Clearly g is CPB_0 and also g is a local E-term on S.

That local strong terms can be detected in EXPTIME is obvious since we can simply examine every n-ary term operation on the input algebra in EXPTIME. □

Note that the proof of the preceding corollary also implies it is EXPTIME-complete to detect a local E-term on S for any S satisfying the condition that for all $(a, b, 0), (a', b', 0) \in S$, we have that $a = a'$.

Corollary 5.1.11. *Let S be any set of $\mathbf{xy}$-matrices of width n for some $n > 0$ such that no $S_i \in S$ has a row of $\mathbf{x}$'s. Testing whether or not an algebra has an S-term is* EXPTIME-*complete.*

Proof. Testing if an algebra has an S-term can certainly be done in EXPTIME since we can simply construct all n-ary term operations on an algebra and check each one individually.

To prove that testing for an S-term is EXPTIME-hard, given any set B with $0 \in B$. Define $g : B^n \to B$ as follows:

$$g(\overline{x}) = \begin{cases} x_0 & \text{if } x_i = x_j \text{ for all } i, j < n \\ 0 & \text{otherwise} \end{cases}.$$

Clearly g is CPB_0, and it is a fairly simple matter to show that g is an S-term. □

6 Decidability of Sequential Term Conditions

6.1 Extended Term Conditions Are Decidable

When considering sequential term conditions (from Definition 2.5.8) it is often a nontrivial matter to determine whether or not the satisfaction of such a condition is even a decidable proposition.

For example, the question of whether or not the presence of a near-unanimity term (see Definition 2.3.17) is decidable was originally raised in [11]. McKenzie proved that given an algebra $\mathbf{A}$ and two elements $x, y \in A$ it is not decidable whether or not $\mathbf{A}$ supports a term which behaves as a near-unanimity term when restricted to the set $\{x, y\}$ (see [25] for the reference). This result was extended by Maróti in [25] to show that it is not decidable whether or not $\mathbf{A}$ supports a term which behaves as a near-unanimity term when restricted to the set $A \setminus \{x, y\}$. In [26], Maróti answered the question by providing a decision procedure which would determine whether or not a finite algebra admitted a near-unanimity term of unspecified arity.

This section presents a modification of Maróti's proof of the decidability of near-unanimity terms, and this modification will allow us to conclude that a wider set of sequential term conditions are similarly decidable.

Definition 6.1.1. Call $\{E_i\}$ an *extended term condition* if it is a sequential strong term condition (See definitions 2.5.8 and 2.5.5) and if there is an $m \times n$ $\mathbf{xy}$-matrix M such that for each i,

$$E_i = \begin{pmatrix} M & X \\ X' & D \end{pmatrix}$$

where X and X' are matrices of the appropriate sizes consisting only of $\mathbf{x}$'s and D is the $i \times i$ matrix with $\mathbf{y}$'s on the diagonal and $\mathbf{x}$'s elsewhere.

Notice that an extended term condition is determined entirely by the underlying $\mathbf{xy}$-matrix M, and so we can refer to it as the M-extended term condition.

Example. t is a k-edge term for some k if and only if t satisfies the $\begin{pmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} \end{pmatrix}$-extended term condition.

The main result of this section will be that it is decidable whether or not a finitely generated clone on a finite set has a term which satisfies a given extended term condition. To this end, fix a finite set A and an extended term condition $\{E_i\}$ with underlying $m \times n$ $\mathbf{xy}$-matrix M.

First we will define a means of characterizing functions in which extended M-terms have a unique and exclusive characteristic, and we will develop a notion of composition of a function with these characteristics. After establishing how these characteristics behave with regard to functional composition, we will define a partial ordering on the set of characteristics in which the unique characteristic of extended M-terms is minimal. We will then distinguish a subset of any clone (which meets a simple assumption) along with its generating set such that:

1. The distinguished subset will contain an extended M-term if there is such a term in the clone, and

2. The functions in the generating set result in a well-behaved set of characteristics.

We will complete the result by proving that the set of minimal elements (with respect to our partial ordering of characteristics) of our distinguished subset's characteristics is computable.

Throughout the results in this section, we will return to the example of the edge term, to lend clarity to the more technical aspects of the proof.

Facts.

- If M has a row of $\mathbf{x}$'s then it can be removed without changing the nature of the extended term condition, so we will assume that M has at least one $\mathbf{y}$ in each row.

- If M has a column of $\mathbf{x}$'s then every clone has an M-term and this problem becomes trivial, so we will assume that M has at least one $\mathbf{y}$ in each column.

- If M has two rows which are identical then one can be removed without changing the nature of the extended term condition, so we will assume that M has no duplicate rows.

- If M has a row of $\mathbf{y}$'s then M can be replaced by the $(m+1)\times(n+1)$ matrix E_1 without changing the nature of the extended term condition, and since E_1 does not have a row of $\mathbf{y}$'s, without loss of generality we can assume that M has at least one $\mathbf{x}$ in each row.

Definition 6.1.2.

- Let ω^+ be the set which contains all finite ordinals and ω, the smallest infinite ordinal.

- Let $\mathcal{B}_A$ be the set of binary operations on A, i.e. $\mathcal{B}_A = \mathcal{O}_A^{(2)}$.

- Let $\mathcal{X}_A \subseteq (\omega^+)^{\mathcal{B}_A}$ denote the set of all $\chi : \mathcal{B}_A \to \omega^+$ such that
  - There is a unique $b \in \mathcal{B}_A$ such that $\chi(b) = \omega$, and
  - $c(\mathbf{x},\mathbf{x}) = b(\mathbf{x},\mathbf{y})$ whenever $\chi(c) > 0$ and $\chi(b) = \omega$.

Definition 6.1.3. For $f \in \mathcal{O}_A^{(k)}$ with $k \geq n$, and $i \geq n-m$ (notice that $n-m$ may be negative) define the ith polymer of f, $f|_i \in \mathcal{B}_A$ to be

$$f|_i(\mathbf{x},\mathbf{y}) = \begin{cases} f(M_{i+m-n}(\mathbf{x},\mathbf{y})\mathbf{x}^{k-n}) & \text{if } i < n \\ f(\mathbf{x}^i\mathbf{y}\mathbf{x}^{k-i-1}) & \text{if } n \leq i < k \\ f(\mathbf{x}^k) & \text{otherwise} \end{cases}$$

Notice that a function f satisfies the M-extended term condition if and only if $f|_i(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for all $n - m \leq i < \omega$.

Definition 6.1.4.

- For simplicity of notation, define $\nu = \{i | n - m \leq i < n\}$.
- Define the characteristic of f to be $\Xi_f = (\alpha_f, \chi_f)$ where $\alpha_f : \nu \to \mathcal{B}_A$ and $\chi_f : \mathcal{B}_A \to \omega^+$ are defined as
$$\alpha_f(i) = f|_i$$
$$\chi_f(b) = |\{i \geq n : f|_i(\mathbf{x}, \mathbf{y}) = b(\mathbf{x}, \mathbf{y})\}|$$
Call α_f the characteristic sequence of f and χ_f the characteristic function of f.
- Let $\Xi_M = (\alpha_M, \chi_M)$ be the characteristic of any term satisfying the M-extended term condition. Then, with $c \in \mathcal{B}_A$ is the projection on the first variable, we have that $\alpha_M(i) = c$ for all $i \in \nu$, and
$$\chi_M(b) = \begin{cases} \omega & \text{if } b = c \\ 0 & \text{otherwise} \end{cases} .$$

Example. When seeking an edge term, we have that $\nu = \{1, 2\}$ and for any $f \in \mathcal{O}_A^{(k)}$, the ith polymer of f is

$$f|_i(\mathbf{x}, \mathbf{y}) = \begin{cases} f(\mathbf{y}\mathbf{y}\mathbf{x}^{k-2}) & \text{if } i = 1 \\ f(\mathbf{y}\mathbf{x}\mathbf{y}\mathbf{x}^{k-3}) & \text{if } i = 2 \\ f(\mathbf{x}^i\mathbf{y}\mathbf{x}^{k-i-1}) & \text{otherwise} \end{cases} .$$

Since we will often be considering characteristics of functions in this section, it is worthwhile to have a characterization of them. Such a characterization is provided by the following lemma.

Lemma 6.1.5. *A pair $(\alpha, \chi) \in (\mathcal{B}_A)^\nu \times (\omega^+)^{\mathcal{B}_A}$ is the characteristic of some function f if and only if:*

- $\chi \in \mathcal{X}_A$*, and*
- $\alpha(i)(\mathbf{x}, \mathbf{x}) = b(\mathbf{x}, \mathbf{y})$ *for all* $i \in \nu$ *where* $\chi(b) = \omega$.

Proof. It is trivial to see that Ξ_f satisfies the listed conditions for any f.

Given (α, χ) satisfying the above conditions, we wish to construct a function $f \in \mathcal{O}_A$ such that $\alpha_f = \alpha$ and $\chi_f = \chi$. By the definition of $\mathcal{X}_A$ there is a unique $b \in \mathcal{B}_A$ such that $\chi(b) = \omega$, therefore $\sum\limits_{c \in \mathcal{B}_A, c \neq b} \chi(c) = k$ is finite so we can choose a sequence $\xi_i \in \mathcal{B}_A$ for $i \geq n$ such that $\chi(c) = |\{i : \xi_i = c\}|$ for each $c \in \mathcal{B}_A$. For each $i \in \nu$, set $\xi_i = \alpha(i)$.

