PRIME MALTSEV CONDITIONS AND CONGRUENCE n-PERMUTABILITY

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Abstract

For $n \geq 2$, a variety $\mathcal{V}$ is said to be congruence n-permutable if every algebra $\mathbf{A} \in \mathcal{V}$ satisfies $\alpha \circ^n \beta = \beta \circ^n \alpha$, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$. Furthermore, given any algebra $\mathbf{A}$ and $k \geq 1$, a k-dimensional Hagemann relation on $\mathbf{A}$ is a reflexive compatible relation $R \subseteq A \times A$ such that $R^{-1} \nsubseteq R \circ^k R$. A famous result of J. Hagemann and A. Mitschke in [18] shows that a variety $\mathcal{V}$ is congruence n-permutable if and only if $\mathcal{V}$ has no member carrying an $(n-1)$-dimensional Hagemann relation: by using this criterion, we provide another Maltsev characterization of congruence n-permutability, equivalent to the well-known Schmidt's ([34]) and Hagemann-Mitschke's ([18]) term-based descriptions.

We further establish that the omission by varieties of certain special configurations of Hagemann relations induces the satisfaction of suitable Maltsev conditions. These omission properties may be used to characterize congruence n-permutable idempotent varieties for some $n \geq 2$, congruence 2-permutable idempotent varieties and congruence 3-permutable locally finite idempotent varieties, yielding that the following are prime Maltsev conditions:

1. congruence n-permutability for some $n \geq 2$ with respect to idempotent varieties;

2. congruence 2-permutability with respect to idempotent varieties;

3. congruence 3-permutability with respect to locally finite idempotent varieties.

Finally, we focus on the analysis of a family of strong Maltsev conditions, which we denote by $\{\mathcal{D}_n : 2 \leq n < \omega\}$, such that any variety $\mathcal{V}$ is congruence n-permutable whenever $\mathcal{D}_n$ is interpretable in $\mathcal{V}$. Among various other properties, we also show that the $\mathcal{D}_n$'s with odd $n \geq 3$ generate decomposable strong Maltsev filters in the lattice of interpretability types.

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Introduction

The purpose of this thesis is to provide a rather deep inspection into the study of congruence n-permutable varieties and related topics, with a special focus on the property of primeness of Maltsev conditions.

Congruence n-permutability describes a singular behavior of the congruences of an algebra, which can be extended to classes of algebras, in particular to varieties. In 1972, E. T. Schmidt provided a characterization in [34] of congruence n-permutable varieties, by proving that having such a property for a variety means realizing some suitable terms that satisfy a suitable finite list of equations (universal algebraists refer to this scenario as the satisfaction of a Maltsev condition). In 1973, J. Hagemann and A. Mitschke discovered and presented in [18] an equivalent characterization for congruence n-permutable varieties, which, over the past few years, has turned out to be the most manageable and useful. When O. Garcia and W. Taylor collected in [15] all the previous literature about Maltsev conditions, in Chapter 5 of the same book they discussed the property of primeness of several of those and finally posed two questions regarding the primeness of congruence modularity and congruence 2-permutability. The former is the well known and so far still unproven *modularity conjecture*, while the latter was shown to be prime in 1983 by S. Tschantz, who wrote up his proof in his unpublished paper [41]. No further generalizations to higher levels of n-permutability, nor alternative proofs of the primeness of congruence 2-permutability, were discovered, until novel partial results were proven years later starting from the beginning of the new millennium. Indeed, in 2001, L. Sequeira proved in [36] that congruence 2 and 3-permutability are prime strong Maltsev conditions with respect to varieties axiomatized by equations involving terms of depth at most 2. Moreover, in 2016 J. Opršal came up with another result in [31], where he has shown that congruence n-permutability for fixed $n \geq 2$ is prime with respect to varieties axiomatized by linear equations. In his paper, Opršal also proved that the strong Maltsev condition of having an n-cube term ($n \geq 2$) for idempotent varieties is a prime condition, providing in fact an alternative primeness argument for idempotent congruence 2-permutability; at the same time, K. Kearnes and A. Szendrei proved in 2016 in [23] the same result on n-cube terms for idempotent varieties. Also, in 2014, M. Valeriote and R. Willard showed that for an idempotent variety, being congruence n-permutable for some $n \geq 2$, is equivalent to containing no 2-element algebra carrying any compatible total order, which in turn implies the primeness of the Maltsev condition itself (notice that this is the case of non-fixed $n \geq 2$). Besides these partial achievements and some new results contained in this thesis, whether congruence n-permutability is a prime strong Maltsev condition, for fixed $n \geq 2$, remains open.

Regarding the further contributions to this area of mathematics that this thesis is meant to expose, we are going to briefly present the topics of every chapter, in order to also highlight the crucial outcomes of our research.

After a quick excursion into the elementary and main notions of set theory, universal algebra and lattice theory in Chapter 1, in Chapter 2 we settle the context where all the topics discussed in this thesis will be taking place: indeed, based essentially on [28], [30], [15], [40] and [39], we define the *Lattice of Interpretability types*, which is the formal environment where the notions of Maltsev conditions and primeness of Maltsev conditions (or better yet Maltsev filters) can be described rigorously.

In Chapter 3 we actually enter the core of the thesis by defining the notion of congruence n-permutability and then showing the details of how E. Schmidt in [34] and later J. Hagemann and A. Mitschke in [18] characterized congruence n-permutability (for fixed $n \geq 2$) as a strong

Maltsev condition. Furthermore, we also present a new Maltsev characterization of congruence n-permutable varieties and the actual interpretations that make this and the two previously mentioned ones equivalent. In addition, in Section 3.2, we define what we have named an *n-dimensional Hagemann relation*, so as to subsequently prove a local version of Hagemann and Mitschke's main result of [18]. That is to say, given a non-congruence n-permutable algebra $\mathbf{A}$ (having fixed $n \geq 2$), it is always possible to build an $(n-1)$-dimensional Hagemann relation in the variety generated by $\mathbf{A}$ via a primitive positive construction over $\mathbf{A}$ (i.e. a set theoretical definition involving primitive positive formulas). Conversely, given an $(n-1)$-dimensional Hagemann relation R carried by an algebra $\mathbf{A}$, there is a procedure to primitively positively build an algebra out of R in the variety generated by $\mathbf{A}$, carrying two congruences failing to be n-permutable.

In Chapter 4, we provide the definition of particular n-dimensional Hagemann relations (after having been inspired by Lemma 2 of [8]), which we have called *n-dimensional special Hagemann relations* and we use these objects to define some omission classes of (interpretability types of) varieties, denoted $\Omega(SHR_n)$. More precisely, in this chapter we prove that such omission classes are Maltsev filters and represent some useful tools that have allowed us in Section 4.2 to characterize the Maltsev filter of idempotent congruence n-permutable varieties for some $n \geq 2$, providing an alternative primeness argument for this Maltsev condition when restricted to idempotent varieties.

In Chapter 5 we gather the most important results of the thesis. In Section 5.1 we establish another argument for proving the primeness of congruence 2-permutability with respect to idempotent varieties: more exactly, we show that idempotent varieties are congruence 2-permutable if and only if they omit 1-dimensional special Hagemann relations. In Section 5.2, we characterize the class of idempotent and locally finite congruence 3-permutable varieties as being the class of varieties omitting other particular configurations of 2-dimensional Hagemann relations, which we have called *2-dimensional generalized special Hagemann relations with middle part*: along with another characterization, these results yield the primeness of congruence 3-permutability with respect to locally finite idempotent varieties. Section 5.3, instead, contains some results regarding particular failures of congruence 4-permutability which we have referred to as *special failures of congruence 4-permutability of genus k* ($k \geq 0$): in the world of idempotent varieties, these special algebras and some properties they have, have induced us to making some considerations and conjectures about the non-primeness of the strong Maltsev condition of congruence 4-permutability.

In Chapter 6, we define a family of strong Maltsev conditions, denoted $\mathcal{D}_n = \mathrm{Mod}(\Delta_n)$, that have turned out to be stronger than congruence n-permutability, in the sense that every variety satisfying Δ_n need be congruence n-permutable. Further properties of these conditions are also proven throughout the chapter, together with the curious fact that the Maltsev filter generated by $\mathcal{D}_n$ is decomposable in the lattice of interpretability types, for odd values of $n \geq 3$.

We conclude this thesis by discussing some potential future results and posing a list of open questions that have naturally emerged while our research on these topics was being conducted.

Besides, we wish to explicitly mention how useful the UACalc software [14], developed by R. Freese, E. Kiss and M. Valeriote, turned out to be as a research tool, especially for building examples that may require long and time-consuming computations.

To close this introduction, we remark and highlight that the results that we have accomplished in this thesis are mostly related to the topic of congruence n-permutability for varieties: we have found a new characterizing Maltsev condition of congruence n-permutable varieties, after we had gotten the inspiration from the shape of some particular relations that we have called *special Hagemann relations*; by using these, in fact, we have built some omission classes and proven that these are Maltsev classes, which, in turn, have allowed us to provide some primeness arguments as far as idempotent congruence 2-permutability and locally finite idempotent congruence 3-permutability are concerned. Furthermore, while studying the aforementioned themes, we ended up considering another family of strong Maltsev conditions that have been proven to be decomposable in some specific cases. These developed techniques and achievements may have further applications in the study of Maltsev conditions and an extension of them could lead to the proof of other interesting primeness properties of some Maltsev classes for which a lot of potential results still need to be brought to light. All of what we have obtained in this thesis suggests the surprising fact that, in spite of having been thoroughly studied for decades, the topic of congruence n-permutability still comes up with a lot of compelling results and the knowledge of it does not look any closer to being eventually saturated.

Chapter 1

Basic notions of Universal Algebra

In this chapter we will present the notation we are going to use throughout the whole thesis, as well as a brief overview of the basic concepts of universal algebra. If one is eager to learn these notions more thoroughly, we highly suggest reading [9] or [28].

The axiomatic set theory this thesis refers to is the extension of ZFC, commonly known as NBG after the three famous mathematicians J. Von Neumann, P. Bernays and K. Gödel, who formulated it and used it. We are going to provide a naive and brief presentation of NBG, without specifying the formal axioms which we will not need explicitly in the next chapters.

The coming sections are not meant to expose a thorough presentation of the topics they contain about set theory and universal algebraic foundations, but they are supposed to provide a short introduction that can hopefully help the reader become familiar with all the objects used later on.

1.1 Basic set theory

We expect the reader already possesses a sufficiently deep knowledge of the fundamental concepts of set, membership ($\in$), inclusion ($\subseteq$), union ($\cup$ or $\bigcup$) and intersection ($\cap$ or $\bigcap$), set difference ($-$), Cartesian product ($\times$) power set and function as well as the fundamental notions of first order logic and the list of ZF axioms. Regarding these notions, the notation we are going to make use of will be the standard one (we mostly refer to [9] and [28]) and will not be specified explicitly unless we consider it necessary. We also wish to point out that we will deal with our objects in a more naive way and not as rigorously as ZFC or NBG would require.

First, recall that, given two sets A and B, A^B denotes the set of all functions having domain B and codomain A. When referring to an element $f \in A^B$, we will frequently write $f : B \to A$ and sometimes call it a *map from B to A*; also, the notations $f : b \rightsquigarrow a$ and $f(b) = a$ are to be considered equivalent. The *image of B through f* is denoted by $f(B)$, meaning the subset of A containing all the elements of the form $f(b)$, for some $b \in B$. On the other hand, the *inverse image of $A' \subseteq A$ through f* is denoted by $f^{-1}(A')$, meaning the subset of B containing all the elements b such that $f(b) \in A'$. If $C \subseteq B$, it is possible to consider the function $f_{|C}$ having domain C and codomain A, called the restriction of f to C. If $B = A$, then A^A also contains the identity map denoted by id_A or simply id.

A crucial object in many parts of the next chapters will be that of a kernel.

Definition 1.1.1. For a function $f : A \to B$, define $\ker f \subseteq A \times A$ as

$$\ker f = \{(a, a') : f(a) = f(a')\}.$$

$\ker f$ is called the *kernel of f*.

For completeness, we also provide the definitions of injectivity, surjectivity and bijectivity.

Definition 1.1.2. Let $f : A \to B$ be a function. We say that

- f is *injective* or *one-to-one*, if $\ker f = \ker id_A$, sometimes denoted by $f : A \hookrightarrow B$;
- f is *surjective* or *onto*, if $f(A) = B$, sometimes denoted by $f : A \twoheadrightarrow B$;
- f is *bijective* or *a bijection*, if it is both injective and surjective, sometimes denoted by $f : A \leftrightarrow B$.

Whenever a function $f \in B^A$ is bijective, then there exists the *inverse of* f in A^B, denoted f^{-1}, satisfying $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$, for all $a \in A$, $b \in B$.

Jumping to another topic, the theory of ordinals and cardinals is taken for granted. In particular, we denote by ω the least infinite cardinal whose elements (natural numbers, also called non-negative integers) are inductively defined by

$$0 := \varnothing;$$

$$n + 1 := n \cup \{n\}.$$

More generally, for any ordinal λ, $\lambda + 1$ denotes the successor ordinal $\lambda \cup \{\lambda\}$. Finally, we denote by $|A|$ the cardinality of a set A.

Regarding the well known axiom of choice, we want to provide an explicit formulation of it, in order to emphasize equivalent results which will be introduced later.

Axiom (Axiom of choice)**.** *For any set C of sets, such that $A \neq \varnothing$, for all $A \in C$, there exists a function $f : C \to \bigcup C$ satisfying $f(A) \in A$, for all $A \in C$.*

We will refer to such a function f as a *choice function on* C.

The axiom of choice has many equivalent statements, one of which is the non-emptiness of any direct product of a collection of sets that we define below as a generalization of the Cartesian product.

Definition 1.1.3. Let I be any set and $C := \{A_i : i \in I\}$ be a family of sets. If $A_i \neq \varnothing$, for every $i \in I$ (resp. $A_j = \varnothing$, for some $j \in I$), then we define the *direct product of* C, denoted $\prod C$ or $\prod_{i \in I} A_i$, as the set of all choice functions on C (resp. as the empty set).

As already mentioned, the axiom of choice ensures that $\prod C \neq \varnothing$, whenever C contains no empty sets. Conveniently, we shall denote the elements of $\prod C$ by the ordered tuples $(a_i : i \in I)$ (assuming to have listed the elements of I), where $a_i \in A_i$, for all $i \in I$ (i.e. we identify each choice function with its image, seen as an ordered list of elements from the A_i's). If $I = \{i_1, \ldots, i_n\}$ for some $n > 1$, then $\prod_{i \in I} A_i$ is denoted by $\prod_{j=i_1}^{i_n} A_j$ or $A_{i_1} \times \cdots \times A_{i_n}$ and its elements are denoted by tuples $(a_{i_1}, \ldots, a_{i_n})$. Moreover, if $A_i = A$, for all $i \in I$ and for some set A, then $\prod_{i \in I} A_i$ is denoted by A^I and is called *direct power of* A: in such a case, indeed, the direct power coincides exactly with the set of all functions from I to A. Whenever I is a finite cardinal $n > 0$, the direct power A^n (or the equivalent $A \times \cdots \times A$, with n factors) represents the set of all tuples of elements of A of the form $(a_0, \ldots, a_{n-1})$: in fact, we might sometimes make an abuse of notation by starting the enumeration from 1, namely by denoting any element of A^n as a tuple of the form $(a_1, \ldots, a_n)$. In later chapters, we will frequently make use of column vectors instead of just row vectors: therefore, the two following notations are to be considered totally equivalent:

$$(a_1, \ldots, a_n) \text{ and } \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Regarding functions and direct products, if $f_i \in B_i^{A_i}$ for $i \in I$, the following function can be naturally defined:

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \to \prod_{i \in I} B_i \;\; : (a_i : i \in I) \rightsquigarrow (f_i(a_i) : i \in I).$$

In particular, if $I = n \geq 1$, $A_i = A$, $B_i = B$ and $f_i = f$, for all $i \in I$ and some A, B, f, then $\prod_{i \in I} f$ is denoted by $f \times \cdots \times f$, with n occurrences of f.

Another natural function which is worth a careful consideration is the projection map: given a direct product $\prod_{i \in I} A_i$, the *projection (map) onto* A_j, for $j \in I$, sometimes called the j^{th} *projection (map)*, is the function usually denoted by $\pi_j : \prod_{i \in I} A_i \to A_j$ satisfying $\pi_j(a_i : i \in I) = a_j$.

Finally, if f is a function from A^n to A, for some integer $n \geq 0$, we will say that f is an *n-ary operation on* A (n is called the *arity of* f, denoted $ar(f)$); for $n = 0$ we might rather use the terminology *constant on* A to mean a 0-ary operation on A, whereas 1-ary and 2-ary operations will be referred to as, respectively, *unary* and *binary operations*. Also, if $f : A^n \to A$ is any operation and $B \subseteq A$, we define $f_{|B} := f_{|B^n}$. Operations on a set A can also be composed, in the sense that if f is an m-ary operation on A and $g_1, \ldots, g_m$ are n-ary operations on A ($m, n \geq 1$), we may define the n-ary operation $f(g_1, \ldots, g_m)$ on A, as follows

$$f(g_1, \ldots, g_m)(\vec{a}) = f(g_1(\vec{a}), \ldots, g_m(\vec{a})),$$

for all $\vec{a} \in A^n$, called the *composition of* f *and* $g_1, \ldots, g_m$. We will use these objects multiple times throughout this thesis.