Then there is a function $f \in \mathcal{O}_A^{(n+k)}$ such that:

$$\begin{aligned} f(M_{i+m-n}(\mathbf{x}, \mathbf{y})\mathbf{x}^{k-n}) &= \xi_i(\mathbf{x}, \mathbf{y}) && \text{if } i \in \nu \\ f(\mathbf{x}^i \mathbf{y} \mathbf{x}^{n+k-i-1}) &= \xi_i(\mathbf{x}, \mathbf{y}) && \text{if } n \leq i < n+k \\ f(\mathbf{x}^{n+k}) &= b(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Clearly then, $\Xi_f = (\alpha, \chi)$. □

Recall the definition of a k-extension of a function from Definition 2.5.7. Also notice that if g' is a k-extension of g for some $k \geq n$, then $\Xi_{g'} = \Xi_g$.

Definition 6.1.6.

- Given $\mathcal{F} \subseteq \mathcal{O}_A$, let $T(\mathcal{F}) = \{\Xi_f : f \in \mathcal{F}\}$ be the set of all characteristics of functions in $\mathcal{F}$, and let $T_A = T(\mathcal{O}_A)$.
- Given $\mathcal{U} \subseteq T_A$, let $X(\mathcal{U})$ be the projection of $\mathcal{U}$ onto the second coordinate, in particular $XT(\mathcal{F}) = \{\chi_f : f \in \mathcal{F}\}$.

  Notice that $XT(\mathcal{O}_A) = \mathcal{X}_A$.

Definition 6.1.7.

- By a composition of $f \in \mathcal{O}_A^{(k)}$ with n-extensions of $g_0, \ldots, g_{k-1} \in \mathcal{O}_A$ we mean an operation of the form $f(g_0', \ldots, g_{k-1}') \in \mathcal{O}_A^{(\ell)}$ where $g_i' \in \mathcal{O}_A^{(\ell)}$ is an n-extension of g_i.
- Say that $\Xi = (\alpha, \chi) \in T_A$ is a composition of $f \in \mathcal{O}_A^{(k)}$ with $\Xi_0, \ldots, \Xi_{k-1} \in T_A$ (where $\Xi_i = (\alpha_i, \chi_i)$) if

  $$\alpha(i) = f(\alpha_0(i), \ldots, \alpha_{k-1}(i))$$

  for all $i \in \nu$, and there is a $\mu : (\mathcal{B}_A)^k \to \omega^+$ such that

  $$\chi(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^k, f(\bar{b}) = c} \mu(\bar{b}) \text{ and}$$

  $$\chi_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^k, b_i = c} \mu(\bar{b})$$

  for every $c \in \mathcal{B}_A$ and every $i < k$.
- Given $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$, let $C_{\mathcal{F}}(\mathcal{G})$ be the set of all possible compositions of operations from $\mathcal{F}$ with n-extensions of operations from $\mathcal{G}$. Given $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq T_A$, let $C_{\mathcal{F}}(\mathcal{U})$ be the set of all possible compositions of operations from $\mathcal{F}$ with characteristics from $\mathcal{U}$.

- Also, inductively define $C_{\mathcal{F}}^{i+1}(\cdot) = C_{\mathcal{F}}(C_{\mathcal{F}}^{i}(\cdot))$ where $C_{\mathcal{F}}^{0}$ is the identity map.

Example. Suppose that we are searching for an edge term and say that g is a 4-ary weak near unanimity term (see Theorem 2.3.11). Letting p be the binary function such that $p(\mathbf{xy}) = \mathbf{x}$, we can see that

$$\alpha_g = (g(\mathbf{yyxx}), g(\mathbf{yxyx})) \text{ and}$$

$$\chi_g(q) = \begin{cases} \omega & \text{if } q = p \\ 1 & \text{if } q(\mathbf{xy}) = g(\mathbf{yxxx}) \\ 0 & \text{otherwise} \end{cases}.$$

Let us now define a 7-ary function f by

$$f(\overline{x}) = g(g(x_0, x_1, x_2, x_3), g(x_0, x_1, x_2, x_4), g(x_0, x_1, x_2, x_5), g(x_0, x_1, x_2, x_6)).$$

Clearly f is a composition of g with 3-extensions of g, and if $r(\mathbf{xy}) = g(\mathbf{yxxx})$ then we can easily see that

$$\alpha_f = (g(\mathbf{yyxx}), g(\mathbf{yxyx})) \text{ and}$$

$$\chi_f(q) = \begin{cases} \omega & \text{if } q = p \\ 4 & \text{if } q(\mathbf{x}, \mathbf{y}) = r(\mathbf{x}, r(\mathbf{x}, \mathbf{y})) \\ 0 & \text{otherwise} \end{cases}.$$

It is also worth noticing that Ξ_f is a composition of g with copies of Ξ_g, and this can be witnessed by $\mu : (\mathcal{B}_A)^4 \to \omega^+$ where

$$\mu(\overline{b}) = \begin{cases} \omega & \text{if } \overline{b} = (p,p,p,p) \\ 1 & \text{if } \overline{b} \text{ is any of } (r,p,p,p), (p,r,p,p), (p,p,r,p), \text{ or } (p,p,p,r) \\ 0 & \text{otherwise} \end{cases}.$$

Next we will show that composition commutes with the reduction to characteristics.

Lemma 6.1.8. *$TC_{\mathcal{F}}(\mathcal{G}) = C_{\mathcal{F}}T(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$.*

Proof. To prove that $TC_{\mathcal{F}}(\mathcal{G}) \subseteq C_{\mathcal{F}}T(\mathcal{G})$, take $f \in \mathcal{F}^{(k)}$ and $g_0, \ldots, g_{k-1} \in \mathcal{G}$ with $h = f(g_0', \ldots, g_{k-1}')$ where $g_i' \in \mathcal{O}_A^{(\ell)}$ is an n-extension of g_i. Then we need to prove that Ξ_h is a composition of f with $\Xi_{g_0}, \ldots, \Xi_{g_{k-1}}$.

For any $i \in \nu$, notice that

$$\begin{aligned} \alpha_h(i) &= h|_i \\ &= f(g_0'|_i, \ldots, g_{k-1}'|_i) \\ &= f(g_0|_i, \ldots, g_{k-1}|_i) \\ &= f(\alpha_{g_0}(i), \ldots, \alpha_{g_{k-1}}(i)). \end{aligned}$$

Now define $\mu : (\mathcal{B}_A)^k \to \omega^+$ as

$$\mu(\overline{b}) = |\{j \geq n : (g_0'|_j, \ldots, g_{k-1}'|_j) = \overline{b}\}|.$$

Then for each $c \in \mathcal{B}_A$,

$$\begin{aligned}\sum_{\bar{b}\in(\mathcal{B}_A)^k, f(\bar{b})=c} \mu(\bar{b}) &= |\{j \geq n : f(g'_0|_j, \ldots, g'_{k-1}|_j) = c\}| \\ &= |\{j \geq n : h|_j = c\}| \\ &= \chi_h(c)\end{aligned}$$

and for each $c \in \mathcal{B}_A$ and each $i < k$,

$$\begin{aligned}\sum_{\bar{b}\in(\mathcal{B}_A)^k, b_i=c} \mu(\bar{b}) &= |\{j \geq n : g'_i|_j = c\}| \\ &= \chi_{g'_i}(c) \\ &= \chi_{g_i}(c)\end{aligned}$$

completing the proof that Ξ_h is a composition of f with $\Xi_{g_0}, \ldots, \Xi_{g_{k-1}}$.

To prove that $C_{\mathcal{F}}T(\mathcal{G}) \subseteq TC_{\mathcal{F}}(\mathcal{G})$, take $\Xi = (\alpha, \chi)$, a composition of $f \in \mathcal{F}^{(k)}$ with $\Xi_{g_0}, \ldots, \Xi_{g_{k-1}}$, where $g_i \in \mathcal{G}^{(\ell_i)}$ and let $\mu : (\mathcal{B}_A)^k \to \omega^+$ witness this composition. Specifically, $\alpha(i) = f(\alpha_{g_0}(i), \ldots, \alpha_{g_{k-1}}(i))$ for all $i \in \nu$ and

$$\chi(c) = \sum_{\bar{b}\in(\mathcal{B}_A)^k, f(\bar{b})=c} \mu(\bar{b}) \text{ and}$$

$$\chi_{g_i}(c) = \sum_{\bar{b}\in(\mathcal{B}_A)^k, b_i=c} \mu(\bar{b})$$

for all $c \in \mathcal{B}_A$ and each $i \in \nu$. To complete this proof we need to find $g'_0, \ldots, g'_{k-1} \in \mathcal{O}_A^{(\ell)}$ for some ℓ such that g'_i is an n-extension of g_i and such that $\Xi_h = \Xi$ where $h = f(g'_0, \ldots, g'_{k-1})$.