A distinguished property of operations deserves a highlighted definition.

Definition 1.1.4. For $n \geq 1$, an n-ary operation f on a set A is said to be *idempotent* if it satisfies, for each $a \in A$,

$$f(a, \ldots, a) = a.$$

Idempotent operations will play a crucial role in some parts of this thesis.

The main reason why we refer to NBG instead of simply ZFC is the fact that we need to deal with classes rather than just sets. The notion of *class* is a generalization of the one of set and all those classes which are not sets are called *proper classes*: in particular, a proper class is a class which is not an element of another class (this prevents Russell's paradox from appearing). For example, we will deal with the class of all sets, classes of algebras and so on. At some point, we will also consider collections of classes, which, with an abuse of language, we will refer to as classes of classes, contradicting the formal definition of proper class itself: in fact, we should be calling these objects *conglomerates*, but we will never use this unfamiliar term in any context whatsoever later on. On the other hand, we might use the term *family* as a synonym of set.

Let us also fix the notation as far as relations between sets are concerned. Given a family of sets[1] $\{A_1, \ldots, A_n\}$ for some $n \geq 1$, an *n-ary relation on* $A_1, \ldots, A_n$ is a subset of $\prod_{i=1}^n A_i$, and whenever $A_i = A$ for all $i \in \{1, \ldots, n\}$ and for some set A, we say the relation is *on* A, or equivalently A *carries* the relation. Moreover, if $B \subseteq A$ and R is an n-ary relation on A, we might sometimes denote by $R_{|B}$ the relation on B defined by $R_{|B} = R \cap B^n$.

If A and B are two sets, and $R \subseteq A \times B$ is a binary relation on A, B, then $R^{-1} \subseteq B \times A$, called the *converse relation of* R, denotes the set defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Moreover, for any $a \in A$, we define

$$a/R = \{b \in B : (a, b) \in R\};$$

and the notation $a \mathrel{R} b$ (resp. $a \not\mathrel{R} b$) is to be considered equivalent to $(a, b) \in R$ (resp. $(a, b) \notin R$): in such a case, we might sometimes say that a *is* (resp. *is not*) *R-related/connected to* b, or similar statements.

In the same setting, we also define, for $C \subseteq A$,

$$C \wr R = \{a/R : a \in C\}.$$

[1]Although the definition of relation and relative notions are given for sets, in Chapter 2 we will use the same notations when dealing with relations on classes (for example the relations of interpretability and equi-interpretability).

Given two relations, it is possible to define suitable operations between them, like $\circ$, $\otimes$ and $\star$. These notions will be further discussed in Chapter 3, especially in Section 3.2; hence we will just give the basic definitions.

Definition 1.1.5. Let $n \geq 2$, $R, T \subseteq A \times A$, $P \subseteq A^n$ and $S \subseteq B \times B$ be relations. Define

$$R \circ T = \{(r,t) : \exists s \in A[(r,s) \in R, (s,t) \in T]\};$$

$$R \otimes S = \left\{ \left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \right) : (a_1, a_2) \in R, (b_1, b_2) \in S \right\};$$

$$R \star P = \left\{ \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} : (a_0, a_1) \in R, (a_1, \ldots, a_n) \in P \right\}.$$

Given a set A, it is always possible to find the following two binary relations on it:

$$0_A = \{(a,a) : a \in A\};$$

$$1_A = A \times A.$$

Among the binary relations on a set, there exist some with properties that deserve special consideration. Such properties are the following for a relation $R \subseteq A \times A$:

- *reflexivity*: $0_A \subseteq R$;
- *symmetry*: $R^{-1} \subseteq R$;
- *antisymmetry*: $R \cap R^{-1} \subseteq 0_A$;
- *transitivity*: $R \circ R \subseteq R$.

Definition 1.1.6. Let $R \subseteq A \times A$ be a binary relation on A.

- R is an *equivalence relation* if it is reflexive, symmetric and transitive;
- R is a *quasi-order* if it is reflexive and transitive;
- R is a *partial order* if it is an antisymmetric quasi order;
- R is a *total order* (or *linear order*) if it is a partial order and $R \cup R^{-1} = A^2$.

Moreover, if R is an equivalence relation, then we call a/R ($a \in A$) an *equivalence R-class* and we define the *quotient of A modulo R*, denoted A/R, as

$$A/R = \{a/R : a \in A\}.$$

Notice that 0_A and 1_A are always equivalence relations on A. Usually, equivalence relations are denoted by lower case Greek letters like α, β, θ, whereas quasi-orders and partial orders are frequently denoted by the symbols $\preceq$ or $\leq$.

It is known that any quasi order induces an equivalence relation: if $\preceq \subseteq A \times A$ is a quasi-order on A, then the relation on A defined by $\theta := \preceq \cap \preceq^{-1}$ is an equivalence relation.

Moreover, equivalence relations on a non-empty set are related to the concept of partition.

Definition 1.1.7. Let A be a non-empty set and $\pi = \{A_i : i \in I\}$ be a family of subsets of A satisfying:

- *non-emptiness*: $A_i \neq \varnothing$, for all $i \in I$;
- *disjointness*: $A_i \cap A_j = \varnothing$, for all $i, j \in I$ with $i \neq j$;
- *covering property*: $\bigcup \pi = A$.

In such a case, π is said to be a *partition of* A.

Given a non-empty set A and an equivalence relation θ on A, it is straightforward to verify that A/θ is a partition of A, sometimes referred to as the *partition induced by* θ. Conversely, if π is a partition of A, then the relation $\theta(\pi)$ defined by

$$(a, b) \in \theta(\pi) \text{ if and only if } \exists P \in \pi[a, b \in P],$$

is an equivalence relation, called the *equivalence relation induced by* π. With this notation, we can easily deduce the following equality

$$A/\theta(\pi) = \pi.$$

Going back to partial orders, a relational structure $\mathbb{P} = \langle P; \leq\rangle$, where $\leq \subseteq P \times P$ is a partial order on P, is called a *partially ordered set* (briefly *poset*); if $\leq$ is a total order on P, then $\mathbb{P}$ is said to be a *totally* (or *linearly*) *ordered set*. Whenever $\mathbb{P} = \langle P; \leq\rangle$ is a poset and $C \subseteq P$ is such that $\mathbb{C} = \langle C; \leq_{|C}\rangle$ is a totally ordered set, then we call $\mathbb{C}$ a *chain of* $\mathbb{P}$.

Furthermore, for a poset $\mathbb{P} = \langle P; \leq\rangle$, $p \in P$ is said to be $\leq$*-maximal for* $Q \subseteq P$ if $p \in Q$ and, for all $q \in Q$, if $p \leq q$, then $p = q$. $\leq$*-minimal* elements are defined dually. Also, a $\leq$*-upper bound* (resp. $\leq$*-lower bound*) *for* Q is an element $r \in P$ such that $q \leq r$ (resp. $r \leq q$), for all $q \in Q$.

Posets play an important role in many branches of mathematics, and are central to the following statement, which is equivalent to the axiom of choice.

Theorem 1.1.1 (Zorn's Lemma)**.** *If $\mathbb{P} = \langle P; \leq\rangle$ is a poset with $P \neq \varnothing$, such that for every chain $\langle C; \leq_{|C}\rangle$ of $\mathbb{P}$ there is a $\leq$-upper bound for C, then there exists a $\leq$-maximal element for P.*

In the next section, we will summarize the elementary themes of universal algebra.

1.2 Elements of universal algebra

The main objects of the field of universal algebra are the so called algebras, meaning certain structures which generalize the concepts of groups, rings, vector spaces, boolean algebras and many others. In the current chapter, we are going to provide a sequence of fundamental definitions, results and observations which are needed for a correct comprehension of the topics. Again, we mention that we mostly refer to [9] and [28].

Let us then begin with the very first basic definition of an algebra.

Definition 1.2.1. We say that $\mathbf{A} = \langle A; F\rangle$ is a *universal algebra*, or simply an *algebra*, if A is a non-empty set, called the *universe of* $\mathbf{A}$, and $F = \{f_i : i \in I\}$, where f_i is an operation on A, called a *basic* or *fundamental operation of* $\mathbf{A}$, for all $i \in I$. Moreover, $i \in I$ is referred to as a *basic operation symbol of* $\mathbf{A}$ and we will call I the *set of operation symbols of* $\mathbf{A}$.

Notice that an algebra is denoted by a bold capital letter, whereas its universe is denoted by the same capital letter in italic. Whenever F contains finitely many operations, for instance $\mathbf{A} = \langle A; \{f_1, \dots, f_n\}\rangle$, we usually drop the brackets $\{,\}$ and write $\mathbf{A} = \langle A; f_1, \dots, f_n\rangle$ instead. Moreover, an algebra is *finite* if its universe is a finite set; otherwise it is *infinite*. Also, an algebra is *idempotent* if all its basic operations are idempotent (see Definition 1.1.4), and a class of algebras is *idempotent* if so is every algebra it contains.

In order to clarify the definition, let us provide an elementary example of an algebra.

Example 1.2.1. A vector space in the standard sense is an example of an algebra. In details, a *vector space over a field* $\mathbf{K}$ with the standard operations of sum, difference and neutral element (denoted, respectively, $+, -, 0$) can be represented as

$$\mathbf{V} = \langle V; \{f_+, f_-, f_0\} \cup \{f_s : s \in K\}\rangle,$$

where $f_+ = +$, $f_- = -$, $f_0 = 0$ and $f_s(v) = sv$ (multiplication by a scalar), for all $v \in V$, $s \in K$.

We wish to point out the difference between operation and operation symbol: in the previous example, the set I is represented by $I = \{+, -, 0\} \cup K$ and the operation is denoted by f_i, for $i \in I$. In other words, if $\mathbf{A} = \langle A; \{f_i : i \in I\}\rangle$ is any algebra, f_i is the name of the operation on A which interprets the symbol of operation i. However, we might sometimes omit the indexed notation and use the name of the symbol as the name of the corresponding operation. For instance, still referring to Example 1.2.1, we might frequently denote a vector space as

$$\mathbf{V} = \langle V; \{+, -, 0\} \cup \{s\cdot : s \in K\}\rangle.$$

More generally, for i a basic operation symbol of $\mathbf{A}$, the standard model theoretical praxis refers to $i^{\mathbf{A}}$ as the corresponding basic operation on A, called the *interpretation of* i *in* $\mathbf{A}$.

Another important notion is the one of type of an algebra.

Definition 1.2.2. Given an algebra $\mathbf{A} = \langle A; \{f_i : i \in I\}\rangle$, the (*similarity*) *type of* $\mathbf{A}$ is the function $\varrho_{\mathbf{A}} : I \to \omega$ (or simply ϱ if $\mathbf{A}$ is clear from the context) defined by $\varrho_{\mathbf{A}}(i) = ar(f_i)$, for all $i \in I$. We call $\varrho(i)$ the *arity of* i and say that i is a $\varrho(i)$-*ary basic operation symbol.*[2]

We wish to point out that two similarity types coincide if and only if they are the same function as a subset of $I \times \omega$: in particular, two algebras with the same similarity types have the same basic operation symbols interpreting basic operations of the same arities. Algebras with the same type are called *similar*.

We use the convention of denoting $\varrho(I)$ as a tuple (with angle brackets) instead of as a set, meaning the type itself of an algebra, without specifying the symbols in I, for which we might sometimes explicitly use the same convention, if necessary. For example, given a group $\mathbf{G} = \langle G; \cdot^{\mathbf{G}}, ^{-1^{\mathbf{G}}}, 1^{\mathbf{G}}\rangle$, we denote its type by $\langle 2, 1, 0\rangle$ and we might sometimes refer to its set of basic operation symbols as $\langle \cdot, ^{-1}, 1\rangle$.

Let us provide some examples of algebras:

Example 1.2.2. A non-empty set is perhaps the example of an algebra with simplest syntax. As a matter of fact, a non-empty set A can be seen as an algebra $\mathbf{A} = \langle A; \varnothing\rangle$ having no basic operations. The class of non-empty sets is denoted by $\mathcal{S}ets$.

Example 1.2.3. A *semigroup* is an algebra $\mathbf{S} = \langle S; \cdot^{\mathbf{S}}\rangle$ of type $\langle 2\rangle$, satisfying associativity, namely for all $r, s, t \in S$

$$r \cdot^{\mathbf{S}} (s \cdot^{\mathbf{S}} t) = (r \cdot^{\mathbf{S}} s) \cdot^{\mathbf{S}} t.$$

The class of semigroups is denoted by $\mathcal{SG}$.

Example 1.2.4. A *semilattice* is an algebra $\mathbf{S} = \langle S; \cdot^{\mathbf{S}}\rangle$ such that $\mathbf{S}$ is a semigroup and $\cdot^{\mathbf{S}}$ is idempotent and commutative (i.e. $r \cdot^{\mathbf{S}} s = s \cdot^{\mathbf{S}} r$, for all $r, s \in S$).

A semilattice naturally carries a partial order. Indeed, if we define

$$R = \{(r, s) \in S^2 : r \cdot^{\mathbf{S}} s = r\},$$

then $\mathbb{S} = \langle S; R\rangle$ is a poset. In literature, a semilattice is usually denoted as $\langle S; \wedge\rangle$ ($\wedge$-*semilattice*) with induced partial order denoted by $\leq$, or $\langle S; \vee\rangle$ ($\vee$-*semilattice*) with induced partial order denoted by $\geq$. The class of $\star$-semilattices is denoted by $\mathcal{S}_\star$, for $\star \in \{\vee, \wedge\}$, or simply $\mathcal{S}$, when no risk of confusion occurs.

The next example deserves a little more attention due to its connection with posets and other reasons that will be motivated later.

Example 1.2.5. A *lattice* is an algebra $\mathbf{A} = \langle A; \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}\rangle$ of type $\langle 2, 2\rangle$ such that $\langle A; \wedge^{\mathbf{A}}\rangle$ and $\langle A; \vee^{\mathbf{A}}\rangle$ are semilattices and satisfying the absorbing laws, i.e. for all $a, b \in A$

$$a \vee^{\mathbf{A}} (a \wedge^{\mathbf{A}} b) = a,$$

$$a \wedge^{\mathbf{A}} (a \vee^{\mathbf{A}} b) = a.$$

[2]We want to remark that the arity of a function symbol corresponds to the arity of the operation it interprets.

Since $\langle A; \wedge^{\mathbf{A}}\rangle$ and $\langle A; \vee^{\mathbf{A}}\rangle$ are semilattices, A carries two partial orders $\leq$ and $\geq$, which, by the absorbing laws, satisfy

$$\leq^{-1} \; = \; \geq,$$

as expected. Thus, it suffices to consider either $\leq$ or $\geq$ without losing any information.

Conversely, given a non-empty poset $\mathbb{P} = \langle P; \leq\rangle$ such that every finite $Q \subseteq P$ has infimum and supremum with respect to $\leq$, if we define

$$p \wedge q = \inf_{\leq}\{p, q\},$$

$$p \vee q = \sup_{\leq}\{p, q\},$$

for all $p, q \in P$, then the algebra $\langle P; \wedge, \vee\rangle$ is a lattice. The class of all lattices is denoted by $\mathcal{L}at$.

Given an algebra $\mathbf{A} = \langle A; F\rangle$, we can consider the least set of operations on A containing F and the projection maps of any arity which is closed under composition: we denote such a set $\mathrm{Clo}(\mathbf{A})$ or $\mathrm{Clo}\,\mathbf{A}$ and its elements are referred to as *term operations of* $\mathbf{A}$. For $n \geq 0$, the set of n-ary terms of $\mathbf{A}$ is denoted by $\mathrm{Clo}_n(\mathbf{A})$ or $\mathrm{Clo}_n\,\mathbf{A}$. Furthermore, we call a *reduct of* $\mathbf{A}$ any algebra $\langle A; G\rangle$, where $G \subseteq \mathrm{Clo}\,\mathbf{A}$. Likewise, for a class of similar algebras K, a *reduct of* K is a class K' of similar algebras whose set of basic operation symbols is a subset of the set of terms derived from the operation symbols of K. For example, for such a class K, we denote by K^{id} the reduct of K having all and only the symbols of idempotent terms of K as basic operation symbols: K^{id} is sometimes called the *full idempotent reduct of* K.

Let us then continue with the fundamental definitions of subalgebras, direct products of algebras and homomorphic images. To do this, we need first to define the notion of compatibility.

Definition 1.2.3. Let A be a non-empty set, $R \subseteq A^n$ be an n-ary relation on A for some $n \geq 1$ and f a k-ary operation on A, for some $k \geq 1$. We say that R is *compatible with* f, or equivalently f is a *polymorphism of* R, if for all $(a_{i,0}, \ldots, a_{i,n-1}) \in R$, $i \in k$, we have

$$\begin{bmatrix} f(a_{0,0}, \ldots, a_{k-1,0}) \\ \vdots \\ f(a_{0,n-1}, \ldots, a_{k-1,n-1}) \end{bmatrix} \in R.$$

The concept of compatibility allows us to define most of the objects we were aiming to.

Definition 1.2.4 (Subalgebra)**.** Let $\mathbf{A}$ be an algebra and B be a subset of A. B is said to be a *subuniverse of* $\mathbf{A}$, written $B \leq \mathbf{A}$, if it is compatible with all the basic operations of $\mathbf{A}$. If B is not empty, and $F = \{f_i : i \in I\}$ is the set of basic operations of $\mathbf{A}$, then the algebra $\mathbf{B} := \langle B; \{f_{i|B} : i \in I\}\rangle$ is said to be a *subalgebra of* $\mathbf{A}$, written $\mathbf{B} \leq \mathbf{A}$.

Given an algebra $\mathbf{A}$ and $X \subseteq A$, since A itself is a subuniverse of $\mathbf{A}$ containing X, it makes sense to consider the least (with respect to inclusion) subuniverse of $\mathbf{A}$ which contains X, denoted by $\mathrm{Sg}^{\mathbf{A}}(X)$, called the *subuniverse of* $\mathbf{A}$ *generated by* X and defined by

$$\mathrm{Sg}^{\mathbf{A}}(X) = \bigcap\{B \leq \mathbf{A} : B \supseteq X\}.$$

An algebra $\mathbf{A}$ is *finitely generated* when there exists a finite $X \subseteq A$ such that $A = \mathrm{Sg}^{\mathbf{A}}(X)$. It is rather straightforward to prove that the property displayed in the next lemma holds

Lemma 1.2.1. *For an algebra* $\mathbf{A}$*,* $X \subseteq A$ *and* $a \in A$*, the following are equivalent*

1. $a \in \mathrm{Sg}^{\mathbf{A}}(X)$
2. *there exist* $n \geq 1$, $\vec{x} \in X^n$ *and* $t^{\mathbf{A}} \in \mathrm{Clo}_n(\mathbf{A})$ *such that* $a = t^{\mathbf{A}}(\vec{x})$.

This lemma turns out to be useful when one needs an actual description of an element in a generated subuniverse and such a description is fairly handy: each element of a generated subuniverse can be obtained by the application of a term operation of the algebra to some generators.

Let us now focus on the concept of direct product of a family of algebras. We have already observed what a direct product of sets is like and the extra step we need to take when dealing with algebras is to define the operations in a proper way.

Definition 1.2.5 (Direct product of algebras)**.** Let $\{\mathbf{A}_i : i \in I\}$ be a family of similar algebras. The *direct product of* $\{\mathbf{A}_i : i \in I\}$, denoted $\prod_{i\in I} \mathbf{A}_i$, is the algebra with universe $\prod_{i\in I} A_i$, similar to each $\mathbf{A}_i$, for $i \in I$, where a basic operation $f^{\prod_{i\in I} \mathbf{A}_i}$ of arity $n \geq 0$ is defined as

$$f^{\prod_{i\in I} \mathbf{A}_i}((a_{0,i} : i \in I), \dots, (a_{n-1,i} : i \in I)) = (f^{\mathbf{A}_i}(a_{0,i}, \dots, a_{n-1,i}) : i \in I),$$

for every symbol of operation f and $(a_{0,i} : i \in I), \dots, (a_{n-1,i} : i \in I) \in \prod_{i\in I} A_i$.

For this object the same conventions as the ones for direct products of sets hold: a direct product of a finite family $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ is denoted by $\prod_{i=1}^{k} \mathbf{A}_i$ or $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$; whereas a *direct power of* $\mathbf{A}$ is denoted by $\mathbf{A}^I$.

Another central concept in universal algebra, generalizing the one of direct product, is the subdirect product.

Definition 1.2.6 (Subdirect product of algebras)**.** Let $\{\mathbf{A}_i : i \in I\}$ be a family of similar algebras and let S be a subuniverse of $\prod_{i\in I} \mathbf{A}_i$. We say that $\mathbf{S}$ is a *subdirect product of* $\{\mathbf{A}_i : i \in I\}$ and write $\mathbf{S} \leq_{sd} \prod_{i\in I} \mathbf{A}_i$, if $\pi_{i|S}$ is onto, for all $i \in I$.

For example, any reflexive subuniverse $R \leq \mathbf{A}^2$, for some algebra $\mathbf{A}$, is always subdirect: as a matter of fact, $\pi_i(R) \supseteq \pi_i(0_A) = A$, for each $i = 0, 1$. Subdirect products will be mentioned again in some coming fundamental results.

Let us then skip to defining the fundamental functions operating between similar algebras.

Definition 1.2.7 (Homomorphism)**.** Let $\mathbf{A}, \mathbf{B}$ be two similar algebras and $h : A \to B$ be any function. We say that h is a *homomorphism* if for all operation symbol f of arity $n \geq 0$,

$$f^{\mathbf{B}}(h(a_1), \dots, h(a_n)) = h(f^{\mathbf{A}}(a_1 \dots, a_n)).$$

In such a case, we write $h : \mathbf{A} \to \mathbf{B}$. Furthermore, if h is injective, it is sometimes called an *embedding*; if it is bijective, it is called an *isomorphism.* If $\mathbf{A}$ and $\mathbf{B}$ are similar algebras such that there exists an isomorphism from $\mathbf{A}$ to $\mathbf{B}$, then we say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* and write $\mathbf{A} \cong \mathbf{B}$.

Fixed any type, the relation of isomorphism $\cong$ is an equivalence relation on the class of all similar algebras. Also, it is not hard to see that if $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $h(A)$ is a subuniverse of $\mathbf{B}$. In particular, if h is surjective, then $\mathbf{B}$ is said to be a *homomorphic image of* $\mathbf{A}$.

We can characterize homomorphic images through special kinds of equivalence relations which we are going to define next, along with the notion of tolerance relation.

Definition 1.2.8. Let $\mathbf{A}$ be an algebra and θ be a reflexive and symmetric relation on A. We say that θ is a *tolerance* (*relation*) *on/of* $\mathbf{A}$ if $\theta \leq \mathbf{A} \times \mathbf{A}$. A transitive tolerance of $\mathbf{A}$ is called a *congruence* (*relation*) *on/of* $\mathbf{A}$. The set of all congruences (resp. tolerances) of $\mathbf{A}$ is denoted by $\mathrm{Con}(\mathbf{A})$ or $\mathrm{Con}\,\mathbf{A}$ (resp. $\mathrm{Tol}(\mathbf{A})$ or $\mathrm{Tol}\,\mathbf{A}$).

Notice that $0_A, 1_A \in \mathrm{Con}\,\mathbf{A}$, for every algebra $\mathbf{A}$.

Likewise for generated subuniverses, if $\mathbf{A}$ is any algebra and X is a binary relation on A, we define the *congruence of* $\mathbf{A}$ *generated by* X, denoted $\mathrm{Cg}^{\mathbf{A}}(X)$, as

$$\mathrm{Cg}^{\mathbf{A}}(X) = \bigcap\{\theta \in \mathrm{Con}\,\mathbf{A} : X \subseteq \theta\}.$$

This is one of the points where we can remark the importance of lattices in universal algebra: indeed, given an algebra $\mathbf{A}$, in $\operatorname{Con}\mathbf{A}$ we can observe that for two congruences α and β, their intersection is still a congruence of $\mathbf{A}$, whereas their union is not; we can then consider instead the congruence generated by their union. More precisely, by defining

$$\alpha \wedge \beta = \alpha \cap \beta,$$

$$\alpha \vee \beta = \mathrm{Cg}^{\mathbf{A}}(\alpha \cup \beta),$$

we obtain that the algebra $\mathbf{Con}(\mathbf{A}) = \langle \operatorname{Con}\mathbf{A}; \wedge, \vee \rangle$ is a lattice, called the *congruence lattice of* $\mathbf{A}$. Observe that 0_A and 1_A are, respectively, the smallest and greatest elements of $\operatorname{Con}\mathbf{A}$ with respect to the order $\leq\, = \,\subseteq$ induced by $\wedge$.

Having defined the quotient of a set modulo an equivalence relation of it (Definition 1.1.6), we can endow such a derived set with a structure of algebra.

Definition 1.2.9 (Quotient of an algebra modulo a congruence)**.** Let $\mathbf{A}$ be an algebra and θ be a congruence of $\mathbf{A}$. We define the algebra $\mathbf{A}/\theta$, called the *quotient of* $\mathbf{A}$ modulo θ, as the algebra of the same type as $\mathbf{A}$, having universe A/θ and such that, for any f n-ary ($n \geq 0$) basic operation symbol of $\mathbf{A}$, a basic operation of $\mathbf{A}/\theta$ is given by

$$f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathbf{A}}(a_1, \ldots, a_n)/\theta,$$

for all $a_1, \ldots, a_n \in A$.

Obviously, the above operation is well defined and it is independent of the choice of the a_i's. Moreover, the reason why the quotient of an algebra is defined right after the notion of homomorphic image is because there is a deep connection between them, as we are going to justify below.

If $\mathbf{A}$ and $\mathbf{B}$ are two similar algebras and $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $\ker h$ turns out to be a congruence of $\mathbf{A}$. Conversely, every congruence of $\mathbf{A}$ is the kernel of a suitable homomorphism: if $\theta \in \operatorname{Con}\mathbf{A}$, then we may define the *θ-quotient map* $\nu_\theta : A \to A/\theta$ as $\nu_\theta(a) = a/\theta$ for all $a \in A$. In fact, ν_θ is a surjective homomorphism and $\ker \nu_\theta = \theta$. A stronger result exists, which is commonly known as the *first homomorphism* (or *isomorphism*) *theorem* and that we refer to as a lemma.

Lemma 1.2.2 (First homomorphism theorem)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be two similar algebras and $h : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. If ν_θ denotes the θ-quotient map $\nu_\theta : \mathbf{A} \to \mathbf{A}/\theta$, then there exists a unique isomorphism $\varphi : \mathbf{A}/\theta \to \mathbf{B}$, such that $\varphi(a/\theta) = h(a)$, for all $a \in A$. In particular, $\ker h = \theta$ and $\mathbf{B} \cong \mathbf{A}/\theta$.*

As a result, we get that every homomorphic image of an algebra $\mathbf{A}$ is isomorphic to a quotient of $\mathbf{A}$ modulo one of its congruences, and precisely modulo the kernel of the homomorphism itself.

In addition, the congruence lattice of the quotient is also deeply related to the congruence lattice of the original algebra, as the following lemma, also known as the *correspondence theorem*, states.

Lemma 1.2.3 (Correspondence theorem)**.** *Let $\mathbf{A}$ be an algebra and $\theta \in \operatorname{Con}\mathbf{A}$. Then, for every congruence of $\mathbf{A}$ $\alpha \geq \theta$, the relation on A/θ defined by*

$$\alpha/\theta = \{(a/\theta, b/\theta) : (a, b) \in \alpha\}$$

is a congruence of $\mathbf{A}/\theta$, and conversely, every congruence of $\mathbf{A}/\theta$ has the form α/θ, for some $\alpha \geq \theta$ in $\operatorname{Con}\mathbf{A}$. In particular,

$$\langle \theta/\leq; \wedge_{|\theta/\leq}, \vee_{|\theta/\leq} \rangle \cong \mathbf{Con}(\mathbf{A}/\theta),$$

via the isomorphism $\alpha \rightsquigarrow \alpha/\theta$.

Notice that $\theta/\leq\, = \{\beta \in \operatorname{Con}\mathbf{A} : \theta \leq \beta\} = \{\beta \in \operatorname{Con}\mathbf{A} : \theta \leq \beta \leq 1_A\} \leq \mathbf{Con}(\mathbf{A})$. As a result, the congruence lattice of a quotient of an algebra modulo a congruence is isomorphic to a suitable sublattice of the congruence lattice of the original algebra. There exists also a connection between subdirect products and congruences of an algebra. Before motivating this claim, let us give a few more definitions.

Definition 1.2.10. Let $\mathbf{S}$ be any algebra. We say that $\mathbf{S}$ is *subdirectly irreducible* if, whenever $\mathbf{S} \leq_{sd} \prod_{i\in I} \mathbf{A}_i$, for some algebras $\mathbf{A}_i$ similar to $\mathbf{S}$ for all $i \in I$, then there exists $i_0 \in I$ such that $\pi_{i_0|S} : \mathbf{S} \to \mathbf{A}_{i_0}$ is an isomorphism.

Examples of subdirectly irreducible algebras are the ones with cardinality 2. In fact, subdirectly irreducible algebras can be characterized by the shape of their congruence lattices, as the next theorem suggests.

Theorem 1.2.1. *For an algebra* $\mathbf{A}$*, the following are equivalent:*

1. *There exists a family of similar algebras* $\{\mathbf{A}_i : i \in I\}$ *and* $\mathbf{A}'$ *such that* $\mathbf{A} \cong \mathbf{A}' \leq_{sd} \prod_{i\in I} \mathbf{A}_i$*;*
2. *There exists* $C \subseteq \operatorname{Con}\mathbf{A}$ *such that* $\bigcap C = 0_A$.

In details, if (2) of Theorem 1.2.1 is satisfied by an algebra $\mathbf{A}$, then there exists an algebra $\mathbf{A}'$ such that

$$\mathbf{A} \cong \mathbf{A}' \leq_{sd} \prod_{\gamma\in C} \mathbf{A}/\gamma,$$

and, in particular, $A' = \{(a/\gamma : \gamma \in C) : a \in A\}$. Therefore, a straightforward corollary is the following

Corollary 1.2.1. *An algebra* $\mathbf{S}$ *is subdirectly irreducible if and only if there exists a unique congruence* $\mu \in \operatorname{Con}\mathbf{S}$ *satisfying* $0_S \prec \mu$.[3]

The congruence μ of $\mathbf{A}$ described in this corollary is called the *monolith of* $\mathbf{A}$. Hence, examples of subdirectly irreducible algebras are all those algebras $\mathbf{A}$ such that $\operatorname{Con}\mathbf{A} = \{0_A, 1_A\}$, where the monolith is exactly 1_A; such algebras are called *simple.*

Similar subdirectly irreducible algebras represent the foundational bricks for the class of algebras having their type; this deep result is due to G. Birkhoff and is stated in the next theorem.

Theorem 1.2.2. *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

At this point, we are ready to focus on collections of algebras which play another important role in universal algebra and in this thesis as well. First, we need to define the operators $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_s$ acting on classes of algebras.

Definition 1.2.11. Let K be a class of similar algebras and define:

$$\mathbf{H}(K) = \{\mathbf{B} : \mathbf{B} \text{ is isomorphic to a homomorphic image of some } \mathbf{A} \in K\};$$

$$\mathbf{S}(K) = \{\mathbf{B} : \mathbf{B} \text{ is isomorphic to a subuniverse of some } \mathbf{A} \in K\};$$

$$\mathbf{P}(K) = \{\mathbf{B} : \mathbf{B} \text{ is isomorphic to a direct product of algebras in } K\};$$

$$\mathbf{P}_s(K) = \{\mathbf{B} : \mathbf{B} \text{ is isomorphic to a subdirect product of algebras in } K\}.$$

If K is such that $\mathbf{H}(K) \subseteq K$, we say that K is *closed under homomorphic images*; if $\mathbf{S}(K) \subseteq K$, then K is *closed under subalgebras* and finally, if $\mathbf{P}(K) \subseteq K$ (resp. $\mathbf{P}_s(K) \subseteq K$), then we say K is *closed under* (*sub*)*direct products.*

A class of similar algebras which is closed under homomorphic images, subalgebras and direct products is called a *variety.* A subclass of a variety K which is a variety is called a *subvariety* of K.

[3]In a poset $\langle P; \leq\rangle$, $p \prec q$ means $p \leq q$, $p \neq q$ and $p/\leq \cap\, q/\geq\, = \{p, q\}$, where $\geq\; = \;\leq^{-1}$.

Varieties are usually denoted by $\mathcal{V}$, $\mathcal{W}$ or other letters in calligraphic font. Moreover, if $\mathbf{O}$ is any operator $\mathbf{H}, \mathbf{S}$ or $\mathbf{P}$ and K is a finite set $\{\mathbf{A}_1, \dots, \mathbf{A}_k\}$, then we write $\mathbf{O}(\mathbf{A}_1, \dots, \mathbf{A}_k)$ instead of $\mathbf{O}(\{\mathbf{A}_1, \dots, \mathbf{A}_k\})$. Notice also that an equivalent definition for $\mathbf{H}$ is

$$\mathbf{H}(K) = \{\mathbf{B} : \mathbf{B} \cong \mathbf{A}/\theta, \text{ for some } \mathbf{A} \in K, \theta \in \operatorname{Con} \mathbf{A}\},$$

due to Lemma 1.2.2; thus, we might sometimes refer to $\mathbf{H}(K) \subseteq K$ as being *closed under quotients*.

For a class K of similar algebras, it makes sense to consider the smallest variety containing K, also referred to as the *variety generated by* K and denoted $\mathbf{V}(K)$. A well known result characterizes the operator $\mathbf{V}$ using the operators $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ as follows

Theorem 1.2.3. *For a class K of similar algebras,*

$$\mathbf{V}(K) = \mathbf{HSP}(K).$$

In other words, any algebra in $\mathbf{V}(K)$ is isomorphic to a quotient of a subalgebra of a direct product of a family of algebras in K.

Moreover, by Birkhoff's Theorem 1.2.2, we also deduce that for a variety $\mathcal{V}$, if $\mathcal{V}_{si}$ denotes the class of subdirectly irreducible algebras of $\mathcal{V}$, then $\mathcal{V} = \mathbf{P}_s(\mathcal{V}_{si})$.

A variety $\mathcal{V}$ is *finitely generated* if $\mathcal{V} = \mathbf{V}(K)$ for some finite K containing finite algebras. Furthermore, a variety $\mathcal{V}$ is *locally finite* if every finitely generated algebra in it is finite.