Let $\zeta : \{j : j \geq n\} \to (\mathcal{B}_A)^k$ be a mapping such that

$$\mu(\bar{b}) = |\{j \geq n : \zeta(j) = \bar{b}\}|$$

for all $\bar{b} \in (\mathcal{B}_A)^k$. Then we have that

$$\begin{aligned}|\{j \geq n : g_i|_j = c\}| &= \chi_{g_i}(c) \\ &= \sum_{\bar{b}\in(\mathcal{B}_A)^k, b_i=c} \mu(\bar{b}) \\ &= |\{j \geq n : \zeta(j)_i = c\}|\end{aligned}$$

for each $i < k$ and each $c \in \mathcal{B}_A$, so for each $i < k$ we can choose a permutation $\sigma'_i : \{j : j \geq n\} \to \{j : j \geq n\}$ such that $g_i|_j = \zeta(\sigma'_i(j))_i$ for all $j \geq n$. For each $i < k$ define $\sigma_i : \omega \to \omega$ as

$$\sigma_i(j) = \begin{cases} j & \text{if } j < n \\ \sigma'_i(j) & \text{otherwise} \end{cases}$$

Letting $\ell = \max\{\sigma_i(j) : i < k, j < \ell_i\}$, then for each $i < k$ the restriction of σ_i to $\{j : j < \ell_i\}$ is an injection into the set $\{i : i < \ell\}$. Define the operations $g'_0, \ldots, g'_{k-1} \in \mathcal{O}_A^{(\ell)}$ as

$$g'_i(x_0, \ldots, x_{\ell-1}) = g(x_{\sigma'_i(0)}, \ldots, x_{\sigma'_i(\ell_i)})$$

Clearly each g'_i is an n-extension of g_i and $g'_i|_j = g_i|_{\sigma_i^{-1}(j)}$, so let $h = f(g'_0, \ldots, g'_{k-1})$. Then for each $c \in \mathcal{B}_A$,

$$\begin{aligned}
\chi_h(c) &= |\{j \geq n : h|_j = c\}| \\
&= |\{j \geq n : f(g'_0|_j, \ldots, g'_{k-1}|_j) = c\}| \\
&= |\{j \geq n : f(g_0|_{\sigma_0^{-1}(j)}, \ldots, g_{k-1}|_{\sigma_{k-1}^{-1}(j)}) = c\}| \\
&= |\{j \geq n : f(\zeta(\sigma_0(\sigma_0^{-1}(j)))_0, \ldots, \zeta(\sigma_{k-1}(\sigma_{k-1}^{-1}(j)))_{k-1}) = c\}| \\
&= |\{j \geq n : f(\zeta(j)) = c\}| \\
&= \sum_{\bar{b} \in (\mathcal{B}_A)^k, f(\bar{b}) = c} \mu(\bar{b}) \\
&= \chi(c)
\end{aligned}$$

Also for each $i \in \nu$ it is clear that

$$\begin{aligned}
\alpha_h(i) &= h|_i \\
&= f(g'_0|_i, \ldots, g'_{k-1}|_i) \\
&= f(g_0|_i, \ldots, g_{k-1}|_i) \\
&= \alpha(i)
\end{aligned}$$

therefore clearly $\Xi_h = \Xi$, completing the proof. □

Definition 6.1.9.

- For $f \in \mathcal{O}_A^{(k)}$ and $i < n$, define $\delta_i(f) \in \mathcal{O}_A^{(n+k-1)}$ as

$$\delta_i(f)(x_0, \ldots, x_{n+k-2}) = f(x_i, x_n, x_{n+1}, \ldots, x_{n+k-2})$$

  and for each $i > 0$, inductively define $\gamma_i(f) \in \mathcal{O}_A^{(ki-i+1)}$ as

$$\gamma_1(f)(x_0, \ldots, x_{k-1}) = f(x_0, \ldots, x_{k-1})$$

$$\gamma_{i+1}(f)(x_0, \ldots, x_{ki+k-i-1}) = f(\gamma_i(f)(x_0, \ldots, x_{ki-i}), x_{ki-i+1}, \ldots, x_{ki+k-i-1})$$

- For $f \in \mathcal{O}_A$, define $\Gamma(f) = \{\gamma_i(f) : i > 0\}$
- For $\mathcal{F} \subseteq \mathcal{O}_A$, define $\Delta(\mathcal{F}) = \{\delta_i(f) : f \in \mathcal{F}, i < n\}$.

Recall (Theorem 2.3.11) that we call $f \in \mathcal{O}_A^{(k)}$ a weak near unanimity operation if f is idempotent and $f(\mathbf{x}^i\mathbf{y}\mathbf{x}^{k-i-1}) = f(\mathbf{x}^j\mathbf{y}\mathbf{x}^{k-j-1})$ for each $i, j < k$.

Lemma 6.1.10. *Let $\mathcal{F} \subseteq \mathcal{O}_A$ and let $g \in \langle \mathcal{F} \rangle^{(k)}$ be a weak near unanimity operation. Then there is a weak near unanimity operation $g' \in \langle \mathcal{F} \rangle^{(\ell)}$ for some ℓ such that $\gamma_2(g')(\mathbf{y}\mathbf{x}^{2\ell-2}) = g'(\mathbf{y}\mathbf{x}^{\ell-1})$.*

Proof. Construct a sequence of functions $g_i \in \langle \mathcal{F} \rangle^{(k^i)}$ as follows.

$$g_1(x_0, \ldots, x_{k-1}) = g(x_0, \ldots, x_{k-1})$$

$$g_{i+1}(x_0, \ldots, x_{k^{i+1}-1}) = g(g_i(x_0, \ldots, x_{k^i-1}), \ldots, g_i(x_{(k-1)k^i}, \ldots, x_{k^{i+1}-1})).$$

Clearly, the associated binary functions, $h_i(\mathbf{x}, \mathbf{y}) = g_i(\mathbf{y}\mathbf{x}^{k^i-1})$, have the property that

$$h_{i+1}(\mathbf{x}, \mathbf{y}) = h_1(\mathbf{x}, h_i(\mathbf{x}, \mathbf{y}))$$

and so we also know that

$$h_{i+j}(\mathbf{x}, \mathbf{y}) = h_i(\mathbf{x}, h_j(\mathbf{x}, \mathbf{y})).$$

By construction, we know that $h_{m!} = h_{2m!}$, and so we can choose $g' = g_{m!}$ with $\ell = k^{m!}$, completing the proof.

$\square$

Lemma 6.1.11. *Let $\mathcal{F} \subseteq \mathcal{O}_A^{(k)}$ and let $g \in \langle \mathcal{F} \rangle^{(k)}$ be a weak near unanimity operation such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Then $\langle \mathcal{F} \rangle$ contains an operation which satisfies the M-extended term condition if and only if $\Xi_M \in \bigcup_{i<\omega} C^i_{\mathcal{F}} T(\Delta\Gamma(g))$.*

Proof. Since $\bigcup_{i<\omega} C^i_{\mathcal{F}} \Delta\Gamma(g) \subseteq \langle \mathcal{F} \rangle$ the reverse implication trivially follows from Lemma 6.1.8.

If $f \in \langle \mathcal{F} \rangle^{(\ell)}$ satisfies the M-extended term condition (necessitating that $\ell \geq n$), then we will construct an operation in $\bigcup_{i<\omega} C^i_{\mathcal{F}} \Delta\Gamma(g)$ which satisfies the M-extended term condition. This is sufficient to prove the lemma since $T(\bigcup_{i<\omega} C^i_{\mathcal{F}} \Delta\Gamma(g)) = \bigcup_{i<\omega} C^i_{\mathcal{F}} T(\Delta\Gamma(g))$ (by Lemma 6.1.8).

- For each $n - m < j < \ell$, let η_j be the $(k-1)$-length sequence of variables $x_{\ell+(j-n+m)(k-1)}$ through $x_{\ell+(j-n+m)(k-1)+k-2}$. Essentially we simply need each η_j to be a sequence of variables with indices at least ℓ such that no two η_j's have any variables in common.
- For each $i < n$, define e_i to be the sequence of variables obtained by concatenating each η_j for which $M^i_{j+m-n} = \mathbf{y}$ where $j \in \nu$. In other words, for each $i < n$ we are going to construct e_i such that if the ith column of M has a $\mathbf{y}$ in row $j + m - n$, the corresponding η_j will be included in e_i.
- For each $n \leq i < \ell$, let $e_i = \eta_i$.
- For each $i < n$ let $m_i = |\{j : M^i_j = \mathbf{y}\}|$.

- For each $i < \ell$ define a function θ_i of arity $k\ell - (n-m)(k-1)$ as follows:

$$\theta_i(\overline{x}) = \begin{cases} \gamma_{m_i}(g)(x_i, e_i) & \text{if } i < n \\ g(x_i, e_i) & \text{if } n \leq i < \ell \end{cases}.$$

- Define a function $h \in \bigcup_{i<\omega} C^i_{\mathcal{F}}\Delta\Gamma(g)$ of arity $k\ell - (n-m)(k-1)$ as follows.

$$h(\overline{x}) = f(\theta_0(\overline{x}), \theta_1(\overline{x}), \ldots, \theta_{\ell-1}(\overline{x})).$$

Notice that $\gamma_{m_i}(g)(x_i, e_i)$ is a function of arity $m_i(k-1)+1$ such that the only variable of $\{x_0, \ldots, x_{n-1}\}$ not simply discarded is x_i. This tells us that $\gamma_{m_i}(g)(x_i, e_i)$ is an n-extension of $\delta_i\gamma_{m_i}(g)$ for each i, and so h is in the relevant set.

We will step aside from the proof for a moment in order to clarify this construction with an example.

Example. If we are searching for an edge term, suppose that $f \in \langle\mathcal{F}\rangle^{(4)}$ is an edge term and g is a 4-ary weak near unanimity term satisfying the specified condition. Then:

- $e_0 = x_4x_5x_6x_7x_8x_9$
- $e_1 = x_4x_5x_6$
- $e_2 = x_7x_8x_9$
- $e_3 = x_{10}x_{11}x_{12}$ and
- $h(\overline{x}) = f(g(g(x_0x_4x_5x_6)x_7x_8x_9), g(x_1x_4x_5x_6), g(x_2x_7x_8x_9), g(x_3x_{10}x_{11}x_{12}))$.

Now let us continue with the proof.

Let p be the binary function defined by $p(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x}^{k-1})$. We will now prove that $h|_i(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for every i. (Cases 2, 4, and 5 are each similar to and simpler than the case they immediately follow.)

Case 1: If $i \in \nu$, then

$$h|_i(\mathbf{x}, \mathbf{y}) = f(M_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}) = \mathbf{x}$$

Proof. First, let us apply the definition of the ith polymer of h.

$$h|_i(\mathbf{x}, \mathbf{y}) = h(M_{i+m-n}(\mathbf{x}, \mathbf{y}), \mathbf{x}^{\ell-n}) = f(a_0, a_1, \ldots, a_{\ell-1})$$

where each a_j is a binary operation in variables $\mathbf{x}$,$\mathbf{y}$.

Notice that every input variable later than x_{n-1} receives an input of $\mathbf{x}$, so $a_j(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for each $j \geq n$. This also tells us that

$$a_j(\mathbf{x}, \mathbf{y}) = \gamma_{m_j}(g)(M^j_{i+m-n}, \mathbf{x}^{m_j(k-1)}) = p(\mathbf{x}, M^j_{i+m-n}) \text{ for each } j < n, \text{ and so}$$

$$h|_i(\mathbf{x}, \mathbf{y}) = f(a_0, a_1, \ldots, a_{\ell-1}) = f(M_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}).$$

□

Case 2: If $n \leq i < \ell$, then

$$h|_i(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}^i, p(\mathbf{x}, \mathbf{y}), \mathbf{x}^{\ell-i-1}) = \mathbf{x}$$

Case 3: If $\ell \leq i < \ell + m(k-1)$ then there is some $j < m$ such that $\ell + j(k-1) \leq i < \ell + (j+1)(k-1)$, and so

$$h|_i(\mathbf{x}, \mathbf{y}) = f(M_j(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}) = \mathbf{x}$$

Proof. Notice first that the only input variable receiving a value of $\mathbf{y}$ is x_i, all others receive a value of $\mathbf{x}$. Also notice that the only η in which x_i appears is η_j, and so for each $b < n$, x_i will appear in e_b exactly if $M^j_{b+m-n} = \mathbf{y}$. Now as in case 1 (above),

$$h|_i(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}^i, \mathbf{y}, \mathbf{x}^{\ell-i-1}) = f(a_0, a_1, \ldots, a_{\ell-1})$$

where each a_b is a binary operation in variables $\mathbf{x}$, $\mathbf{y}$. We know that for each $b < n$, $a_b(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, M_{b+m-n})$ and for each $b \geq n$, $a_b(\mathbf{x}, \mathbf{y}) = \mathbf{x}$, therefore

$$h|_i(\mathbf{x}, \mathbf{y}) = f(a_0, a_1, \ldots, a_{\ell-1}) = f(M^j_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}).$$

□

Case 4: For $m \leq j < \ell - n + m$ if $\ell + j(k-1) \leq i < \ell + (j+1)(k-1)$, then

$$h|_i(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}^j, p(\mathbf{x}, \mathbf{y}), \mathbf{x}^{\ell-i-1}) = \mathbf{x}$$

Case 5: If $i \geq k\ell - (n-m)(k-1)$, then

$$h|_i(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}^\ell) = \mathbf{x}$$

Therefore h satisfies the M-extended term condition. □

The preceding lemma gives us a distinguished subset of $\langle \mathcal{F} \rangle$, namely $\bigcup\limits_{i<\omega} C^i_{\mathcal{F}} \Delta\Gamma(g)$ which will contain an extended M-term if the clone does. This distinguished subset is generated by $\Delta\Gamma(g)$, a set for which the characteristics of members are fairly easy to specify (this will be done in the proof of Lemma 6.1.16). Now we will define a partial ordering on characteristics and focus on proving that it is decidable whether or not $\Xi_M \in \bigcup\limits_{i<\omega} C^i_{\mathcal{F}} T(\Delta\Gamma(g))$ through calculating its minimal elements with respect to this partial ordering.

Definition 6.1.12.

- For each $k > 1$, define a partial order $\sqsubseteq_k$ on ω^+ such that 0 and ω are each comparable only with themselves, and for all positive a, b $a \sqsubseteq_k b$ if and only if $a \leq b$ and $k | b - a$.

- Acting coordinatewise, we can extend this partial order to one on $\mathcal{X}_A$, and we can extend this to a partial order on T_A by saying that $(\alpha, \chi) \sqsubseteq_k (\alpha', \chi')$ if and only if $\alpha = \alpha'$ and $\chi \sqsubseteq_k \chi'$.
- For $\mathcal{U} \subseteq T_A$, let $F_k(\mathcal{U})$ denote the order filter (upward closed set) with respect to $\sqsubseteq_k$ generated by $\mathcal{U}$.
- For $\mathcal{U} \subseteq \mathcal{X}_A$, let $F_k(\mathcal{U})$ denote the order filter (upward closed set) with respect to $\sqsubseteq_k$ generated by $\mathcal{U}$.

Notice that ω^+ with partial order $\sqsubseteq_k$ consists of exactly k infinite chains and 2 isolated points. Clearly ω^+ is well-founded under $\sqsubseteq_k$ (i.e. it has no strictly decreasing infinite sequence and no infinite incomparable sequence), hence any finite power of ω^+ is also well-founded under $\sqsubseteq_k$. Since $\mathcal{X}_A \subseteq (\omega^+)^{\mathcal{B}_A}$, $\mathcal{X}_A$ is well-founded under $\sqsubseteq_k$ and so is T_A (since it is simply a finite union of copies of $\mathcal{X}_A$).

Lemma 6.1.13. *Let $k > 0$, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq T_A$. Then $F_k C_{\mathcal{F}}(\mathcal{U}) \subseteq C_{\mathcal{F}} F_k(\mathcal{U})$ and $C_{\mathcal{F}} F_k(\mathcal{U})$ is an order filter.*

Proof. Take $(\alpha, \chi) \in C_{\mathcal{F}}(\mathcal{U})$ with $\chi \sqsubseteq_k \chi'$. Then (α, χ) is a composition of some $f \in \mathcal{F}^{(\ell)}$ with characteristics $(\alpha_i, \chi_i) \in \mathcal{U}$, $i < \ell$, and so there is a $\mu : (\mathcal{B}_A)^\ell \to \omega^+$ such that

$$\chi(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, f(\bar{b}) = c} \mu(\bar{b})$$

$$\chi_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, b_i = c} \mu(\bar{b})$$

We will show that (α, χ') is a composition of f with characteristics in $F_k(\mathcal{U})$.

Let D be the set of all $d \in \mathcal{B}_A$ such that $\chi(d) \neq \chi'(d)$. By the definition of $\sqsubseteq_k$ we know that $0 < \chi(d) < \chi'(d) < \omega$ and $k | (\chi'(d) - \chi(d))$ for all $d \in D$. For each $d \in D$ pick $\bar{b}_d \in (\mathcal{B}_A)^\ell$ such that $f(\bar{b}_d) = d$ and $0 < \mu(\bar{b}_d) < \omega$. Define $\mu' : (\mathcal{B}_A)^\ell \to \omega^+$ as follows

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) + \chi'(d) - \chi(d) & \text{if } \bar{b} = \bar{b}_d \text{ for some } d \in D \\ \mu(\bar{b}) & \text{otherwise} \end{cases}$$

Clearly we can see that

$$\chi'(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, f(\bar{b}) = c} \mu'(\bar{b})$$

and so we can use μ' to define $\chi'_i : \mathcal{B}_A \to \omega^+$ for each $i < \ell$ as

$$\chi'_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, b_i = c} \mu'(\bar{b})$$

It immediately follows that $\chi_i \sqsubseteq_k \chi_i'$ for each $i < \ell$ and so $(\alpha_i, \chi_i) \sqsubseteq_k (\alpha_i, \chi_i')$, demonstrating that $(\alpha_i, \chi_i') \in F_k(\mathcal{U})$. To complete this part of the proof we need only notice that μ' was constructed to witness the fact that (α, χ') is a composition of f with (α_i, χ_i'), $i < \ell$.

To show that $C_{\mathcal{F}}F_k(\mathcal{U})$ is an order filter, notice that

$$F_k C_{\mathcal{F}} F_k(\mathcal{U}) \subseteq C_{\mathcal{F}} F_k F_k(\mathcal{U}) = C_{\mathcal{F}} F_k(\mathcal{U}) \subseteq F_k C_{\mathcal{F}} F_k(\mathcal{U})$$

□

Lemma 6.1.14. *Let $k > 0$ and let $\mathcal{F} \subseteq \mathcal{O}_A$, $\mathcal{U} \subseteq T_A$ be finite sets. Then the $\sqsubseteq_k$-minimal elements of $C_{\mathcal{F}}F_k(\mathcal{U})$ can be effectively computed.*

Proof. Let (α, χ) be an arbitrary minimal element of $C_{\mathcal{F}}F_k(\mathcal{U})$. Then (α, χ) is a composition of $f \in \mathcal{F}^{(\ell)}$ with some characteristics $(\alpha_0, \chi_0), \ldots, (\alpha_{\ell-1}, \chi_{\ell-1}) \in F_k(\mathcal{U})$ witnessed by a mapping $\mu : (\mathcal{B}_A)^\ell \to \omega^+$. Notice that f and μ uniquely determine χ and $\chi_0, \ldots, \chi_{\ell-1}$, and similarly that f, μ and $\alpha_i \in (\mathcal{B}_A)^\nu$, $i < \ell$ uniquely determine (α, χ) and $(\alpha_0, \chi_0), \ldots, (\alpha_{\ell-1}, \chi_{\ell-1})$.

Since $(\mathcal{B}_A)^\ell$ is finite, $(\omega^+)^{(\mathcal{B}_A)^\ell}$ is well-founded under $\sqsubseteq_k$ and so we may assume that μ is minimal among mappings which witness the fact that (α, χ) is a composition of f with elements of $F_k(\mathcal{U})$.

Define $p = \max(\{k\} \cup \{\chi'(b) : \chi' \in X(\mathcal{U}), b \in \mathcal{B}_A, \chi'(b) \neq \omega\})$, which is a natural number dependent only on k and $\mathcal{U}$.