On the one hand, Theorem 1.2.3 provides us with a semantic description of a variety. On the other hand, a syntactic characterization can be also provided, and what we need for that is the notion of equation of some fixed type.

Therefore, let K be a class of similar algebras and let I and σ be, respectively, the set of basic operation symbols and the type of the algebras in K: we will refer to them as the *set of basic operation symbols* and the *type of* K, respectively. Moreover, a term in the language of K will be simply called a *term of* K. Given two terms p, q of K having the same arity, an *equation* (*of type* σ) is a word of the form

$$p \approx q.$$

Given an equation $p \approx q$ of type σ (resp. a set of equations Σ of type σ), we say that an algebra $\mathbf{A}$ *satisfies* $p \approx q$ (resp. Σ), or equivalently $p \approx q$ (resp. Σ) *holds/is valid in* $\mathbf{A}$, denoted $\mathbf{A} \models p \approx q$ (resp. $\mathbf{A} \models \Sigma$) if $p^{\mathbf{A}} = q^{\mathbf{A}}$ (resp. $s^{\mathbf{A}} = r^{\mathbf{A}}$, for all $s \approx r \in \Sigma$)[4]. Likewise, a class of similar algebras K *satisfies* $p \approx q$ (resp. Σ), or equivalently $p \approx q$ (resp. Σ) *holds/is valid in* K, denoted $K \models p \approx q$ (resp. $K \models \Sigma$) if so does/is $\mathbf{A}$, for every $\mathbf{A} \in K$. Let us then define

$$\operatorname{Mod}(\Sigma) = \{\mathbf{A} : \mathbf{A} \models \Sigma\},$$

called the *class of models of* Σ (or equivalently we might sometimes say that Σ *axiomatizes* $\operatorname{Mod}(\Sigma)$), and

$$\operatorname{Th}(K) = \{p \approx q : K \models p \approx q\},$$

called the *equational theory of* K, for a set of equations Σ and a class of similar algebras K. Thus, we are ready to state the famous *HSP theorem* due to G. Birkhoff.

Theorem 1.2.4 (HSP theorem)**.** *For any class of similar algebras K,*

$$\mathbf{HSP}(K) = \operatorname{Mod}(\operatorname{Th}(K)).$$

In particular, any variety is the class of models of a certain equational theory and vice versa.

This theorem gives a syntactic description of varieties and will be implicitly invoked almost anytime we will be dealing with such classes.

Other important objects in the study of varieties are free algebras, which we define as follows.

[4]Using the usual model theoretic notation, given an equation $p \approx q$, this definition is equivalent to saying that $\mathbf{A}$ is a model of the sentence $\forall x_1 \dots \forall x_k (p(x_1, \dots, x_k) = q(x_1, \dots, x_k))$, where k is the arity of the two terms.

Definition 1.2.12. Let $\mathbf{A}$ be an algebra of a certain type, K be a class of algebras of the same type and $X \subseteq A$. We say that $\mathbf{A}$ is a *free algebra in/of K over X*, or equivalently $\mathbf{A}$ is *freely generated in K by X*, if

- $\mathbf{A} \in K$;
- $A = \mathrm{Sg}^{\mathbf{A}}(X)$;
- For every algebra $\mathbf{B} \in K$, if $g : X \to B$ is any function, then there exists a (unique) homomorphism $\gamma_g : \mathbf{A} \to \mathbf{B}$ extending g, i.e. such that $\gamma_g(a) = g(a)$, for all $a \in X$.

We usually denote a free algebra in K over X by $\mathbf{F}_K(X)$.

Whenever the class K is a variety, the free algebra $\mathbf{F}_K(X)$ exists, for any non-empty set X. Moreover, for any class K of similar algebras, if $|X| = |Y|$ for some sets X, Y, then $\mathbf{F}_K(X) \cong \mathbf{F}_K(Y)$. The latter fact allows us to consider *the* free algebra of a class K over X, for some fixed set X.

Another interesting fact is that for a variety $\mathcal{V}$ and for some non-empty set X, if $\mathcal{V}$ contains an algebra $\mathbf{A}$ generated by Y and $|Y| \leq |X|$, then $\mathbf{A} \in \mathbf{H}(\mathbf{F}_{\mathcal{V}}(X))$; more generally $\mathcal{V} = \mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(\omega))$.

Also, it is a known fact that the universe of the free algebra of a variety $\mathcal{V}$ over a given set X is essentially the set of terms of $\mathcal{V}$ with variables in X, subject to the equations in the equational theory of $\mathcal{V}$ (more precisely, it is a quotient of the set of terms with variables in X modulo a suitable congruence); for this reason, when dealing with $F_{\mathcal{V}}(X)$, we will use the following description (with an abuse of notation due to the omission of the notation for congruence classes)

$$F_{\mathcal{V}}(X) = \{t(x_1, \ldots, x_n) : t \text{ is a term of arity } n \leq |X| \text{ of } \mathcal{V}, x_1, \ldots, x_n \in X\}.$$

For example, an element of the free algebra of the variety of abelian groups $\mathcal{AB}$ over the set $\{x, y\}$ is $x \cdot y$, which is syntactically different from $y \cdot x$, but coincides with it in $F_{\mathcal{AB}}(\{x, y\})$: hence, in this scenario, we would write $x \cdot y = y \cdot x \in F_{\mathcal{AB}}(\{x, y\})$.

Finally, any free algebra of a variety has the strong property of satisfying all and only the equations valid in the variety itself. The following lemma formalizes this claim.

Lemma 1.2.4. *Let $\mathcal{V}$ be a variety, X and Y be disjoint non-empty sets and let $p, q \in F_{\mathcal{V}}(X \cup Y)$ satisfy*

$$(p(\vec{x}, \vec{y}), q(\vec{x}, \vec{y})) \in \mathrm{Cg}^{\mathbf{F}_{\mathcal{V}}(X \cup Y)}(Y \times Y),$$

for some $\vec{x} \in X^n$, $\vec{y} \in Y^m$, for $n, m \geq 0$ Then, $\mathcal{V}$ satisfies $p(\vec{x}, y, \ldots, y) \approx q(\vec{x}, y, \ldots, y)$, for some $y \in Y$.

In particular, $\mathcal{V}$ satisfies $p \approx q$ if and only if $\mathbf{F}_{\mathcal{V}}(X \cup Y)$ does.

This lemma will be invoked both implicitly and explicitly several times, especially when we will present some standard techniques relative to Maltsev conditions.

What has been introduced so far can be considered sufficient to face all the topics that the reader may encounter in the next chapters.

To close this elementary overview of the main concepts of universal algebra, we dedicate the next section to describing a few properties of lattices which will be sometimes mentioned later.

1.3 Elementary notions of lattice theory

Lattices have a central role in universal algebra because they show up naturally in many settings, one of which can be considered the most important, meaning the case of the congruence lattice of any algebra. Therefore, it is worthwhile analyzing a few other aspects of these objects, even just with a superficial approach.

We have already provided the definition of lattice in Example 1.2.5 as an algebra satisfying certain conditions. Next, we will give an equivalent formulation of the definition, based on an axiomatic description of the variety $\mathcal{L}at$.

Definition 1.3.1. A *lattice* is any member of the variety $\mathcal{L}at$ with basic operation symbols $\langle \wedge, \vee \rangle$ and type $\langle 2, 2 \rangle$, axiomatized by the following equations

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z;$$
$$x \wedge x \approx x;$$
$$x \wedge y \approx y \wedge x;$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z;$$
$$x \vee x \approx x;$$
$$x \vee y \approx y \vee x;$$
$$x \wedge (x \vee y) \approx x;$$
$$x \vee (x \wedge y) \approx x.$$

Given a lattice $\mathbf{A} = \langle A; \wedge^{\mathbf{A}}, \vee^{\mathbf{A}} \rangle$, we will feel free to omit $^{\mathbf{A}}$ and we will then refer to that as $\langle A; \wedge, \vee \rangle$.

Moreover, we have already mentioned in the previous section that a lattice $\mathbf{A}$ is equipped with two natural partial orders, which we will denote respectively by $\leq$ and $\geq$, defined by

$$\leq := \{(a, b) \in A^2 : a \wedge b = a\},$$
$$\geq := \{(a, b) \in A^2 : a \vee b = a\},$$

satisfying

$$\leq^{-1} \;\; = \;\; \geq .$$

It is further worth mentioning two subvarieties of $\mathcal{L}at$ which deserve a particular consideration. These varieties are defined below.

Definition 1.3.2. A lattice $\mathbf{M}$ is said to be *modular* if it satisfies the modular law, i.e.

$$a \geq b \text{ implies } a \wedge (b \vee c) = b \vee (a \wedge c).$$

At first sight, the modularity of a lattice is a non-equational expression, but we can prove that the class $\mathcal{M}od$ of modular lattices is a variety and, in particular,

$$\mathcal{M}od = \mathrm{Mod}(\mathrm{Th}(\mathcal{L}at) \cup \{\varepsilon\}),$$

where ε is the following equation:

$$x \vee (y \wedge (x \vee z)) \approx (x \vee y) \wedge (x \vee z).$$

R. Dedekind discovered a very useful characterization of modular lattices, making use of the smallest non-modular finite lattice, sometimes called the *pentagon* and denoted by $\mathbf{N}_5$, represented below

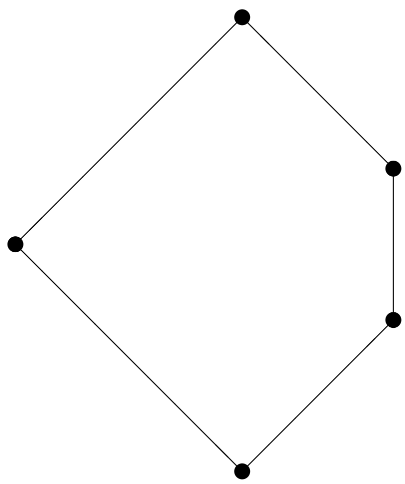

Figure 1.1: The non-modular lattice $\mathbf{N}_5$

Dedekind's result claims that

Theorem 1.3.1. *A lattice* $\mathbf{A}$ *is modular if and only if* $\mathbf{N}_5 \notin \mathbf{S}(\mathbf{A})$.

A special kind of modular lattice is a distributive lattice, defined as follows

Definition 1.3.3. A lattice $\mathbf{D}$ is *distributive*, if for all $a, b, c \in D$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

In fact, it turns out that the two above equalities are equivalent, namely any lattice satisfying the former also satisfies the latter and viceversa. Also, distributivity is a condition expressible via equations, which is to say the class of distributive lattices is a variety, denoted by $\mathcal{D}$. Likewise for Theorem 1.3.1, there exists a characterization of distributivity, namely

Theorem 1.3.2. *A lattice* $\mathbf{A}$ *is distributive if and only if* $\mathbf{N}_5, \mathbf{M}_3 \notin \mathbf{S}(\mathbf{A})$.

We have denoted by $\mathbf{M}_3$ the famous lattice of 5 elements called the *diamond*, represented in the next figure:

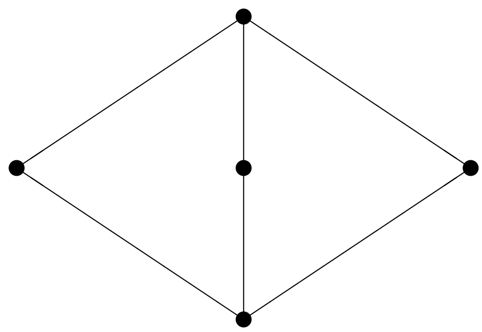

Figure 1.2: The non-distributive lattice $\mathbf{M}_3$

We could keep listing other properties of lattices besides modularity and distributivity, but we decide to stop here since we will just mention these two throughout the thesis.

On the other hand, we need to introduce some particular sublattices which will be implicitly central for the topic of Maltsev conditions. These objects are called filters and are defined as follows.

Definition 1.3.4. Given a lattice $\mathbf{A}$ and a non-empty subset $F \subseteq A$, we say that F is a *filter of* $\mathbf{A}$, if, for all $a, b \in A$

- if $a \in F$ and $a \leq b$, then $b \in F$;
- if $a, b \in F$, then $a \wedge b \in F$.

Notice that a filter of a lattice $\mathbf{A}$ is always a subuniverse of $\mathbf{A}$. Moreover, given $a \in A$, the set $a/\leq$ is also a filter of $\mathbf{A}$, called the *principal filter generated by* a. Some properties of filters that will be mentioned several times are the ones of primeness, decomposability and their negations. We define these notions below.

Definition 1.3.5. Let F be a filter of a lattice $\mathbf{A}$. We say that F is

- *prime*, if for all $a, b \in F$, whenever $a \vee b \in F$, then $a \in F$ or $b \in F$;
- *decomposable*, if there exist two distinct filters $D, E \subseteq A$, such that $F \subsetneq D$, $F \subsetneq E$ and $F = D \cap E$;
- *indecomposable*, if it is not decomposable.

Lattice theory is a widely developed field and we invite the reader to read [7] and [16] for further and deeper explanations. For the purpose of this document, we consider the so far defined concepts sufficient for a basic understanding of what is going to be presented next.

Chapter 2

The lattice of interpretability types of varieties

In 1974, W.D. Neumann defined in [30] a rather natural relation in the class of all varieties, which was thoroughly studied and analyzed later on by O. C. Garcia and W. Taylor in [15]. This chapter will be dealing with the fundamental aspects of the lattice of interpretability types, as well as one of the main notions in universal algebra which is that of Maltsev condition.

2.1 Definition and properties of the lattice L

Before providing the formal definition of the above mentioned relation, let us consider a few examples that might help figure out the setting we are going to present.

Consider the variety $\mathcal{V}$ of type $\langle 3,2,1\rangle$ and basic operation symbols $\langle p,f,u\rangle$, whose axiomatization is given by the following list of equations:

$$\begin{aligned} p(x,y,u(x)) &\approx u(y);\\ u(f(x,y)) &\approx p(x,x,y);\\ f(x,f(x,y)) &\approx f(f(x,y),y). \end{aligned}$$

On the other hand, consider the variety $\mathcal{W}$ of type $\langle 1,2,3\rangle$ with basic operation symbols $\langle v,g,q\rangle$, axiomatized by:

$$\begin{aligned} q(x,y,v(x)) &\approx v(y);\\ v(g(x,y)) &\approx q(x,x,y);\\ g(x,g(x,y)) &\approx g(g(x,y),y). \end{aligned}$$

Informally, we could claim that the varieties $\mathcal{V}$ and $\mathcal{W}$ coincide, but in fact they do not, because the names of the respective functional symbols of basic operations are different. If we associate bijectively p to q, f to g and u to v via a function φ (and its inverse φ^{-1}), then we can claim that "up to φ" the two varieties are the same.

The next example will describe a slightly different case, which is not as intuitive as the previous one.

Let $\mathcal{G}$ be the variety of type $\langle 2,1,0\rangle$ and basic operation symbols $\langle p,u,e\rangle$, whose axioms are

$$\begin{aligned} p(p(x,y),z) &\approx p(x,p(y,z));\\ p(x,e) &\approx x;\\ p(e,x) &\approx x;\\ p(x,u(x)) &\approx e;\\ p(u(x),x) &\approx e. \end{aligned}$$

Given the previous example and relative observation, we can say that "up to some function", the variety $\mathcal{G}$ coincides with the variety of groups, where p, u, e get associated respectively to the usual symbols $\cdot, ^{-1}, 1$, for the multiplicative notation of groups.

Likewise, consider the variety ([28]) $\mathcal{DG}$ of type $\langle 2 \rangle$ with basic operation symbol $\langle d \rangle$ axiomatized by:

$$d(d(x,z), d(y,z)) \approx d(x,y);$$
$$d(d(x,x), d(d(y,y),y)) \approx y.$$

The members of this variety are called *division groups*; indeed, such a name is justified by the fact that the varieties $\mathcal{G}$ and $\mathcal{DG}$ are essentially the same, in the following sense (which will be soon defined formally):

- for $\mathbf{G} \in \mathcal{G}$, the algebra $\mathbf{H} := \langle G; d^{\mathbf{H}} \rangle$ is a division group, where the operation $d^{\mathbf{H}}$ is defined on G as $d^{\mathbf{H}}(a,b) = p^{\mathbf{G}}(a, u^{\mathbf{G}}(b))$, for all $a, b \in G$;
- for $\mathbf{D} \in \mathcal{DG}$, the algebra $\mathbf{E} := \langle D; p^{\mathbf{E}}, u^{\mathbf{E}}, e^{\mathbf{E}} \rangle \in \mathcal{G}$, where the operations $p^{\mathbf{E}}, u^{\mathbf{E}}, e^{\mathbf{E}}$ are defined on D as
$$e^{\mathbf{E}} = d^{\mathbf{D}}(a,a);$$
$$u^{\mathbf{E}}(a) = d^{\mathbf{D}}(d^{\mathbf{D}}(a,a),a);$$
$$p^{\mathbf{E}}(a,b) = d^{\mathbf{D}}(a, d^{\mathbf{D}}(d^{\mathbf{D}}(a,a),b));$$
for all $a, b \in D$.

In these two examples we have emphasized the fact that two varieties can be considered the same up to some (so far undefined) equivalence if the members of one can "simulate" members of the other and viceversa. The next and last example will show that such a "simulation" may occur in one direction only.