Claim: For all $\bar{b} \in (\mathcal{B}_A)^\ell$, if $\mu(\bar{b}) > p$ then $\mu(\bar{b}) = \omega$.

Proof. To get a contradiction, assume that $p < \mu(\bar{c}) < \omega$ for some $\bar{c} \in (\mathcal{B}_A)^\ell$. Define $\mu' : (\mathcal{B}_A)^\ell \to \omega^+$ as

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) - k & \text{if } \bar{b} = \bar{c} \\ \mu(\bar{b}) & \text{otherwise} \end{cases}$$

and define χ' and $\chi_0', \ldots, \chi_{\ell-1}'$ as

$$\chi'(d) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, f(\bar{b}) = d} \mu'(\bar{b})$$

and

$$\chi_i'(d) = \sum_{\bar{b} \in (\mathcal{B}_A)^\ell, b_i = d} \mu'(\bar{b})$$

Observe that $\mu'(\bar{c}) = \mu(\bar{c}) - k > p - k \geq 0$.

We will now argue that $\mu' \sqsubseteq_k \mu$ and that μ' also witnesses that (α, χ) is a composition of f with elements of $F_k(\mathcal{U})$, contradicting the minimality of μ.

First we must argue that $(\alpha_i, \chi_i') \in F_k(\mathcal{U})$ for each $i < \ell$. Clearly $\chi_i'(b) = \chi_i(b)$ for all $b \neq c_i$ and all $i < \ell$, so let us now consider the value of $\chi_i'(c_i)$.

Case 1: $\chi_i'(c_i) = \omega$, in which case $\chi_i(c_i) = \omega$ as well, hence $(\alpha_i, \chi_i') = (\alpha_i, \chi_i) \in F_k(\mathcal{U})$.

Case 2: $\chi'_i(c_i) = \chi_i(c_i) - k$ and so $\chi'_i(c_i) \geq \mu(\overline{c}) - k > p - k \geq 0$ and so $\chi'_i \in \mathcal{X}_A$. Since $(\alpha_i, \chi_i) \in F_k(\mathcal{U})$ there is a characteristic $(\alpha_i, \chi''_i) \in \mathcal{U}$ such that $\chi''_i \sqsubseteq_k \chi_i$. By choice of p we have that $\chi''_i(c_i) \leq p < \mu(\overline{c}) \leq \chi_i(c_i)$ and so $\chi''_i(c_i) \leq \chi_i(c_i) - k$. Therefore $\chi''_i \sqsubseteq_k \chi'_i$ and so $(\alpha_i, \chi'_i) \in F_k(\mathcal{U})$.

Analogously, $\chi'(b) = \chi(b)$ for all $b \neq f(\overline{c})$, and either $\chi'(f(\overline{c})) = \omega = \chi(f(\overline{c}))$ or $\chi'(f(\overline{c})) = \chi(f(\overline{c})) - k > p - k \geq 0$, hence $\chi' \sqsubseteq_k \chi$. Since $(\alpha_i, \chi'_i) \in F_k(\mathcal{U})$ we get that $(\alpha, \chi') \in C_{\mathcal{F}}F_k(\mathcal{U})$. From the minimality of χ we get that $\chi' = \chi$ and so μ' contradicts the minimality of μ among representations of χ. □

Therefore, the following algorithm will calculate all the $\sqsubseteq_k$-minimal elements of $C_{\mathcal{F}}F_k(\mathcal{U})$.

Algorithm.
Input: Natural number $k > 0$ and finite sets A, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq T_A$.
Output: All the $\sqsubseteq_k$-minimal elements of $C_{\mathcal{F}}F_k(\mathcal{U})$.

(1) Set $R = \emptyset$

(2) For each $f \in \mathcal{F}$ (say $f \in \mathcal{F}^{(\ell)}$) do:

  (a) Set $p = \max(\{k\} \cup \{\chi'(b) : \chi' \in X(\mathcal{U}), b \in \mathcal{B}_A, \chi'(b) \neq \omega\})$

  (b) For each $\mu : (\mathcal{B}_A)^\ell \to \{0, 1, \ldots, p, \omega\}$ do:

    (i) For each $c \in \mathcal{B}_A$ calculate:

$$\chi(c) = \sum_{\overline{b} \in (\mathcal{B}_A)^\ell, f(\overline{b}) = c} \mu(\overline{b})$$

    (ii) For each $c \in \mathcal{B}_A$ and each $i < \ell$ calculate:

$$\chi_i(c) = \sum_{\overline{b} \in (\mathcal{B}_A)^\ell, b_i = c} \mu(\overline{b})$$

    (iii) For each $\overline{\alpha} \in ((\mathcal{B}_A)^\nu)^\ell$ do:
    If all $(\alpha_i, \chi_i) \in F_k(\mathcal{U})$ then $R := R \cup \{(f(\overline{\alpha}), \chi)\}$

(3) The minimal elements of R are the minimal elements of $C_{\mathcal{F}}F_k(\mathcal{U})$.

□

Lemma 6.1.15. *Let $k > 0$ and let $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq T_A$ be finite sets. Then $\bigcup_{i \in \omega} C^i_{\mathcal{F}}F_k(\mathcal{U})$ is an order filter with respect to $\sqsubseteq_k$ and its minimal elements can be effectively computed.*

Proof. By Lemmas 6.1.13 and 6.1.14, $C^i_{\mathcal{F}}F_k(\mathcal{U})$ is an order filter for each $i > 0$ and its minimal elements can be effectively computed. If we let $\mathcal{U}_i$ for $i < \omega$ be defined as the set of minimal elements of $\bigcup_{j\leq i} C^j_{\mathcal{F}}F_k(\mathcal{U})$ then the minimal elements of $\bigcup_{i\in\omega} C^i_{\mathcal{F}}F_k(\mathcal{U})$ will be the minimal elements of $\bigcup_{i\in\omega} \mathcal{U}_i$.

Since T_A is well-founded under $\sqsubseteq_k$, the increasing (under inclusion) sequence of filters $\bigcup_{j<i} C^j_{\mathcal{F}}F_k(\mathcal{U})$ must eventually stabilize and so the sequence of $\mathcal{U}_i$'s must also eventually stabilize. Since we can calculate each $\mathcal{U}_i$, we simply continue to do so until we reach some ℓ such that $\mathcal{U}_\ell = \mathcal{U}_{\ell-1}$, then the minimal elements of $\bigcup_{i\in\omega} C^i_{\mathcal{F}}F_k(\mathcal{U})$ will be the minimal elements of $\bigcup_{i<\ell} \mathcal{U}_i$, a finite set whose elements we will already have computed. □

Lemma 6.1.16. *Let $g \in \mathcal{O}^{(k)}_A$ be a weak near unanimity operation such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$, then $T\Delta\Gamma(g) = F_{k-1}T\Delta(\{g\})$.*

Proof. We will simply calculate the characteristics of every $\delta_i(\gamma_j(g))$ and notice that they form an order filter with respect to $\sqsubseteq_{k-1}$ whose minimal elements are exactly $T\Delta(\{g\})$.

First, let us pick out $p, q \in \mathcal{B}_A$ such that $p(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}\mathbf{x}^{k-1})$ and $q(\mathbf{x}, \mathbf{y}) = \mathbf{x}$. Then for every $i < n$, $j > 0$ and $\ell \in \nu$ let $g' = \delta_i(\gamma_j(g))$ and it is clear that

$$\alpha_{g'}(\ell) = \begin{cases} p & \text{if } M^i_\ell = \mathbf{y} \\ q & \text{otherwise} \end{cases},$$

$$\alpha_{\delta_n(\gamma_j(g))}(\ell) = q$$

and, with regard to the characteristic functions, for every $c \in \mathcal{B}_A$ it is clear that

$$\chi_{g'}(c) = \begin{cases} \omega & \text{if } c = q \\ j(k-1) & \text{if } c = p \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\chi_{\delta_n(\gamma_j(g))}(c) = \begin{cases} \omega & \text{if } c = q \\ j(k-1)+1 & \text{if } c = p \\ 0 & \text{otherwise} \end{cases}.$$

These characteristics clearly satisfy the requirements, completing the proof. □

Recall that if a clone contains a function which satisfies the M-extended term condition, then it will also contain a weak near unanimity term and so Lemma 6.1.10 says that it will also have a weak near unanimity term g satisfying $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$.

Theorem 6.1.17. *Given a finite set $\mathcal{F} \subseteq \mathcal{O}_A$, it is decidable whether or not $\langle\mathcal{F}\rangle$ contains an operation satisfying the M-extended term condition.*

Proof. First, it is decidable whether or not $\langle A, \mathcal{F}\rangle$ omits type 1 (see Theorem 2.3.11). If it does not omit type 1 then there cannot be any term which satisfies the M-extended term condition and so we can halt with that answer, otherwise there must be a weak near unanimity term in $\langle \mathcal{F}\rangle$ and so there must be a weak near unanimity term $g \in \langle \mathcal{F}\rangle^{(k)}$ such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$ (by Lemma 6.1.10). Clearly we can calculate the minimal elements of $F_{k-1}T\Delta(\{g\})$ (given in the proof of Lemma 6.1.16), and so by Lemma 6.1.15 we can compute the minimal elements of $\bigcup_{i\in\omega} C^i_{\mathcal{F}}F_{k-1}T\Delta(\{g\})$, call this set $\mathcal{U}$. Since Ξ_M is minimal with respect to $\sqsubseteq_{k-1}$, Lemma 6.1.11 tells us that $\langle \mathcal{F}\rangle$ will contain a term which satisfies the M-extended term condition if and only if $\Xi_M \in \mathcal{U}$. □

Algorithm.
Input: finite sets A and $\mathcal{F} \subseteq \mathcal{O}_A$
Output: Whether or not $\langle \mathcal{F}\rangle$ contains an operation satisfying the M-extended term condition.

(1) Determine whether or not $\langle A, \mathcal{F}\rangle$ omits type 1. If it does not, then $\langle \mathcal{F}\rangle$ does not contain the desired operation.

(2) Search for a weak near unanimity term $g \in \langle \mathcal{F}\rangle^{(k)}$ such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Since $\langle A, \mathcal{F}\rangle$ omits type 1 we are guaranteed to find such a term.

(3) Calculate $\mathcal{U}$, the set of minimal elements of $\bigcup_{i\in\omega} C^i_{\mathcal{F}}F_{k-1}T\Delta(\{g\})$ (Lemma 6.1.15 explains how to do this).

(4) $\langle \mathcal{F}\rangle$ contains the desired operation if and only if $\Xi_M \in \mathcal{U}$ (this is given by Lemmas 6.1.11 and 6.1.16).

6.2 Partial Runtime Calculations

Now we will address the time and space taken by the algorithms used in this section. Unfortunately, the upperbound we calculate for the complexity of the final algorithm is not one which is reasonable to express with precision.

Let us begin with a few definitions, and some simplifying notation.

Definition 6.2.1.

- Define a function $\Phi : \mathcal{X}_A \to \omega$ by

$$\Phi(\chi) = \sum_{c\in\mathcal{B}_A, \chi(c)\neq\omega} \chi(c)$$

- A function $\mu : (\mathcal{B}_A)^r \to \omega^+$ used for calculating compositions of basic functions with characteristics will be called a composition function, and we can extend the definition of Φ above to allow for its application to such μ.

- For brevity, write $f(q) = \exp O(g(q))$ instead of $\log f(q) = O(g(q))$.
- Abbreviate $\log\log(q)$ as $\log^2(q)$ and $\exp\exp(q)$ as $\exp^2(q)$.

Facts.

- If $f \in \mathcal{O}_A^{(k)}$ then $\Phi(\chi_f) = |\{i : f|_i(\mathbf{x}, \mathbf{y}) \neq f(\mathbf{x}^k)\}| \leq k$
- For all $\chi \in \mathcal{X}_A$, $\chi : \mathcal{B}_A \to \{0, \ldots, \Phi(\chi), \omega\}$.
- Regarding characteristic functions...
  - A characteristic function χ takes $O(|A|^{|A|^2} \log(\Phi(\chi)))$ space.
  - Given $k > 0$ and characteristic functions χ, χ' with $\Phi(\chi') > \Phi(\chi)$, determining whether or not $\chi \sqsubseteq_k \chi'$ takes $O(|A|^{|A|^2}(\log(\Phi(\chi')))^2)$ time.
  - Evaluating a characteristic function at a specified binary operation takes $O(|A|^2 \log(|A|))$ time.
- Given $f \in \mathcal{F}^{(r)}$ and $\overline{c} \in (\mathcal{B}_A)^r$, calculating $f(\overline{c})$ takes $O(r|A|^2(\log(|A|))^2)$ time.
- A composition function $\mu : (\mathcal{B}_A)^r \to \{0, \ldots, \ell, \omega\}$ takes $O(|A|^{r|A|^2} \log(\ell))$ space.
- If $\mu : (\mathcal{B}_A)^r \to \{0, \ldots, \ell, \omega\}$ witnesses the fact that $\chi \in \mathcal{X}_A$ is a composition of $f \in \mathcal{O}_A^{(r)}$ with $\chi_0, \ldots, \chi_{r-1} \in \mathcal{X}_A$, then $\Phi(\chi), \Phi(\chi_i) \leq |A|^{r|A|^2}\ell$.

Recall 6.2.2. A glossary of variables and letters for this section. Newly defined variables are underlined.

- m is the number of rows of $\mathbf{xy}$-matrix M.
- n is the number of columns of $\mathbf{xy}$-matrix M.
- ν is the index set of characteristic sequences. Also, $|\nu| = n$.
- A is a finite set. Let $a = |A|$ for the sake of brevity.
- $\mathcal{F}$ is a finite set of basic operations on A. For simplicity, assume that $\mathcal{F} \subseteq \mathcal{O}_A^{(r)}$ and that $|\mathcal{F}| = b$.

Lemma 6.2.3. *If a finite algebra* $\mathbf{A}$ *generates a variety which omits type* 1*, then it supports a weak near unanimity term of arity less than* $2|A|$.

Proof. By [2], if $\mathbf{A}$ omits type 1 then it supports a cyclic term of every prime arity greater than $|A|$ (a cyclic term is any term which is invariant under cyclic permutation of its variables). Clearly every cyclic term is a weak near unanimity term. There must be a prime number p with $|A| \leq p < 2|A|$, so $\mathbf{A}$ must support a weak near unanimity term of arity $p < 2|A|$. $\square$

The following is a list of simplifications which will be used in the course of estimating the time and space taken by the upcoming algorithms. Each simplification replaces a formula with a larger-valued formula (at least in the limit) which is easier to read.

- $q^k(\log(q))^\ell \to q^{k+\ell}$
- $q! \to q^q$
- $O(2^{f(q)}) \to \exp O(f(q))$
- $b \leq a^{a^r}$

First, we will need to determine the complexity of the algorithm to perform the composition of characteristic functions.

Algorithm. Compositional construction of characteristic functions.
Input: $f \in \mathcal{O}_A^{(r)}$ and $\mu : (\mathcal{B}_A)^r \to \{0, \ldots, \ell, \omega\}$
Output: $r+1$ characteristic functions $\chi, \chi_0, \ldots, \chi_{r-1}$ such that χ is a composition of f with $\chi_0, \ldots, \chi_{r-1}$, and this composition is witnessed by μ.
Notation: $Comp(f, \mu)$

1. Initialize all $r+1$ characteristic functions to 0.
   Takes $\exp O(a^3 \log(r))$ time.
2. For each $\overline{c} \in (\mathcal{B}_A)^r$ do:
   Storing the index takes $O(ra^3)$ space and the loop repeats $\exp O(ra^3)$ times.
   (a) For each $i < r$ do:
   Storing the index takes $O(\log(r))$ space and the loop repeats $O(r)$ times.
      i. Increase $\chi_i(c_i)$ by $\mu(\overline{c})$
      Takes $O(ra^3 \log(\ell))$ time.
   (b) Calculate $f(\overline{c})$
   Uses $O(a^3)$ space and takes $O(ra^4)$ time.
   (c) Increase $\chi(f(\overline{c}))$ by $\mu(\overline{c})$
   Takes $O(ra^3 \log(\ell))$ time.
3. Return $\chi, \chi_0, \ldots, \chi_{r-1}$.
   At no time does the space taken by these variables exceed $\exp O(ra^3 \log(\ell))$

Time: $\exp O(ra^3 \log(\ell))$
Space: $\exp O(ra^3 \log(\ell))$

Notice that if μ witnesses the fact that χ is a composition of $f \in \mathcal{O}_A^{(r)}$ with $\chi_0, \ldots, \chi_{r-1}$, then $\Phi(\mu) \leq \sum_{i<r} \Phi(\chi_i) \leq r \max_{i<r} \Phi(\chi_i)$.

Algorithm. This is the algorithm at the end of the proof of Lemma 6.1.14.
Input: A number $k > 0$ and finite sets $\mathcal{U} \subseteq T_A$ and $\mathcal{F}$.
Output: The $\sqsubseteq_k$-minimal elements of $C_{\mathcal{F}} F_k(\mathcal{U})$.
Notation: $MinComp(k, \mathcal{U}, \mathcal{F})$
Note: For simplicity, let $p = \max(\{k\} \cup \{\Phi(\chi) : \chi \in X(\mathcal{U})\})$, and say $q = |\mathcal{U}|$.
Note: Let Min denote the function which returns the $\sqsubseteq_k$-minimal elements of its input set. We can easily bound the runtime of this function by noticing that we need use at most $|R|^2$ comparisons to calculate $Min(R)$.

1. Set $R = \mathcal{U}$.
2. Calculate p, as defined above.
   Takes $O(\log(p))$ space and $\exp O(a^3 \log(p) \log(q))$ time.
3. For each $f \in \mathcal{F}$ and each $\mu : (\mathcal{B}_A)^r \to \{0, \ldots, p, \omega\}$ do:
   Storing the index takes $\exp^2 O(r \log(a) \log^3(p))$ space and the loop repeats $\exp^2 O(ra^3 \log^2(p))$ times
   (a) Calculate $\chi, \chi_0, \ldots, \chi_{r-1} = Comp(f, \mu)$
   Takes $\exp O(ra^3 \log(p))$ space and $\exp O(ra^3 \log(p))$ time.
   (b) For each $\overline{\alpha} \in ((\mathcal{B}_A)^\nu)^r$ do:
   Storing the index takes $O(nra^3)$ space and the loop repeats $\exp O(nra^3)$ times.
      i. If all $(\alpha_i, \chi_i) \in F_k(\mathcal{U})$ then set $R = R \cup \{(f(\overline{\alpha}), \chi)\}$.
      Each membership test in $F_k(\mathcal{U})$ takes $\exp O(na^3 \log(r) \log(q) \log^2(p))$ time and there are r such tests. If all these tests are passed, then we must calculate $f(\overline{\alpha})$ (taking $O(ra^4)$ time).
4. return $Min(R)$.
   Notice that for each (α, χ) in R at any point during this algorithm, $\Phi(\chi) \leq rp$, and so

$$|R| \leq a^{na^2} (a^{a^2})^{\frac{1}{(a^{a^2})!(a^{a^2}-1)!}(rp + a^{a^2} + \frac{(a^{a^2})(a^{a^2}-3)}{4})^{a^{a^2}-1} + 1}$$

   (see [33] for details). Calculating $Min(R)$ will therefore need $O(|R|^2)$ comparisons taking $O(a^{a^2} r^2 p^2)$ time each.
   To simplify notice that $|R| = \exp^3 O(a^3 \log^2(n) \log^2(p) \log^2(r))$.
   Therefore this step will take at most $\exp^3 O(a^3 \log^2(n) \log^2(p) \log^2(r))$ time and R will use at most $\exp^3 O(a^3 \log^2(n) \log^2(p) \log^2(r))$ space.