Let $\mathcal{B}$ be the variety of Boolean algebras, i.e. the variety of type $\langle 2,2,1,0,0 \rangle$ with basic operation symbols $\langle \wedge, \vee, ', 0, 1 \rangle$, where $\wedge$ and $\vee$ satisfy the equations of distributive lattices, plus

$$x \wedge 0 \approx 0;$$
$$x \vee 1 \approx 1;$$
$$(x \wedge y)' \approx x' \vee y';$$
$$(x \vee y)' \approx x' \wedge y';$$
$$x \wedge x' \approx 0;$$
$$x \vee x' \approx 1;$$
$$(x')' \approx x;$$
$$0' \approx 1;$$
$$1' \approx 0.$$

Moreover, let $\mathcal{L}at$ be the variety of lattices. Since each Boolean algebra is in particular a lattice by definition, every Boolean algebra can "simulate" a lattice. Nevertheless, the converse does not hold. As a matter of fact, it is known that any finite Boolean algebra has cardinality 2^n, for some integer $n \geq 0$. If the inverse "simulation" held, then all lattices could be considered as Boolean algebras when carrying the suitable term operations as fundamental operations, and in particular the 3-element lattice. This is impossible because of the previous cardinality argument.

At this point, we are ready to give the formal definition of the concept of *interpretation.*

Definition 2.1.1 ([28],[15])**.** Let $\mathcal{V}$ and $\mathcal{W}$ be any two varieties having as sets of basic operation symbols, respectively, F and G. An *interpretation of* $\mathcal{V}$ in $\mathcal{W}$ is a function ι from F to the set of terms of $\mathcal{W}$, such that the following statements hold:

1. For any n-ary $f \in F$ $(n > 0)$, $\iota(f)$ is an n-ary term of $\mathcal{W}$;
2. For any 0-ary $e \in F$, $\iota(e)$ is a unary term of $\mathcal{W}$ such that $\mathcal{W} \models \iota(e)(x) \approx \iota(e)(y)$;

3. For any $\mathbf{A} \in \mathcal{W}$, the algebra $\mathbf{A}^{(\iota)} := \langle A; \{\iota(f)^{\mathbf{A}} : f \in F\}\rangle \in \mathcal{V}$.

We say that $\mathcal{V}$ is *interpretable* in $\mathcal{W}$, and write $\mathcal{V} \preceq \mathcal{W}$, if there exists an interpretation of $\mathcal{V}$ in $\mathcal{W}$.

With this notation, the examples provided before Definition 2.1.1 can be rephrased by saying that $\mathcal{V} \preceq \mathcal{W} \preceq \mathcal{V}$, $\mathcal{G} \preceq \mathcal{DG} \preceq \mathcal{G}$ and $\mathcal{L} \preceq \mathcal{B} \not\preceq \mathcal{L}$.

Notice that $\preceq$ is a quasi-order on the class of all varieties and hence it induces the following equivalence relation.

Definition 2.1.2. Two varieties $\mathcal{V}$ and $\mathcal{W}$ are *equi-interpretable* if

$$\mathcal{V} \preceq \mathcal{W} \preceq \mathcal{V}.$$

If $\mathcal{V}$ and $\mathcal{W}$ are equi-interpretable, we write $\mathcal{V} \equiv \mathcal{W}$.

The relation of equi-interpretability partitions the class of all varieties in $\equiv$-classes, each of which will be called an *interpretability type of varieties.* The usual notation $\mathcal{V}/\!\equiv$ will be replaced by $\overline{\mathcal{V}}$.

If $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ are two interpretability types, a natural partial order can be defined between them:

Definition 2.1.3. For any varieties $\mathcal{V}, \mathcal{W}$

$$\overline{\mathcal{V}} \leq \overline{\mathcal{W}} \text{ if and only if } \mathcal{V} \preceq \mathcal{W}.$$

If K denotes the class of all varieties, the structure whose universe is the quotient of K modulo $\equiv$, $\mathbb{L} = \langle K/\!\equiv; \leq\rangle$, is a poset which in fact turns out to be a complete lattice, in the sense stated in the next theorem.

Theorem 2.1.1. *Let J be a subclass of $L = K/\!\equiv$. If J is a set, then $\mathbb{J} = \langle J; \leq\rangle$ has both infimum and supremum.*

Thus, $\mathbf{L} = \langle L; \vee, \wedge\rangle$, where $\vee = \sup_{\leq}\{\cdot,\cdot\}$ and $\wedge = \inf_{\leq}\{\cdot,\cdot\}$, is a lattice, called the *lattice of interpretability types of varieties.*

In order to have an explicit description of the operations of join and meet, we need the following definition.

Definition 2.1.4 ([15])**.** Let $\mathcal{V} = \mathrm{Mod}(\Sigma)$ and $\mathcal{W} = \mathrm{Mod}(\Delta)$ be two varieties having types, respectively, σ and δ, and set of basic operation symbols S and D. Suppose further that $S \cap D = \varnothing$[1]. Define

$$\mathcal{V} \amalg \mathcal{W} = \mathrm{Mod}(\Sigma \cup \Delta),$$

having type $\sigma \cup \delta$, set of basic operation symbols $S \cup D$, and

$$\mathcal{V} \otimes \mathcal{W} = \mathrm{Mod}(\Gamma),$$

having type $\sigma \cup \delta \cup \{2\}$ and set of basic operation symbols $S \cup T \cup \{\cdot\}$, where Γ is the following set of equations:

$$\begin{aligned}
x \cdot x &\approx x; \\
(x \cdot y) \cdot (u \cdot v) &\approx x \cdot v; \\
p(x_1 \cdot y_1, \dots, x_n \cdot y_n) &\approx p(x_1, \dots, x_n) \cdot p(y_1, \dots, y_n) \qquad \text{for } n\text{-ary } p \in S \cup D; \\
d(x_1, \dots, x_n) \cdot y &\approx x_1 \cdot y \qquad \text{for } n\text{-ary } d \in D; \\
x \cdot s(y_1, \dots, y_n) &\approx x \cdot y_1 \qquad \text{for } n\text{-ary } s \in S; \\
p \cdot y &\approx q \cdot y \qquad \text{for } p \approx q \in \Sigma; \\
x \cdot a &\approx x \cdot b \qquad \text{for } a \approx b \in \Delta.
\end{aligned}$$

[1] We can assume this without loss of generality; indeed, if $S \cap D \neq \varnothing$, we can define $F_D = \{f_d : d \in D\}$ in such a way that $S \cap F_D = \varnothing$ and $f_d^{\langle B; F_D\rangle} = d^{\mathbf{B}}$, for all $\mathbf{B} \in \mathcal{W}$. The variety $\mathcal{W}'$ having F_D as set of basic operation symbols is then equi-interpretable to $\mathcal{W}$.

$\mathcal{V} \amalg \mathcal{W}$ is called the *coproduct* of $\mathcal{V}$ *and* $\mathcal{W}$; $\mathcal{V} \otimes \mathcal{W}$ is called the (*varietal*) *product of* $\mathcal{V}$ *and* $\mathcal{W}$.

The next theorem will justify the above definition.

Theorem 2.1.2 ([15]). *For any varieties* $\mathcal{V}, \mathcal{W}$,

$$\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}};$$

$$\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}.$$

Thus, the varieties defined in Definition 2.1.4 play the role of least upper bound and greatest lower bound for varieties in the lattice $\mathbf{L}$, up to equi-interpretability. However, that definition only expresses the equational theory of the two varieties, and we would like to have a concrete characterization of the models they contain. Therefore, we are going to give next a semantic description of $\mathcal{V} \otimes \mathcal{W}$ and of $\mathcal{V} \amalg \mathcal{W}$. For this purpose, we need the following objects defined.

Definition 2.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two (non necessarily similar) algebras. The *non-indexed product of* $\mathbf{A}$ *and* $\mathbf{B}$, denoted $\mathbf{A} \otimes \mathbf{B}$, is the algebra

$$\langle A \times B; \{f \otimes g : f \in \mathrm{Clo}_n \mathbf{A}, g \in \mathrm{Clo}_n \mathbf{B}, n \geq 0\}\rangle,$$

where $f \otimes g$ is defined via the n-ary term operations f and g of, respectively, $\mathbf{A}$ and $\mathbf{B}$ by

$$(f \otimes g)((a_1, b_1), \ldots, (a_n, b_n)) = (f(a_1, \ldots, a_n), g(b_1, \ldots, b_n)).$$

By putting together some results from [39],[40] and [15], we can summarize the properties of $\mathcal{V} \otimes \mathcal{W}$ in the following statement:

Theorem 2.1.3. *Given two varieties* $\mathcal{V}$ *and* $\mathcal{W}$,

$$\mathcal{V} \otimes \mathcal{W} \equiv \{\mathbf{A} \otimes \mathbf{B} : \mathbf{A} \in \mathcal{V}, \mathbf{B} \in \mathcal{W}\}.$$

More specifically, whenever $\mathbf{A} \otimes \mathbf{B} \in \mathcal{V} \otimes \mathcal{W}$,

1. *if* $\mathbf{C} \leq \mathbf{A} \otimes \mathbf{B}$, *then there exist* $\mathbf{P} \leq \mathbf{A}$ *and* $\mathbf{Q} \leq \mathbf{B}$ *such that* $\mathbf{C} \cong \mathbf{P} \otimes \mathbf{Q}$;
2. *if* $\theta \in \mathrm{Con}(\mathbf{A} \otimes \mathbf{B})$, *then there exist* $\alpha \in \mathrm{Con}(\mathbf{A})$ *and* $\beta \in \mathrm{Con}(\mathbf{B})$ *such that* $\theta = \alpha \otimes \beta$, *which in turn yields:*
3. *if* $\mathbf{C} \in \mathbf{H}(\mathbf{A} \otimes \mathbf{B})$, *then there exist* $\mathbf{P} \in \mathbf{H}(\mathbf{A})$ *and* $\mathbf{Q} \in \mathbf{H}(\mathbf{B})$ *such that* $\mathbf{C} \cong \mathbf{P} \otimes \mathbf{Q}$;
4. *if also* $\mathbf{C} \otimes \mathbf{D} \in \mathcal{V} \otimes \mathcal{W}$, *then*

$$(\mathbf{A} \otimes \mathbf{B}) \times (\mathbf{C} \otimes \mathbf{D}) \cong (\mathbf{A} \times \mathbf{C}) \otimes (\mathbf{B} \times \mathbf{D}).$$

For characterizing the operation of coproduct $\amalg$ (see [40]), we first need the following definition.

Definition 2.1.6. Let $\mathbf{A} = \langle A; \{f_i : i \in I\}\rangle$ and $\mathbf{B} = \langle B; \{f_j : j \in J\}\rangle$ be two algebras such that $|A| = |B|$, and let $s : A \to B$ be any bijection. Suppose further that $I \cap J = \varnothing$.

We define the *amalgamation of* $\mathbf{A}$ *and* $\mathbf{B}$ *relative to* s, denoted $\mathbf{A} \amalg_s \mathbf{B}$, to be the algebra $\langle A; \{h_i : i \in I \cup J\}\rangle$, such that for every $i \in I \cup J$, if h_i has arity $n \geq 0$, and $a_1, \ldots, a_n \in A$,

$$h_i(a_1, \ldots, a_n) = \begin{cases} f_i(a_1, \ldots, a_n) & \text{if } i \in I; \\ s^{-1}(f_i(s(a_1), \ldots, s(a_n))) & \text{if } i \in J. \end{cases}$$

Unlike the non-indexed product, the operation of amalgamation is not defined on every pair of algebras, for there are the restrictions of having disjoint sets of basic operation symbols and equal cardinality. Yet, as one could expect, this is exactly the operation that we need to describe the coproduct of varieties, as the next theorem states, of which we omit the proof that can be deduced from the results of [40].

Theorem 2.1.4. *For any two varieties $\mathcal{V}, \mathcal{W}$, whose sets of basic operation symbols are, respectively, I and J,*

$$\mathcal{V} \amalg \mathcal{W} \equiv \{\mathbf{A} \amalg_s \mathbf{B} : \mathbf{A} \in \mathcal{V}, \mathbf{B} \in \mathcal{W}, |A| = |B|, s : A \leftrightarrow B\}.$$

In particular, if $\mathbf{C} = \langle C; \{f_i : i \in I \cup J\}\rangle \in \mathcal{V} \amalg \mathcal{W}$, then $\langle C; \{f_i : i \in I\}\rangle \in \mathcal{V}$ and $\langle C; \{f_i : i \in J\}\rangle \in \mathcal{W}$. In fact:

$$\langle C; \{f_i : i \in I\}\rangle \amalg_{id} \langle C; \{f_i : i \in J\}\rangle = \mathbf{C}.$$

Moreover, for $\mathbf{A} \amalg_s \mathbf{B} \in \mathcal{V} \amalg \mathcal{W}$, we get:

1. *$C \leq \mathbf{A} \amalg_s \mathbf{B}$ if and only if $C \leq \mathbf{A}$ and $s(C) \leq \mathbf{B}$;*
2. *$\theta \in \mathrm{Con}(\mathbf{A} \amalg_s \mathbf{B})$ if and only if $\theta \in \mathrm{Con}(\mathbf{A})$ and $(s \times s)(\theta) \in \mathrm{Con}(\mathbf{B})$;*
3. *$(\mathbf{A} \amalg_s \mathbf{B})^2 = \mathbf{A}^2 \amalg_{s \times s} \mathbf{B}^2$.*

In the lattice $\mathbf{L}$, for $J \subseteq L$ any sub*set* of L, we could also define the $\leq$-supremum and $\leq$-infimum of J, respectively $\bigvee J$ and $\bigwedge J$, by generalizing Definition 2.1.4, Definition 2.1.5 and Theorem 2.1.2. However, since we are going to mention these objects just in a specific scenario, for our purpose we will only need to present a syntactic definition for the coproduct of a family of varieties, whereas we will define the product in a semantic flavor. For more details, refer to [40] and [15].

Definition 2.1.7 (Generalization of Definition 2.1.5)**.** Let $\kappa > 0$ be any cardinal and $U = \{\mathbf{A}_i : i < \kappa\}$ a family of algebras. The *non-indexed product of* U, denoted $\bigotimes_{i<\kappa} \mathbf{A}_i$, is the algebra

$$\left\langle \prod_{i<\kappa} A_i; \left\{ \bigotimes_{i<\kappa} f_i : f_i \in \mathrm{Clo}_n \mathbf{A}_i, n \geq 0 \right\} \right\rangle,$$

where the fundamental operations are defined for all $n \geq 0$ and $\vec{a}^1, \ldots, \vec{a}^n \in \prod_{i<\kappa} A_i$ [where $\vec{a}^j = (a_i^j : i < \kappa)$], as

$$\left(\bigotimes_{i<\kappa} f_i \right) (\vec{a}^1, \ldots, \vec{a}^n) = (f_i(a_i^1, \ldots, a_i^n) : i < \kappa).$$

This construction will be used as part of the next definition

Definition 2.1.8. Let $\kappa > 0$ be a cardinal and $U = \{\mathcal{V}_i : i < \kappa\}$ a family of varieties, such that for each $i < \kappa$, $\mathcal{V}_i = \mathrm{Mod}(\Sigma_i)$ has type σ_i and set of basic operation symbols F_i. Moreover, assume $F_i \cap F_j = \varnothing$, for $i, j < \kappa$, with $i \neq j$. Define the *coproduct of* U, denoted $\coprod_{i<\kappa} \mathcal{V}_i$, the variety of type $\bigcup_{i<\kappa} \sigma_i$ with set of basic operation symbols $\bigcup_{i<\kappa} F_i$ axiomatized by $\bigcup_{i<\kappa} \Sigma_i$, namely

$$\coprod_{i<\kappa} \mathcal{V}_i = \mathrm{Mod}\left(\bigcup_{i<\kappa} \Sigma_i \right).$$

Moreover, define the (*varietal*) *product of* U, denoted $\bigotimes_{i<\kappa} \mathcal{V}_i$, as

$$\bigotimes_{i<\kappa} \mathcal{V}_i = \mathbf{HSP}\left(\left\{ \bigotimes_{i<\kappa} \mathbf{A}_i : \mathbf{A}_i \in \mathcal{V}_i \right\} \right).$$

And to confirm the expectations we have

Theorem 2.1.5. *In the lattice $\mathbf{L}$, for any cardinal $\kappa > 0$ and $\{\overline{\mathcal{V}}_i : i < \kappa\} \subseteq L$,*

$$\bigvee_{i<\kappa} \overline{\mathcal{V}}_i = \overline{\coprod_{i<\kappa} \mathcal{V}_i};$$

$$\bigwedge_{i<\kappa} \overline{\mathcal{V}}_i = \overline{\bigotimes_{i<\kappa} \mathcal{V}_i}.$$

Another important property of the lattice $\mathbf{L}$ is the fact that it has both least and greatest elements. Notice that if we look at the class $\mathcal{S}ets$ (see Example 1.2.2) as the class of models of $\{x \approx x\}$, with no basic operation symbols, we deduce that such a class is a variety. Furthermore, notice that $\mathcal{S}ets$ is trivially interpretable in any other variety $\mathcal{V}$, yielding that, for every variety $\mathcal{V}$, $\overline{\mathcal{S}ets} \leq \overline{\mathcal{V}}$, and hence $\overline{\mathcal{S}ets}$ is the bottom element of $\mathbf{L}$. A curious fact is that the interpretability type of $\mathcal{S}ets$ contains the variety of semigroups $\mathcal{SG}$: in this setting, up to equi-interpretability, the variety of sets coincides with the variety of semigroups. As a matter of fact, we can always interpret the only binary basic operation symbol of $\mathcal{SG}$ in one of the two only binary terms of $\mathcal{S}ets$, that is either the first projection π_1 or the second projection map π_2: because these maps satisfy the associative law, every set along with π_i ($i \in \{1,2\}$) as a fundamental operation is a semigroup, showing that $\mathcal{SG} \preceq \mathcal{S}ets$.