Time: $\exp^3 O(a^3 \log(r) \log^2(n) \log^2(p))$
Space: $\exp^3 O(a^3 \log(r) \log^2(n) \log^2(p))$

Algorithm. The main algorithm of this section.
Input: Finite sets A and $\mathcal{F} \subseteq \mathcal{O}_A$, and $m \times n$ $\mathbf{xy}$-matrix M
Output: Whether or not $\langle \mathcal{F} \rangle$ contains a term satisfying the M-extended term condition.
Note: As stated before, assume that $\mathcal{F} \subseteq \mathcal{O}_A^{(r)}$.

1. Determine whether or not $\langle \mathcal{F} \rangle$ omits type 1. If not, return False.
   This can be done by constructing the 2-generated free algebra in $V(\mathbf{A})$ and searching for a Siggers term, taking $\exp O(a^3)$ space and time (see Theorem 2.3.11)

2. Search for a weak near unanimity term $g \in \langle \mathcal{F} \rangle^{(k)}$ such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$.
   In order to do this, we can simply construct the free algebra in $V(\mathbf{A})$ on $2a$ generators (taking $\exp^2 O(a^2 \log^2(r))$ time and $\exp^2 O(a^2)$ space), find a weak near unanimity term (taking $\exp^2 O(a^2)$ time), and compose it with itself as in Lemma 6.1.10 (taking $\exp^3 O(a^2)$ time and resulting in a function g of arity at most $(2a)^{a^a}$ which takes at most $\exp^3 O(a^2)$ space.

3. Calculate $\mathcal{U}_0 = T\Delta(\{g\})$.
   Taking $\exp^2 O(a^3)$ space and time. Notice that every $\chi \in XT\Delta(\{g\})$ has $\Phi(\chi) \leq (2a)^{a^a}$.

4. For each $i \geq 0$, do:
   (a) Let $\mathcal{U}_{i+1} = MinComp(arity(g) - 1, \mathcal{U}_i, \mathcal{F})$.
   For each i, let $\varphi_i = \max\{\Phi(\chi) : \chi \in X(\mathcal{U}_i)\}$. We can then see that this step takes $\exp^3 O(a^3 \log(r) \log^2(n) \log^2(\varphi_i))$ space and time.
   (b) If $\mathcal{U}_{i+1} = \mathcal{U}_i$, stop looping and let $\mathcal{U} = \mathcal{U}_i$.
   This comparison takes no additional space and $\exp^3 O(a^3 \log(r) \log^2(n) \log^2(\varphi_i))$ time.

5. If $\Xi_M \in \mathcal{U}$ then $\langle \mathcal{F} \rangle$ has the the desired function, otherwise not.

Notice that at the beginning of step 4, there was no indication of when the iteration would cease. From [14] we can derive an upperbound on the number of iterations the algorithm requires before termination, though the calculation is intricate if one wishes to be specific. For our purposes it suffices to say that the best known upperbound on the number of iterations is not primitive recursive. More specifically the number of iterations of the main loop of the preceding algorithm will be bounded above by a function in $\mathcal{F}_{1+a^{a^2}}$ of the Fast Growing Hierarchy (as used in [14] among other places). It is unknown whether or not this upperbound for the runtime of this algorithm is the best feasible, but no further significant improvements can be made on the structure of $\langle T_A, \sqsubseteq_{k-1} \rangle$ alone as the bounds presented in [14] are tight.

6.3 Other Considerations

Since there are infinitely many matrices which will define equivalent extended term conditions, it is worth attempting to classify extended term conditions based on non-equivalence. While this task will certainly not be completed here, we can begin it with a few simple observations.

Lemma 6.3.1. *Given $M \in M_{m\times n}(\{\mathbf{x},\mathbf{y}\})$ and algebra $\mathbf{A}$, if $\mathbf{A}$ supports a near unanimity term then it supports an extended M-term.*

Proof. Suppose that f is a k-ary near unanimity term on $\mathbf{A}$ with $k \geq 3$. Define $f' : A^{m+k} \to A$ to be:

$$f'(x_0, x_1, \ldots, x_{m+k-1}) = f(x_m, x_{m+1}, \ldots, x_{m+k-1})$$

Let E_k be the $(m+k)\times(n+k)$ matrix which is kth in the sequence defining an extended M-term, then it is clear that f' is a strong E_k-term, and so f' is an extended M-term. □

Lemma 6.3.2. *[6] If an algebra $\mathbf{A}$ supports a strong E-term for some E, then $\mathbf{A}$ supports an edge term.*

In [24] a complete characterization is given of those strong term conditions whose possession is equivalent to that of a near unanimity term, and it is natural to ask whether or not there are any extended term conditions equivalent neither to an edge term nor a near unanimity term. While no comprehensive classification of extended term conditions is evident, the following lemma demonstrates that such a classification must have infinitely many equivalence classes.

For each $k \geq 3$ let M_k be the $k\times k$ matrix with $\mathbf{x}$'s on the diagonal and $\mathbf{y}$'s elsewhere.

Lemma 6.3.3. *Let V be a vector space over a field F of positive characteristic p. Then for each $k \geq 3$, V supports an extended M_k-term if and only if $p|(k-1)$.*

Proof. Let f be an extended M_k-term over V of minimal arity, say that f has arity $n \geq k$. We can write f in the form

$$f(x_0, x_1, \ldots, x_{n-1}) = a_0x_0 + a_1x_1 + \ldots + a_{n-1}x_{n-1}$$

for some $a_i \in F$, since every term in a vector space can be written in this way (exercise). Suppose that $n > k$. Then for any $\mathbf{v} \neq \mathbf{0} \in V$, we know that $\mathbf{0} = f(\mathbf{0}^{n-1}\mathbf{v}) = a_0\mathbf{0} + a_1\mathbf{0} + \ldots + a_{n-2}\mathbf{0} + a_{n-1}\mathbf{v}$ and so $a_{n-1} = 0$. Let us then define $g(x_0, x_1, \ldots, x_{n-2}) = a_0x_0 + a_1x_1 + \ldots + a_{n-2}x_{n-2}$. g is clearly a term operation on V and it also must be an extended M_k-term of arity $n-1$, contradicting our assumption of minimality. This proves that V supports an extended M_k-term if and only if V supports a strong M_k-term and so $n = k$.

Now for each $i < k$ and each $\mathbf{v} \neq \mathbf{0} \in V$ we know that $\mathbf{v} = f(\mathbf{0}^i\mathbf{v}\mathbf{0}^{k-i-1}) = a_0\mathbf{0} + a_1\mathbf{0} + \ldots + a_{i-1}\mathbf{0} + a_i\mathbf{v} + a_{i+1}\mathbf{0} + \ldots + a_{n-1}\mathbf{0}$ and so $a_i = 1$. We also know that $\mathbf{0} = f(\mathbf{0}\mathbf{v}^{k-1}) = (k-1)\mathbf{v}$ and so $k - 1 = 0$ in F, i.e. $p|(k-1)$. From this we know that if V supports a strong M_k-term then $p|(k-1)$.

Clearly if $p|(k-1)$ then $f(x_0, \ldots, x_{k-1}) = \sum_{i<k} x_i$ is a strong M_k-term, completing the proof. □

Lemma 6.3.2 tells us that every such vector space supports an edge term (in fact every vector space supports a Mal'cev term) and no vector space supports a near unanimity term, as they are not congruence distributive.Therefore we can conclude that for each prime p, the possession of an extended M_{p+1}-term is equivalent to neither the possession of an edge term nor the possession of a near unanimity term. Furthermore if $p \neq q$ are primes then the possession of an extended M_{p+1}-term is not equivalent to the possession of an extended M_{q+1}-term.

It is worth asking whether or not there is likely to be an improvement in complexity if we assume that our algebra is idempotent. In fact one such result (specific to edge terms) already exists; it demonstrates that detection of an edge operation on an idempotent algebra is in co-NP.

Theorem 6.3.4. *[28] For a finite set of idempotent operations $\mathcal{F}$ on a finite set A, $\langle \mathcal{F} \rangle$ contains an edge operation if and only if for every subalgebra S of $\mathbf{A}$ and every nonempty subset D of S there is an $f \in \mathcal{F}^{(n)}$ (for some $n > 1$) and an $i < n$ such that $f(S^iDS^{n-i-1}) \nsubseteq D$.*

In order to demonstrate that detection of an edge term operation is in co-NP therefore, we need only provide a subalgebra S of $\mathbf{A}$ and a nonempty subset D of S (both of which of course have cardinality less than that of A) and it is easy to check that for every basic operation $f \in \mathcal{F}^{(n)}$ and every $i < n$, $f(S^iDS^{n-i-1}) \subseteq D$

7 Conclusion

Let us conclude with an overview of the current state of algorithmic complexity with respect to solving those problems dealt with in this document. In each case the problem we aim to solve is, when given a finite algebra, to determine whether or not the variety generated by that algebra satisfies the given condition. We will divide these problems into classes based on what is known about their complexity in both the general and idempotent cases.

Let $\mathbf{A}$ represent the input algebra for each of the following problems.

- If $\mathbf{A}$ is idempotent, detecting each of the following can be done in polynomial time, otherwise their detections are each EXPTIME-complete.
  - A weak near unanimity term of unspecified arity. (3.2.11 and 5.1.9)
  - Weak near unanimity terms of all but finitely many arities. (3.2.12 and 5.1.8)
  - A Siggers term. (3.2.11 and 5.1.8)
  - A sequence of Jónsson terms of unspecified length. (3.2.4 and 5.1.6)
  - A sequence of Hagemann-Mitschke terms of unspecified length. (3.2.10 and 5.1.7)
  - A sequence of Gumm terms of unspecified length. (3.2.5 and 5.1.6)
  - A sequence of Day terms of unspecified length. (3.2.5 and 5.1.6)
  - A sequence of terms of unspecified length guaranteeing omission of types 1 and 5. (3.2.10 and 5.1.8)
  - A sequence of Hobby-McKenzie terms of unspecified length. (3.2.13 and 5.1.8)
  - A sequence of terms of unspecified length guaranteeing omission of types 1, 2, 4 and 5. (3.2.14 and 5.1.8)
  - A local strong E-term, where E is a fixed $\mathbf{xy}$-matrix containing a column with exactly one $\mathbf{y}$. (3.1.3 and 5.1.10)
- If $\mathbf{A}$ is idempotent detecting each of the following can be done in polynomial time, otherwise their detections can be done in EXPTIME, but no relevant hardness result exists.
  - For fixed $n > 2$, an n-ary near unanimity term. (3.1.6)
  - A Mal'cev term. (3.1.7)
  - For fixed $k > 1$, a k-edge term. (3.1.7)
  - A Pixley term. (3.1.8)
  - A strong E-term, where E is a fixed $\mathbf{xy}$-matrix satisfying the DCC. (3.1.5)

- Detection of each of the following is EXPTIME-complete if $\mathbf{A}$ is not idempotent, and there is no known improvement if $\mathbf{A}$ is idempotent.
  - For fixed $n > 2$, an n-ary weak near unanimity term. (5.1.9)
  - For fixed $n > 2$, a sequence of Jónsson terms of length n. (5.1.6)
  - For fixed $n > 2$, a sequence of Hagemann-Mitschke terms of length n. (5.1.7)
  - For fixed $n > 2$, a sequence of Gumm terms of length n.
  - For fixed $n > 2$, a sequence of Day terms of length n.
  - For fixed $n > 2$, a sequence of Hobby-McKenzie terms of length n. (5.1.8)
  - For fixed $n > 2$, a sequence of terms of length n guaranteeing omission of types 1 and 5. (5.1.8)
  - For fixed $n > 2$, a sequence of terms of length n guaranteeing omission of types 1, 2, 4 and 5. (5.1.8)
  - Any other fixed non-sequential term condition which can be satisfied by a constant-projection blend (this includes all strictly weak term conditions), other than a Siggers term. (5.1.5)
- Detection of a k-edge term for unspecified k is in co-NP if $\mathbf{A}$ is idempotent, otherwise it is known to be decidable. (6.3.4 and 6.1.17)
- Detection of an extended M-term (for M either fixed or part of the input) is decidable, though no improvement is known if $\mathbf{A}$ is idempotent (near unanimity terms are included here). (6.1.17)
- Detection of a properly sequential term condition which is satisfiable by constant-projection blends is EXPTIME-hard if $\mathbf{A}$ is not idempotent, but nothing else is known about them (this includes all properly sequential weak term conditions), other than those already mentioned. (5.1.5)
- Nothing is known about the detection of a properly sequential term condition which is not satisfiable by constant-projection blends.
- Detection of a local constant term on $S \subseteq A$ where S is part of the input is constant-time if $\mathbf{A}$ is idempotent, can be done in polynomial time if S is a subuniverse of $\mathbf{A}$ and otherwise is complete for EXPTIME(or PSPACE, depending on the arities of the basic functions of $\mathbf{A}$). (4.1.4, 4.2.6, and 4.2.10)

As we can see a great deal is known about this wide class of decision problems, though there is significant work yet to be done. For example there is no term condition for which it is known that detection cannot be accomplished in polynomial time on an idempotent algebra. Likewise there is no (idempotent) term condition for which it is known that detection is possible in less than EXPTIME in the general case. Furthermore no term condition is known to need more than EXPTIME to detect.

In future, we would like to see hardness results for some strong term condition, improvements for detection of weak term conditions in the idempotent case, and lowering of the upperbound on detection of extended term conditions, if such things are possible (it appears that Theorem 6.3.4 by Ralph McKenzie would be a good place to begin in this last endeavour). A relevant definition of local Hagemann-Mitschke terms would seem to be worth attempting.

References

[1] Authorship undecided, Private communication from Dr. Matt Valeriote: Dept. of Math & Stats, McMaster University. On Maltsev Conditions Associated with Omitting Certain Types of Local Structures, 2011.

[2] Libor Barto and Marcin Kozik. Absorbing subalgebras, cyclic terms and the constraint satisfaction problem. To Appear, 2011.

[3] Libor Barto and Marcin Kozik. Private communication from Dr. Libor Barto: Dept. of Math & Stats, McMaster University, 2011.

[4] Clifford Bergman, David Juedes, and Giora Slutzki. Computational complexity of term-equivalence. *Internat. J. Algebra Comput.*, 9(1):113–128, 1999.

[5] George M. Bergman. On the existence of subalgebras of direct products with prescribed d-fold projections. *Algebra Universalis*, 7(3):341–356, 1977.

[6] Joel Berman, Paweł Idziak, Petar Marković, Ralph McKenzie, Matthew Valeriote, and Ross Willard. Varieties with few subalgebras of powers. *Trans. Amer. Math. Soc.*, 362(3):1445–1473, 2010.

[7] V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras. I, II. *Kibernetika (Kiev)*, (3):1–10; ibid. 1969, no. 5, 1–9, 1969.

[8] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput.*, 34(3):720–742 (electronic), 2005.

[9] Andrei A. Bulatov and Matthew Valeriote. Recent results on the algebraic approach to the CSP. In *Complexity of Constraints*, pages 68–92, 2008.

[10] Stanley Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[11] B. A. Davey, L. Heindorf, and R. McKenzie. Near unanimity: an obstacle to general duality theory. *Algebra Universalis*, 33(3):428–439, 1995.

[12] Alan Day. A characterization of modularity for congruence lattices of algebras. *Canad. Math. Bull.*, 12:167–173, 1969.

[13] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. *SIAM J. Comput.*, 28(1):57–104 (electronic), 1999.

[14] Diego Figueira, Santiago Figueira, Sylvain Schmitz, and Philippe Schnoebelen. Ackermann and primitive-recursive bounds with Dickson's lemma. *CoRR*, abs/1007.2989, 2010.

[15] Ralph Freese and Matthew A. Valeriote. On the complexity of some Maltsev conditions. *Internat. J. Algebra Comput.*, 19(1):41–77, 2009.

[16] O. C. García and W. Taylor. The lattice of interpretability types of varieties. *Mem. Amer. Math. Soc.*, 50(305):v+125, 1984.

[17] David Geiger. Closed systems of functions and predicates. *Pacific J. Math.*, 27:95–100, 1968.

[18] H. Peter Gumm. Geometrical methods in congruence modular algebras. *Mem. Amer. Math. Soc.*, 45(286):viii+79, 1983.

[19] Joachim Hagemann and A. Mitschke. On n-permutable congruences. *Algebra Universalis*, 3:8–12, 1973.

[20] David Hobby and Ralph McKenzie. *The structure of finite algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1988.

[21] Bjarni Jónsson. Algebras whose congruence lattices are distributive. *Math. Scand.*, 21:110–121 (1968), 1967.

[22] Keith A. Kearnes. Congruence permutable and congruence 3-permutable locally finite varieties. *J. Algebra*, 156(1):36–49, 1993.

[23] Dexter Kozen. Lower bounds for natural proof systems. In *18th Annual Symposium on Foundations of Computer Science (Providence, R.I., 1977)*, pages 254–266. IEEE Comput. Sci., Long Beach, Calif., 1977.

[24] Petar Marković and Ralph McKenzie. Few subpowers, congruence distributivity and near-unanimity terms. *Algebra Universalis*, 58(2):119–128, 2008.

[25] Miklós Maróti. On the (un)decidability of a near-unanimity term. *Algebra Universalis*, 57(2):215–237, 2007.

[26] Miklós Maróti. The existence of a near-unanimity term in a finite algebra is decidable. *J. Symbolic Logic*, 74(3):1001–1014, 2009.

[27] Miklós Maróti and Ralph McKenzie. Existence theorems for weakly symmetric operations. *Algebra Universalis*, 59(3-4):463–489, 2008.

[28] Miklós Maróti and Ralph McKenzie. Finitely related clones and algebras with cube terms. preprint, 2011.

[29] Ralph McKenzie. Private communication from Dr. Matt Valeriote: Dept. of Math & Stats, McMaster University, 2010.

[30] Ralph McKenzie and Japheth Wood. The type set of a variety is not computable. *Internat. J. Algebra Comput.*, 11(1):89–130, 2001.

[31] Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor. *Algebras, lattices, varieties. Vol. I.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987.

[32] Alden F. Pixley. Functionally complete algebras generating distributive and permutable classes. *Math. Z.*, 114:361–372, 1970.

[33] G. J. Rieger. Über Partitionen. *Math. Ann.*, 138:356–362, 1959.

[34] Mark H. Siggers. A strong Mal'cev condition for locally finite varieties omitting the unary type. *Algebra Universalis*, 64(1-2):15–20, 2010.

[35] Michael Sipser. *Introduction to the theory of computation: second edition*. PWS Pub., Boston, 2 edition, 2006.