On the other hand, since any variety $\mathcal{V}$ contains the variety $\mathcal{T} = \mathrm{Mod}(\{x \approx y\})$ of trivial algebras (one element algebras) as a subvariety, we have that $\mathcal{V} \preceq \mathcal{T}$, where the interpretation is the identity map. In addition, all trivial varieties are equi-interpretable, yielding that $\overline{\mathcal{T}}$ is the top element of $\mathbf{L}$.

As a further observation, there is a thoroughly studied class of varieties which is worth a special consideration; that is to say the class of idempotent varieties. In this setting, if $\mathcal{V}$ and $\mathcal{W}$ are two idempotent varieties, then their coproduct $\mathcal{V} \amalg \mathcal{W}$ is also idempotent because it is axiomatized by equations only involving idempotent terms. Likewise for the product $\mathcal{V} \otimes \mathcal{W}$, since the only extra basic operation symbol $\cdot$ satisfies $x \cdot x \approx x$, it is idempotent. Therefore, we can well define the class of interpretability types of idempotent varieties

$$L^{id} := \{\overline{\mathcal{V}} : \text{ there exists } \mathcal{W} \text{ such that } \mathcal{V} \equiv \mathcal{W} \equiv \mathcal{W}^{id}\}.$$

Such a subclass of L is, by the previous observations, a sublattice of $\mathbf{L}$, being closed under $\vee$ and $\wedge$. In addition, since $\mathcal{S}ets$ and $\mathcal{T}$ are idempotent varieties, we get that $\overline{\mathcal{S}ets}, \overline{\mathcal{T}} \in L^{id}$, making this class a $0,1$-sublattice of $\mathbf{L}$.

2.2 Maltsev conditions

The lattice of interpretability types $\mathbf{L}$ is the natural environment for the central universal algebraic concept of Maltsev condition.

Informally, a (strong) Maltsev condition M is a finite set of function symbols $\{t_1, \ldots, t_k\}$ along with a finite set of equations Σ involving them. Given such an M, a variety $\mathcal{V}$ satisfies M, if there exist some terms of $\mathcal{V}$, say $t_1', \ldots, t_k'$, such that t_i and t_i' have the same arity ($1 \leq i \leq k$) and $\mathcal{V}$ satisfies Σ', where Σ' is obtained by replacing each t_i by t_i' in every equation in Σ. This idea can be made more rigorous within the lattice $\mathbf{L}$.

We say that a variety is *finitely presentable* if it has finitely many basic operation symbols and it is axiomatized by finitely many equations. Here are the definitions we need:

Definition 2.2.1. In the lattice of interpretability types $\mathbf{L}$:

1. A *strong Maltsev condition* is the interpretability type of a finitely presentable variety $\mathcal{M}$, that is to say $\overline{\mathcal{M}}$.
2. A *Maltsev condition* is a countably infinite set of strong Maltsev conditions, namely $\{\overline{\mathcal{M}}_n : n < \omega\}$, such that $\overline{\mathcal{M}}_{n+1} \leq \overline{\mathcal{M}}_n$, for every $n < \omega$.

We observe that this definition does not consider a strong Maltsev condition as a particular case of a Maltsev condition, because of two facts. Firstly, a strong Maltsev condition has been defined as an interpretability type, while a Maltsev condition as a set. Secondly, we have required that a Maltsev condition be a countably infinite set. We could have overcome this essential difference by defining a Maltsev condition as a countable set of interpretability types $\{\overline{\mathcal{M}}_n : n < \omega\}$, with $\overline{\mathcal{M}}_{n+1} \leq \overline{\mathcal{M}}_n$, for all $n < \omega$, and in the specific case of $\overline{\mathcal{M}}_{n+1} = \overline{\mathcal{M}}_n$, for all $n < \omega$, say that such a Maltsev condition is strong. Under this point of view, a strong Maltsev condition would have been a singleton containing the interpretability type of a finitely presentable variety. Nonetheless, we prefer the statement displayed in Definition 2.2.1, which seems more intuitive and expressive. Although this definition does not formally identify a strong Maltsev condition as a particular form of Maltsev condition, we will feel free, in some contexts, to refer to both strong and not strong Maltsev conditions as just Maltsev conditions.

Definition 2.2.2. Let $\mathcal{V}$ be a variety, $\overline{\mathcal{M}}$ a strong Maltsev condition and $M = \{\overline{\mathcal{M}}_n : n < \omega\}$ a Maltsev condition.

1. We say that $\mathcal{V}$ *satisfies* $\overline{\mathcal{M}}$, if $\overline{\mathcal{M}} \leq \overline{\mathcal{V}}$.
2. We say that $\mathcal{V}$ *satisfies* M, if there exists $n < \omega$ such that $\overline{\mathcal{M}}_n \leq \overline{\mathcal{V}}$.

A given (strong) Maltsev condition defines naturally a class of interpretability types,

Definition 2.2.3. Let $H \subseteq L$ (recall L is the class of all interpretability types) be a subclass.

1. H is a *strong Maltsev class* if there exists a strong Maltsev condition $\overline{\mathcal{M}}$ with
$$H = \{\overline{\mathcal{V}} : \overline{\mathcal{M}} \leq \overline{\mathcal{V}}\}.$$
2. H is a *Maltsev class* if there exists a Maltsev condition $\{\overline{\mathcal{M}}_n : n < \omega\}$ with
$$H = \{\overline{\mathcal{V}} : \exists n(\overline{\mathcal{M}}_n \leq \overline{\mathcal{V}})\}.$$

Notice that a strong Maltsev class defined as in Definition 2.2.3 is actually a principal filter generated by the interpretability type of the finitely presentable variety $\mathcal{M}$. On the other hand, a Maltsev class is a nested union of countably many strong Maltsev classes. This fact justifies the name these objects are frequently called with: *Maltsev filters* (see also [36]).

Maltsev conditions have played a main role in universal algebra during the past decades, and one of the reasons is that the presence of terms constrained to satisfy certain equations in a variety $\mathcal{V}$ often affects the behavior or characterizes some peculiar properties of the members of the variety itself, as will be clarified later on.

Let us provide some examples of (strong) Maltsev conditions that are often met in this area of study. The coming example may be seen as the very first strong Maltsev condition to have been considered ([25]).

Example 2.2.1. Let $\mathcal{CP}_2$ be the variety of type $\langle 3 \rangle$ with basic operation symbol $\langle p_1 \rangle$, axiomatized by the following two equations:

$$\begin{aligned} x &\approx p_1(x, y, y); \\ p_1(x, x, y) &\approx y. \end{aligned}$$

If a variety $\mathcal{V}$ is such that $\mathcal{CP}_2 \preceq \mathcal{V}$ via the interpretation ι (equivalently, $\overline{\mathcal{CP}}_2 \leq \overline{\mathcal{V}}$), we say that $\mathcal{V}$ has a *Maltsev term*, that is exactly the term $\iota(p_1) := t$, with respect to which $\mathcal{V} \models \{x \approx t(x, y, y), t(x, x, y) \approx y\}$.

If a variety $\mathcal{V}$ has a Maltsev term t, then every algebra $\mathbf{A} \in \mathcal{V}$ has permuting congruences, i.e. for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$,
$$\alpha \circ \beta = \beta \circ \alpha.$$
The proof of this fact is elementary but really meaningful. As a matter of fact, let $(a, c) \in \alpha \circ \beta$, which means there exists $b \in A$ with $a \ \alpha \ b \ \beta \ c$. Therefore,

$$a = t^{\mathbf{A}}(a, b, b) \ \beta \ t^{\mathbf{A}}(a, b, c) \ \alpha \ t^{\mathbf{A}}(b, b, c) = c.$$

The term t allows us to find the element $d = t^{\mathbf{A}}(a, b, c)$ which $\beta \circ \alpha$-connects a and c.

Surprisingly, it turns out that if the congruences of every algebra of a variety $\mathcal{V}$ permute, then $\mathcal{CP}_2 \preceq \mathcal{V}$, namely $\mathcal{V}$ has a Maltsev term. Indeed, consider in $\mathcal{V}$ the algebra freely generated by $\{x, y, z\}$, call it $\mathbf{F}$. By assumption, $\mathbf{F}$ has permuting congruences, in particular, since $x \operatorname{Cg}^{\mathbf{F}}(x, z)$ $z \operatorname{Cg}^{\mathbf{F}}(z, y) \ y$, there must exist $t \in F$ (hence t is a ternary term), such that

$$x \ \operatorname{Cg}^{\mathbf{F}}(y, z) \ t(x, y, z) \ \operatorname{Cg}^{\mathbf{F}}(x, y) \ y,$$

which yields that $t^{\mathbf{F}}(x, y, y) = x$ and $t^{\mathbf{F}}(x, x, y) = y$. Because these identities hold in the free algebra $\mathbf{F}$, they must hold in every member of the variety, proving that t is a Maltsev term for $\mathcal{V}$.

The procedure just described is again rather elementary but it has inspired a lot of deep mathematical results, in a way that it has been considered as a sort of standard template when one deals with Maltsev conditions.

The next example introduces a Maltsev condition which was described through terms by Alan Day in [10] and is called *congruence modularity*.

Example 2.2.2. For $n > 1$, define $\mathcal{CM}_n$ to be the variety of type $\langle \overbrace{4, \ldots, 4}^{n+1 \text{ times}} \rangle$ and basic operation symbols $\langle m_0, \ldots, m_n \rangle$, whose axioms are:

$$
\begin{aligned}
x &\approx m_0(x, y, z, w); \\
m_i(x, y, y, z) &\approx m_{i+1}(x, y, y, z) \text{ for odd } i; \\
m_i(x, x, y, y) &\approx m_{i+1}(x, x, y, y) \text{ for even } i; \\
m_n(x, y, z, w) &\approx w; \\
m_i(x, y, y, x) &\approx x \text{ for every } 0 \leq i \leq n.
\end{aligned}
$$

If a variety $\mathcal{V}$ is such that for some $n > 1$, $\mathcal{CM}_n \leq \mathcal{V}$, then the interpreted terms of $\mathcal{V}$ are called *Day terms*.

Using analogous techniques as the ones shown in Example 2.2.1, Day proved in [10] that the class of varieties having Day terms is exactly the class of varieties whose members have modular congruence lattices (congruence modular varieties).

As already mentioned before, these two examples show that some Maltsev conditions are able to heavily affect the behavior of congruences of the algebras in the varieties, even though this is not the only feasible case. Other Maltsev conditions will be investigated in this thesis in the next chapters.

It is also possible to analyze some Maltsev conditions in an "indirect" way, which means without necessarily having access to the terms and the equations that axiomatize the finitely presented varieties defining the condition itself. The tools to do this have been provided by Walter Taylor in [39], and are presented in the following criterion.

Theorem 2.2.1. *Let H be a non-empty class of interpretability types.*
H is a [strong] Maltsev class if and only if the following independent statements hold for H:

1. *For varieties $\mathcal{V}, \mathcal{W}$, if $\mathcal{W} \equiv \mathcal{V}$ and $\overline{\mathcal{V}} \in H$, then $\overline{\mathcal{W}} \in H$;*
2. *For varieties $\mathcal{V}, \mathcal{W}$, if $\mathcal{W}$ is a subvariety of $\mathcal{V}$ and $\overline{\mathcal{V}} \in H$, then $\overline{\mathcal{W}} \in H$;*
3. *For varieties $\mathcal{V}_i$, $i < \kappa$, where $\kappa > 0$ is a cardinal, if $\{\overline{\mathcal{V}}_i : i < \kappa\} \subseteq H$ and κ is finite [arbitrarily large], then*
$$\overline{\bigotimes_{i<\kappa} \mathcal{V}_i} = \bigwedge_{i<\kappa} \overline{\mathcal{V}}_i \in H;$$
4. *For every set of equations Σ, if $\overline{\mathrm{Mod}(\Sigma)} \in H$ and has type σ, then there exists a finite subset $\Delta \subseteq \Sigma$ such that $\overline{\mathrm{Mod}(\Delta)} \in H$ and has type σ.*

The first statement in the necessary condition of Theorem 2.2.1 guarantees the unambiguity of considering H as a class of interpretability types of varieties, instead of just varieties; the second and third basically express the defining lattice theoretic properties of a filter, namely closure with respect to upper bounds and (arbitrary) infima. The fourth condition, instead, is related to the fact that a (strong) Maltsev filter requires the presence of finitely presentable varieties.

As a last discussion for this section, we want to point out that some obvious questions may arise when studying a Maltsev class, especially if we view it as a filter. For instance, we can wonder whether a given Maltsev filter is prime, not prime, decomposable or not.

The next definition will just rephrase these well known notions from pure lattice theory (we did recall them in Definition 1.3.5), in our specific context.

Definition 2.2.4. We say that a strong Maltsev condition $\overline{\mathcal{M}}$ is *prime* (respectively *decomposable*) if the strong Maltsev filter $\{\overline{\mathcal{V}} : \overline{\mathcal{M}} \leq \overline{\mathcal{V}}\}$ is prime (respectively decomposable).

We say that a Maltsev condition $\{\overline{\mathcal{M}}_n : n < \omega\}$ is *prime* (respectively *decomposable*) if the Maltsev filter $\{\overline{\mathcal{V}} : \exists n(\overline{\mathcal{M}}_n \leq \overline{\mathcal{V}})\}$ is prime (respectively decomposable).

For a strong Maltsev condition $\overline{\mathcal{M}}$ we could equivalently say that it is prime if, whenever $\overline{\mathcal{M}} \leq \overline{\mathcal{V}} \vee \overline{\mathcal{W}}$, then $\overline{\mathcal{M}} \leq \overline{\mathcal{V}}$ or $\overline{\mathcal{M}} \leq \overline{\mathcal{W}}$; likewise, it is decomposable if $\overline{\mathcal{M}} = \overline{\mathcal{V}} \vee \overline{\mathcal{W}}$, for some varieties $\mathcal{V}$ and $\mathcal{W}$ such that $\overline{\mathcal{V}} \lneq \overline{\mathcal{M}}$ and $\overline{\mathcal{W}} \lneq \overline{\mathcal{M}}$.

In [15] the authors conjectured that the strong Maltsev condition of congruence permutability (see Example 2.2.1) and the Maltsev condition of congruence modularity (see Example 2.2.2) are both prime. The first conjecture was proven in [41], but it has never been published. The second conjecture, also known as *Taylor's modularity conjecture* is still open, although partial results have been achieved in [36], [27], [5], [4], and more recently in [31].

For the rest of this thesis, we will be dealing with other primeness questions concerning other specific Maltsev conditions.

We invite the reader to pay attention to the following note about a change in the notation that will occur in the next chapters. After having specified in details what is the difference between a variety and its interpretability type and having supported with some examples and the definitions of interpretation and equi-interpretability, from this point on, we will no longer use the notation $\overline{\mathcal{V}}$ to refer to as the interpretability type of $\mathcal{V}$, nor the expression $\mathcal{V} \preceq \mathcal{W}$. Instead, when dealing with a variety $\mathcal{V}$, we will replace $\overline{\mathcal{V}}$ with just $\mathcal{V}$, intending it as a variety up to equi-interpretability, and hence $\mathcal{V} \preceq \mathcal{W}$ will be replaced, with an abuse of notation, by $\mathcal{V} \leq \mathcal{W}$. Also, by an *idempotent variety* $\mathcal{V}$ we will simultaneously mean the interpretability type of $\mathcal{V}$ containing an idempotent variety; likewise, as a *locally finite variety*, we will simultaneously mean an interpretability type containing a locally finite variety. Due to these observations, we will also feel free to treat L and L^{id} as the classes of all varieties and idempotent varieties, respectively.

Therefore, a strong Maltsev class induced by (the interpretability type of) $\mathcal{M}$, will be denoted by $\{\mathcal{V} : \mathcal{M} \leq \mathcal{V}\}$, and analogously a Maltsev class induced by the Maltsev condition $\{\mathcal{M}_n : n < \omega\}$ will be denoted by $\{\mathcal{V} : \exists n(\mathcal{M}_n \leq \mathcal{V})\}$.

Chapter 3

An overview of congruence n-permutable varieties

The leading character of this thesis is definitely going to be the Maltsev condition of congruence n-permutability, which will be defined and analyzed across this chapter. We have already mentioned permuting congruences in Example 2.2.1, and that condition can be generalized to higher occurrences of the composition symbol and to any binary relation.

Definition 3.0.1. Let A be a nonempty set and $R, S \subseteq A \times A$ be binary relations on A. We define the *iterated composition* $R \circ^n S$, for $n \geq 1$, inductively as follows:

$$R \circ^1 S = R;$$

$$R \circ^{n+1} S = R \circ (S \circ^n R).$$

Notice that this is a set theoretic notion which in fact turns out to be one of the most useful operations on binary subuniverses of algebras.

The operation $\circ$ is associative on the set of reflexive relations on a nonempty set A and this allows us to omit the use of brackets; for example, if $n > 2$, $R \circ^n S$ can be written as $R \circ S \circ R \circ \ldots$, where there are n alternating factors R and S, or equivalently $n-1$ occurrences of $\circ$. Moreover, for R, S reflexive, we observe straightforwardly that $R, S \subseteq R \circ S$, and if $R \subseteq S$, with S transitive, then $R \circ S = S$.

Notice further that in general $R \circ S \neq S \circ R$, even when R and S are equivalence relations, and in particular when they are congruence relations of an algebra. With this latter fact in mind, we are then ready to provide the definition of n-permutability.

Definition 3.0.2. Let $\mathbf{A}$ be an algebra, $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ and $n > 1$. We say that α, β *n-permute*, or are *n-permutable* if

$$\alpha \circ^n \beta = \beta \circ^n \alpha.$$

An algebra $\mathbf{A}$ is (*congruence*) *n-permutable*, if any two congruences of $\mathbf{A}$ n-permute. A variety is (*congruence*) *n-permutable* if every algebra in it is.

Notice that if an algebra is congruence n-permutable, for some $n > 1$, then it is congruence k-permutable for every $k \geq n$. Likewise for a variety. Moreover, given an algebra $\mathbf{A}$, while in general the following holds:

$$\alpha \vee \beta = \bigcup_{k<\omega} \alpha \circ^k \beta,$$

for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, in the case of $\mathbf{A}$ being congruence n-permutable ($n \geq 2$), we can in fact describe the join of two congruences as a finite union, or more precisely as

$$\alpha \vee \beta = \alpha \circ^n \beta = \beta \circ^n \alpha.$$

In the next section we will present three equivalent Maltsev conditions that describe congruence n-permutability for varieties.

3.1 Maltsev conditions for congruence n-permutability

As already mentioned, (the interpretability types of) congruence n-permutable varieties form a Maltsev class. In this section we will show that there are at least four Maltsev conditions equivalent to congruence n-permutability and these are obtained under different perspectives. The first of these conditions is also historically the first to have been discovered by E.T. Schmidt [34] in 1969, followed by the one discovered by J. Hagemann and A. Mitschke [18] in 1973. The third condition is instead part of our new contribution to this field, whereas the fourth is presented in [21].

Let us then begin with Schmidt's theorem which produces a sequence of $(n+1)$-ary terms for every congruence n-permutable variety.

Theorem 3.1.1 ([34])**.** *Let $n > 1$ and $\mathcal{V}$ a variety. Then the following are equivalent:*

1. *$\mathcal{V}$ is congruence n-permutable;*
2. *$\mathcal{S}_n \leq \mathcal{V}$, where $\mathcal{S}_n$ is the variety of type $\langle \overbrace{n+1, \ldots, n+1}^{n+1 \text{ times}} \rangle$ and basic operation symbols $\langle s_0, \ldots, s_n \rangle$ axiomatized by:*

$$
\begin{aligned}
x_0 &\approx s_0(x_0, \ldots, x_n);\\
s_i(x_0, x_0, x_2, x_2, \ldots) &\approx s_{i+1}(x_0, x_0, x_2, x_2, \ldots) \text{ for odd } i;\\
s_i(x_0, x_1, x_1, x_3, x_3, \ldots) &\approx s_{i+1}(x_0, x_1, x_1, x_3, x_3, \ldots) \text{ for even } i;\\
s_n(x_0, \ldots, x_n) &\approx x_n;
\end{aligned}
$$

Proof. Let us first prove that (2) $\Rightarrow$ (1).

Without loss of generality, call $s_0, \ldots, s_n$ the terms of $\mathcal{V}$ which satisfy the displayed equations. Let further $\mathbf{A} \in \mathcal{V}$ with $\alpha, \beta \in \operatorname{Con} \mathbf{A}$. We aim to prove that α and β n-permute.

Consider $(a_0, a_n) \in \alpha \circ^n \beta$, which is to say

$$
a_0 \ \alpha \ a_1 \ \beta \ a_2 \ \alpha \ \cdots \begin{cases} \beta \ a_n & \text{if } n \text{ is even;} \\ \alpha \ a_n & \text{if } n \text{ is odd.} \end{cases}
$$

More precisely, the cases listed below occur:

$$
\text{For even } n: \begin{cases} a_i \ \alpha \ a_{i+1} & \text{for even } 0 \leq i \leq n-2; \\ a_i \ \beta \ a_{i+1} & \text{for odd } 0 \leq i \leq n-1. \end{cases}
$$

$$
\text{For odd } n: \begin{cases} a_i \ \alpha \ a_{i+1} & \text{for even } 0 \leq i \leq n-1; \\ a_i \ \beta \ a_{i+1} & \text{for odd } 0 \leq i \leq n-2. \end{cases}
$$

Using the terms s_i's and their properties we need to produce a sequence of elements that $\beta \circ^n \alpha$-connect a_0 to a_n.

First, assume n is even. For even $0 \leq i \leq n-2$, it is straightforward to check that the following $\beta \circ \alpha$-connection holds:

$$
\begin{gathered}
e_i := s_i^{\mathbf{A}}(\overbrace{a_i}^{x_0}, \ldots, \overbrace{a_i, a_i}^{x_{i-1}, x_{i-1}}, \overbrace{a_{i+1}, a_{i+1}}^{x_{i+1}, x_{i+1}}, \ldots, \overbrace{a_{i+1}, a_{i+1}}^{x_{n-1}, x_{n-1}}) \\
\shortparallel \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_i}^{x_0}, \ldots, \overbrace{a_i, a_i}^{x_{i-1}, x_{i-1}}, \overbrace{a_{i+1}, a_{i+1}}^{x_{i+1}, x_{i+1}}, \ldots, \overbrace{a_{i+1}, a_{i+1}}^{x_{n-1}, x_{n-1}}) \\
\beta \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_i}^{x_0}, \ldots, \overbrace{a_i, a_i}^{x_{i-1}, x_{i-1}}, \overbrace{a_{i+1}, a_{i+2}}, \ldots, \overbrace{a_{i+2}, a_{i+2}}) \\
\alpha \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_{i+1}, a_{i+1}}^{x_0, x_0}, \ldots, \overbrace{a_{i+1}, a_{i+1}}^{x_i, x_i}, \overbrace{a_{i+2}, a_{i+2}}^{x_{i+2}, x_{i+2}}, \ldots, \overbrace{a_{i+2}, a_{i+2}}^{x_{n-2}, x_{n-2}}, \overbrace{a_{i+2}}^{x_n}) =: o_{i+1}.
\end{gathered}
$$

Likewise, for odd $1 \leq i \leq n-1$, the following $\alpha \circ \beta$-connection holds:

$$\begin{array}{c}
o_i := s_i^{\mathbf{A}}(\overbrace{a_i, a_i}^{x_0,x_0}, \dots, \overbrace{a_i, a_i}^{x_i,x_i}, \overbrace{a_{i+1}, a_{i+1}}^{x_{i+2},x_{i+2}}, \dots, \overbrace{a_{i+1}}^{x_n}) \\
\shortparallel \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_i, a_i}^{x_0,x_0}, \dots, \overbrace{a_i, a_i}^{x_i,x_i}, \overbrace{a_{i+1}, a_{i+1}}^{x_{i+2},x_{i+2}}, \dots, \overbrace{a_{i+1}}^{x_n}) \\
\alpha \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_i, a_i}^{x_0,x_0}, \dots, \overbrace{a_i, a_i}^{x_i,x_i}, \overbrace{a_{i+1}, a_{i+2}}, \dots, \overbrace{a_{i+2}}^{x_n}) \\
\beta \\
s_{i+1}^{\mathbf{A}}(\overbrace{a_{i+1}}^{x_0}, \overbrace{a_{i+1}, a_{i+1}}^{x_1,x_1}, \dots, \overbrace{a_{i+1}, a_{i+1}}^{x_i,x_i}, \overbrace{a_{i+2}, a_{i+2}}^{x_{i+2},x_{i+2}}, \dots, \overbrace{a_{i+2}, a_{i+2}}^{x_n,x_n}) =: e_{i+1}.
\end{array}$$

With this helpful notation, we get that

$$a_0 = e_0 \ \beta \circ \alpha \ o_1 \ \alpha \circ \beta \ \cdots \ e_{n-2} \ \beta \circ \alpha \ o_{n-1} = a_n,$$

which means $(a_0, a_n) \in (\beta \circ \alpha) \circ^{n-1} (\alpha \circ \beta) = \beta \circ^n \alpha$, as needed.

For n odd, an analogous reasoning, which we omit, yields the same result.

For proving that (1) $\Rightarrow$ (2), let us assume that $\mathcal{V}$ is congruence n-permutable. In particular, the algebra $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(\{x_0, \dots, x_n\})$ is congruence n-permutable. The strategy now is to consider two suitable congruences which will produce the desired terms when forced to n-permute. Let us distinguish the even and odd cases of n.

If n is even, define:

$$\alpha = \mathrm{Cg}^{\mathbf{F}}(\{(x_i, x_{i+1}) : 0 \leq i \leq n-2 \text{ even}\}) = \mathrm{Cg}^{\mathbf{F}}(\{(x_0, x_1), \dots, (x_{n-2}, x_{n-1})\});$$

$$\beta = \mathrm{Cg}^{\mathbf{F}}(\{(x_i, x_{i+1}) : 1 \leq i \leq n-1 \text{ odd}\}) = \mathrm{Cg}^{\mathbf{F}}(\{(x_1, x_2), \dots, (x_{n-1}, x_n)\}).$$

These two congruences are built in such a way that

$$x_0 \ \alpha \ x_1 \ \beta \ x_2 \cdots \alpha \ x_{n-1} \ \beta \ x_n,$$

namely $(x_0, x_n) \in \alpha \circ^n \beta$. Since α and β n-permute, there exist $s_1, \dots, s_{n-1} \in F$ such that

$$x_0 \ \beta \ s_1(x_0, \dots, x_n) \ \alpha \ \cdots \beta \ s_{n-1}(x_0, \dots, x_n) \ \alpha \ x_n.$$

If we set $s_0(x_0, \dots, x_n) = x_0$ and $s_n(x_0, \dots, x_n) = x_n$, then the previous expression can be summarized as:

$$s_i(x_0, \dots, x_n) \ \beta \ s_{i+1}(x_0, \dots, x_n) \text{ for even } 0 \leq i \leq n-2;$$

$$s_i(x_0, \dots, x_n) \ \alpha \ s_{i+1}(x_0, \dots, x_n) \text{ for odd } 1 \leq i \leq n-1.$$

The definition of α and β along with the above relationships force the variety to satisfy:

$$s_i(x_0, x_1, x_1, \dots, x_{n-1}, x_{n-1}) \approx s_{i+1}(x_0, x_1, x_1, \dots, x_{n-1}, x_{n-1}) \text{ for even } 0 \leq i \leq n-2;$$

$$s_i(x_0, x_0, \dots, x_{n-1}, x_{n-1}, x_n) \approx s_{i+1}(x_0, x_0, \dots, x_{n-1}, x_{n-1}, x_n) \text{ for odd } 1 \leq i \leq n-1.$$

If n is odd, the definitions of α and β are slightly different:

$$\alpha = \mathrm{Cg}^{\mathbf{F}}(\{(x_i, x_{i+1}) : 0 \leq i \leq n-1 \text{ even}\}) = \mathrm{Cg}^{\mathbf{F}}(\{(x_0, x_1), \dots, (x_{n-1}, x_n)\});$$

$$\beta = \mathrm{Cg}^{\mathbf{F}}(\{(x_i, x_{i+1}) : 1 \leq i \leq n-2 \text{ odd}\}) = \mathrm{Cg}^{\mathbf{F}}(\{(x_1, x_2), \dots, (x_{n-2}, x_{n-1})\}).$$

Even in this case, $(x_0, x_n) \in \alpha \circ^n \beta$, and hence, by n-permutability, there exist $s_0 = \pi_0, s_1, \dots$ $\dots, s_{n-1}, s_n = \pi_n \in F$, with

$$s_i(x_0, \dots, x_n) \ \beta \ s_{i+1}(x_0, \dots, x_n) \text{ for even } 0 \leq i \leq n-1;$$

$$s_i(x_0, \ldots, x_n) \ \alpha \ s_{i+1}(x_0, \ldots, x_n) \text{ for odd } 1 \leq i \leq n-2.$$

Again, we can deduce that the variety $\mathcal{V}$ satisfies

$$s_i(x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1}, x_n) \approx s_{i+1}(x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1}, x_n) \text{ for even } 0 \leq i \leq n-1;$$

$$s_i(x_0, x_0, \ldots, x_{n-1}, x_{n-1}) \approx s_{i+1}(x_0, x_0, \ldots, x_{n-1}, x_{n-1}) \text{ for odd } 1 \leq i \leq n-2.$$

In either case, $\mathcal{S}_n \leq \mathcal{V}$, as desired. □

A direct consequence of (the proof of) Theorem 3.1.1 is the following.

Corollary 3.1.1. *For $n \geq 2$, a variety $\mathcal{V}$ is congruence n-permutable if and only if the $(n+1)$-generated free algebra in $\mathcal{V}$ is congruence n-permutable.*

If a variety $\mathcal{V}$ is such that $\mathcal{S}_n \leq \mathcal{V}$, for some $n \geq 2$, as in Theorem 3.1.1(2), the terms of $\mathcal{V}$ that interpret $s_0, \ldots, s_n$ are called *Schmidt terms.*

Notice that if a variety $\mathcal{V}$ is such that $\mathcal{S}_2 \leq \mathcal{V}$, with Schmidt terms p_0, p_1, p_2, then p_0 and p_2 are projections and p_1 is a Maltsev term (see Example 2.2.1). Indeed, Schmidt's characterization in Theorem 3.1.1 is the most natural generalization of the one discovered by Maltsev, since it deals with suitable congruences as the definition of n-permutability suggests.

Despite being the most natural, Schmidt's condition is not the most convenient, in the sense that it involves a sequence of $n+1$ $(n+1)$-ary terms. In fact, J. Hagemann and A. Mitschke found out later ([18]) that congruence n-permutability as a Maltsev condition can be described using a sequence of $n+1$ (or rather $n-1$ if we do not consider the projections, as we will not) 3-ary terms, meaning that it is possible to fix the arity of the terms regardless of the value of n. In order to do this, they provided a useful characterization of congruence n-permutable varieties using some nice properties of reflexive compatible relations that such varieties may contain. That characterization is presented in the next theorem.

Theorem 3.1.2 ([18])**.** *Let $n > 1$. For a variety $\mathcal{V}$, the following are equivalent:*

1. *$\mathcal{V}$ is congruence n-permutable;*
2. *For any $\mathbf{A} \in \mathcal{V}$, and every subuniverse $R \leq \mathbf{A} \times \mathbf{A}$,*

$$\textit{if } R \textit{ is reflexive, then } R^{-1} \subseteq R \circ^{n-1} R;$$

3. *$\mathcal{HM}_n \leq \mathcal{V}$, where $\mathcal{HM}_n$ is the variety of type $\langle \overbrace{3, \ldots, 3}^{n-1 \text{ times}} \rangle$ and basic operation symbols $\langle p_1, \ldots, p_{n-1} \rangle$, axiomatized by:*

$$\begin{aligned} x &\approx p_1(x, y, y); \\ p_i(x, x, y) &\approx p_{i+1}(x, y, y) \textit{ for } 1 \leq i \leq n-1; \\ p_{n-1}(x, x, y) &\approx y. \end{aligned}$$

Proof. For the rest of the proof, let $n > 1$ be fixed.

(1) ⇒ (2): If $\mathcal{V}$ is congruence n-permutable, then by Theorem 3.1.1, $\mathcal{V}$ has Schmidt terms, call them $s_0, \ldots, s_n$. Let $\mathbf{A}$ be any algebra in $\mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$ a reflexive subuniverse. For every $a, b \in A$, if $(a, b) \in R$, we want to prove that $(b, a) \in R \circ^{n-1} R$.

For $1 \leq i \leq n-1$, define the following elements of A:

$$c_i = s_i^{\mathbf{A}}(\overbrace{a, \ldots, a}^{i \text{ times}}, b \ldots, b);$$

$$d_i = s_i^{\mathbf{A}}(\overbrace{a, \ldots, a}^{i+1 \text{ times}}, b \ldots, b).$$

The equations that the s_i's satisfy, yield that $c_i = d_{i-1}$ for all $2 \leq i \leq n-1$. Moreover, since R is reflexive, $(a,a) \in R$ and $(b,b) \in R$, and hence the $n-1$ pairs $(d_1, c_1), \dots, (d_{n-1}, c_{n-1}) \in R$. Therefore,

$$b = d_{n-1} \ R \ c_{n-1} = d_{n-2} \ \cdots \ c_2 = d_1 \ R \ c_1 = a.$$

The $n-1$ occurrences of R show that $(b,a) \in R \circ^{n-1} R$.

(2) $\Rightarrow$ (3): Assume $\mathcal{V}$ satisfies (2).

Consider in $\mathcal{V}$, the free algebra generated by $\{x,y\}$, call it $\mathbf{F}$, and define the subuniverse R of $\mathbf{F}^2$ as follows:

$$R = \mathrm{Sg}^{\mathbf{F}^2}(\{(x,x),(x,y),(y,y)\}).$$

Because R contains the pairs (x,x) and (y,y), it is reflexive. Furthermore, since $(y,x) \in R^{-1}$, by assumption $(y,x) \in R \circ^{n-1} R$, i.e. $y =: u_{n-1} \ R \ u_{n-2} \ R \ \cdots \ R \ u_1 \ R \ u_0 := x$, for some $u_1, \dots, u_{n-2} \in F$.

For $1 \leq i \leq n-1$, call $p_i^{\mathbf{F}}$ the 3-ary term operation of $\mathbf{F}$ such that

$$(u_i, u_{i-1}) = p_i^{\mathbf{F}}((x,x),(x,y),(y,y)) = (p_i^{\mathbf{F}}(x,x,y), p_i^{\mathbf{F}}(x,y,y)).$$

By expanding the above equality, we get

$$\begin{aligned} y &= p_{n-1}^{\mathbf{F}}(x,x,y); \\ p_i^{\mathbf{F}}(x,y,y) &= u_i = p_{i-1}^{\mathbf{F}}(x,x,y) \text{ for } 1 < i < n-1; \\ p_1^{\mathbf{F}}(x,y,y) &= x. \end{aligned}$$

These equalities are valid in the free algebra $\mathbf{F}$ and hence they must be satisfied by the whole variety $\mathcal{V}$, via the $\mathcal{V}$-terms $p_1, \dots, p_{n-1}$, proving that $\mathcal{HM}_n \leq \mathcal{V}$.

(3) $\Rightarrow$ (1): Suppose $\mathcal{V}$ has terms satisfying the equations displayed in the statement (3), and suppose without loss of generality that these terms are $p_1, \dots, p_{n-1}$.

Let $\mathbf{A}$ be any algebra of $\mathcal{V}$, and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$.

Let $(a_0, a_n) \in \alpha \circ^n \beta$ and call $a_1, \dots, a_{n-1} \in A$ some elements such that $a_i \ \alpha \ a_{i+1}$, for even i, while $a_i \ \beta \ a_{i+1}$, for odd i. Using these elements, define for $1 \leq i \leq n-1$

$$c_i = p_i^{\mathbf{A}}(a_{i-1}, a_i, a_i);$$

$$c_n = p_n^{\mathbf{A}}(a_{n-1}, a_n, a_n);$$

where p_n is just the third projection (this choice is notationally convenient). Notice that $c_1 = a_0$ and $c_n = a_n$ If i is even, then

$$c_i = p_i^{\mathbf{A}}(a_{i-1}, a_i, a_i) \ \alpha \ p_i^{\mathbf{A}}(a_{i-1}, a_i, a_{i+1}) \ \beta \ p_i^{\mathbf{A}}(a_i, a_i, a_{i+1}) = p_{i+1}^{\mathbf{A}}(a_i, a_{i+1}, a_{i+1}) = c_{i+1}.$$

Similarly, if i is odd, then

$$c_i = p_i^{\mathbf{A}}(a_{i-1}, a_i, a_i) \ \beta \ p_i^{\mathbf{A}}(a_{i-1}, a_i, a_{i+1}) \ \alpha \ p_i^{\mathbf{A}}(a_i, a_i, a_{i+1}) = p_{i+1}^{\mathbf{A}}(a_i, a_{i+1}, a_{i+1}) = c_{i+1}.$$

Therefore,

$$a_0 = c_1 \ \beta \circ \alpha \ c_2 \ \alpha \circ \beta \ \cdots \ c_n = a_n$$

with $n-1$ alternating factors $\beta \circ \alpha$ and $\alpha \circ \beta$. This proves that $(a_0, a_n) \in (\beta \circ \alpha) \circ^{n-1} (\alpha \circ \beta) = \beta \circ^n \alpha$, concluding the proof. $\square$

If a variety $\mathcal{V}$ is such that $\mathcal{HM}_n \leq \mathcal{V}$, for some $n \geq 2$, the terms interpreting $p_1, \dots, p_n$ are called *Hagemann-Mitschke terms.*

Again, notice that for $n = 2$, a term interpreting p_1 is a Maltsev term. Hagemann-Mitschke's condition generalizes the one of Maltsev by increasing the number of terms but not the arity. We remark that Hagemann-Mitschke terms, as observed in the proof of Theorem 3.1.2, are obtained by looking at congruence n-permutability for a variety as a condition on reflexive subuniverses instead of the defining condition on congruences. More specifically, the argument of the proof shows that it is necessary and sufficient to analyze the 2-generated free algebra in order to verify whether a variety is congruence n-permutable, as formally stated below

Corollary 3.1.2. *For $n \geq 2$, a variety $\mathcal{V}$ is congruence n-permutable if and only if $R^{-1} \subseteq R \circ^{n-1} R$, where R is the subuniverse of $\mathbf{F}_{\mathcal{V}}(\{x,y\}) \times \mathbf{F}_{\mathcal{V}}(\{x,y\})$ generated by $\{(x,x),(x,y),(y,y)\}$.*

Nevertheless, in a variety $\mathcal{V}$, the relation R mentioned in Corollary 3.1.2 is such that if $\mathcal{V}$ is, for example, not n-permutable for any $n > 1$, then $(y,x) \notin R \circ^k R$, for all $k \geq 1$. This is to say R is not specifically capturing the failure of congruence k-permutability, for a fixed value of $k > 1$. In order to have this peculiar requirement met, we need to consider a family of relations, that we are going to present soon, which in fact will allow us to find another Maltsev condition for congruence n-permutability.

Theorem 3.1.3. *Let $n > 1$ and $\mathcal{V}$ a variety. The following statements are equivalent:*

1. *$\mathcal{V}$ is congruence n-permutable;*
2. *$\mathcal{X}_n \leq \mathcal{V}$, where $\mathcal{X}_n$ is the variety of type $\langle\overbrace{k_n, \ldots, k_n}^{n-1 \text{ times}}\rangle$, with $k_n = \frac{n^2+5n+2}{2}$, and basic operation symbols $\langle t_1, \ldots, t_{n-1}\rangle$, axiomatized by:*

$$\begin{aligned} x_0 &\approx t_1(\vec{v}); \\ t_i(\vec{w}) &\approx t_{i+1}(\vec{v}) \text{ for } 1 \leq i < n-1; \\ t_{n-1}(\vec{w}) &\approx x_n; \end{aligned}$$

where $\vec{v}, \vec{w} \in \{x_0, \ldots, x_n\}^{k_n}$ are defined as

$$\vec{v} = (\underbrace{x_0, \ldots, x_n}_{n+1 \text{ times}}, \underbrace{x_0, \ldots, x_n}_{n+1 \text{ times}}, \underbrace{x_1, \ldots, x_n}_{n \text{ times}}, \underbrace{x_2, \ldots, x_n}_{n-1 \text{ times}}, \ldots\ldots, \underbrace{x_{n-1}, x_n}_{2 \text{ times}});$$

$$\vec{w} = (\underbrace{x_0, \ldots, x_0}_{n+1 \text{ times}}, \underbrace{x_1, \ldots, x_1}_{n+1 \text{ times}}, \underbrace{x_2, \ldots, x_2}_{n \text{ times}}, \underbrace{x_3, \ldots, x_3}_{n-1 \text{ times}}, \ldots\ldots, \underbrace{x_n, x_n}_{2 \text{ times}}).$$

Proof. Fix any integer $n > 1$.

(1) $\Rightarrow$ (2): Let $\mathcal{V}$ be a congruence n-permutable variety and call $\mathbf{F}$ the free algebra in $\mathcal{V}$ generated by $X := \{x_0, \ldots, x_n\}$. Define the following set of pairs of variables:

$$V = (\{x_0, x_1\} \times X) \cup \bigcup_{i=2}^{n} (\{x_i\} \times \{x_{i-1}, \ldots, x_n\}).$$

V is a binary relation on the set X, where $x_0/V = x_1/V = X$ and for $2 \leq i \leq n$, $x_i/V = X - \{x_0, \ldots, x_{i-2}\}$, therefore

$$|V| = 2(n+1) + \sum_{i=2}^{n}(n-i+2) = 2(n+1) + \sum_{i=2}^{n} i = 2(n+1) + \frac{n(n+1)}{2} - 1 = \frac{n^2+5n+2}{2}.$$

Call $k_n := |V|$. Consider now the subuniverse R of $\mathbf{F}^2$ defined by

$$R = \mathrm{Sg}^{\mathbf{F}^2}(V).$$

Since V contains (x_i, x_i) for all $0 \leq i \leq n$, R is reflexive. Hence, by Theorem 3.1.2(2), we have that $R^{-1} \subseteq R \circ^{n-1} R$, and because $(x_0, x_n) \in R$ by definition of V, we deduce that there must exist $u_1, \ldots, u_{n-2} \in F$ such that

$$x_n =: u_{n-1} \; R \; u_{n-2} \; R \; \cdots \; R \; u_1 \; R \; u_0 := x_0.$$

Let $t_i^{\mathbf{F}}$ ($1 \leq i \leq n-1$) be the k_n-ary term operation of $\mathbf{F}$ such that $(u_i, u_{i-1}) \in t_i^{\mathbf{F}}(V)$. Also, if $\vec{v}$ and $\vec{w}$ are defined as in the statement (2), then we have that

$$V = \bigcup_{i=1}^{k_n} \{(\pi_i(\vec{w}), \pi_i(\vec{v}))\}.$$

Therefore, if we order the pairs in V according to $\vec{v}$ and $\vec{w}$, we have that

$$\begin{aligned} x_n &= t^{\mathbf{F}}_{n-1}(\vec{w}); \\ t^{\mathbf{F}}_i(\vec{v}) &= u_i = t^{\mathbf{F}}_{i-1}(\vec{w}) \text{ for } 2 \leq i \leq n-1; \\ t^{\mathbf{F}}_1(\vec{v}) &= x_0. \end{aligned}$$

These identities being satisfied in $\mathbf{F}$ yields that $\mathcal{V}$ satisfies them, proving that $\mathcal{X}_n \leq \mathcal{V}$.

(2) $\Rightarrow$ (1): Assume $\mathcal{V}$ has terms $t_1, \ldots, t_{n-1}$ satisfying (2).

Let $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$ be a reflexive subuniverse. For any $(a,b) \in R$, we are going to define some elements of A using the terms $t_1, \ldots, t_{n-1}$ in the following way.

For $1 \leq i \leq n-1$, let

$$d_i = t^{\mathbf{A}}_i(\overbrace{a,\ldots,a}^{n+1 \text{ times}}, \overbrace{a,\ldots,a}^{n+1 \text{ times}}, \overbrace{a,\ldots,a}^{n \text{ times}}, \ldots, \overbrace{a,\ldots,a}^{n-i+2 \text{ times}}, \overbrace{b,\ldots,b}^{n-i+1 \text{ times}}, \ldots, \overbrace{b,b}^{2 \text{ times}});$$

$$\begin{aligned} e_i = t^{\mathbf{A}}_i(&\overbrace{\underbrace{a,\ldots,a}_{i \text{ times}}, b,\ldots,b}^{n+1 \text{ times}}, \overbrace{\underbrace{a,\ldots,a}_{i \text{ times}}, b,\ldots,b}^{n+1 \text{ times}}, \overbrace{\underbrace{a,\ldots,a}_{i-1 \text{ times}}, b,\ldots,b}^{n \text{ times}}, \ldots \\ &\ldots, \overbrace{\underbrace{a}_{1 \text{ time}}, b,\ldots,b}^{n-i+2 \text{ times}}, \overbrace{b,\ldots,b}^{n-i+1 \text{ times}}, \ldots, \overbrace{b,b}^{2 \text{ times}}). \end{aligned}$$

Since $(a,a),(b,b) \in R$ (R is reflexive) and $(a,b) \in R$, we observe that $(d_i, e_i) \in R$, for $1 \leq i \leq n-1$. Moreover, notice that the equations satisfied by the t_i's yield

$$d_{n-1} = t^{\mathbf{A}}_{n-1}(\underbrace{\overbrace{a,\ldots,a}^{n+1 \text{ times}}}_{x_0,\ldots,x_0}, \underbrace{\overbrace{a,\ldots,a}^{n+1 \text{ times}}}_{x_1,\ldots,x_1}, \underbrace{\overbrace{a,\ldots,a}^{n \text{ times}}}_{x_2,\ldots,x_2}, \ldots, \underbrace{\overbrace{a,a,a}^{3 \text{ times}}}_{x_{n-1},x_{n-1},x_{n-1}}, \underbrace{\overbrace{b,b}^{2 \text{ times}}}_{x_n,x_n}) = b;$$

$$\text{for } 2 \leq i \leq n-1,$$

$$\begin{aligned} e_i = t^{\mathbf{A}}_i(&\underbrace{\overbrace{a,\ldots,a,b,\ldots,b}^{n+1 \text{ times}}}_{x_0,\ldots,x_{i-1},x_i,\ldots,x_n}, \underbrace{\overbrace{a,\ldots,a,b,\ldots,b}^{n+1 \text{ times}}}_{x_0,\ldots,x_{i-1},x_i,\ldots,x_n}, \underbrace{\overbrace{a,\ldots,a,b,\ldots,b}^{n \text{ times}}}_{x_1,\ldots,x_{i-1},x_i,\ldots,x_n}, \ldots \\ &\ldots, \underbrace{\overbrace{a,b,\ldots,b}^{n-i+2 \text{ times}}}_{x_{i-1},x_i,\ldots,x_n}, \underbrace{\overbrace{b,\ldots,b}^{n-i+1 \text{ times}}}_{x_i,\ldots,x_n}, \ldots, \underbrace{\overbrace{b,b}^{2 \text{ times}}}_{x_{n-1},x_n}) = \end{aligned}$$

$$= t^{\mathbf{A}}_{i-1}(\underbrace{\overbrace{a,\ldots,a}^{n+1 \text{ times}}}_{x_0,\ldots,x_0}, \underbrace{\overbrace{a,\ldots,a}^{n+1 \text{ times}}}_{x_1,\ldots,x_1}, \underbrace{\overbrace{a,\ldots,a}^{n \text{ times}}}_{x_2,\ldots,x_2}, \ldots, \underbrace{\overbrace{a,\ldots,a}^{n-i+3 \text{ times}}}_{x_{i-1},\ldots,x_{i-1}}, \underbrace{\overbrace{b,\ldots,b}^{n-i+2 \text{ times}}}_{x_i,\ldots,x_i}, \ldots, \underbrace{\overbrace{b,b}^{2 \text{ times}}}_{x_n,x_n}) = d_{i-1};$$

$$e_1 = t^{\mathbf{A}}_1(\underbrace{\overbrace{a,b,\ldots,b}^{n+1 \text{ times}}}_{x_0,\ldots,x_n}, \underbrace{\overbrace{a,b,\ldots,b}^{n+1 \text{ times}}}_{x_0,\ldots,x_n}, \underbrace{\overbrace{b,\ldots,b}^{n \text{ times}}}_{x_1,\ldots,x_n}, \ldots, \underbrace{\overbrace{b,b}^{2 \text{ times}}}_{x_{n-1},x_n}) = a.$$

Therefore,

$$b = d_{n-1} \ R \ e_{n-1} = d_{n-2} \ R \ \cdots \ R \ e_2 = d_1 \ R \ e_1 = a,$$

with $n-1$ occurrences of R, proving that $(b,a) \in R \circ^{n-1} R$.

By the arbitrariness of R and $(a,b) \in R$, Theorem 3.1.2(2) guarantees that $\mathcal{V}$ is congruence n-permutable, completing the proof. $\square$

The terms of a variety $\mathcal{V}$ witnessing that $\mathcal{X}_n \leq \mathcal{V}$, for some $n > 1$, will be called *$\mathcal{X}_n$-terms*.

This new Maltsev condition is not particularly convenient as far as the terms are concerned, because the arity of the $n-1$ $\mathcal{X}_n$-terms grows polynomially in n, according to the formula $k_n = \frac{n^2+5n+2}{2}$, as shown in Theorem 3.1.3. Nevertheless, we have presented it also because we want to emphasize the fact that the relation R used in the proof of Theorem 3.1.3 to find the $\mathcal{X}_n$-terms, can characterize congruence n-permutability. Indeed,

Corollary 3.1.3. *For $n > 1$ and $\mathcal{V}$ a variety, $\mathcal{V}$ is congruence n-permutable if and only if $R_n^{-1} \subseteq R_n \circ^{n-1} R_n$, where R_n is the subuniverse of $\mathbf{F}^2_{\mathcal{V}}(\{x_0, \ldots, x_n\})$ generated by*

$$V_n = (\{x_0, x_1\} \times \{x_0, \ldots, x_n\}) \cup \bigcup_{i=2}^{n} (\{x_i\} \times \{x_{i-1}, \ldots, x_n\}).$$

Moreover, unlike the relation R of Corollary 3.1.2, for any $n > 1$ the relation R_n of Corollary 3.1.3 satisfies $(x_n, x_0) \in R_n \circ^n R_n$, independently of $\mathcal{V}$ being or not congruence $(n+1)$-permutable.

So far, we have shown that congruence n-permutability for a variety, having fixed $n > 1$, is equivalent to any one of the Maltsev conditions presented in Theorems 3.1.1, 3.1.2, 3.1.3: this means that the varieties $\mathcal{S}_n$, $\mathcal{HM}_n$ and $\mathcal{X}_n$ are equi-interpretable. The next theorem is only a recapitulation of the three theorems mentioned above, in whose proof we will show how the mutual interpretations between the varieties $\mathcal{S}_n$, $\mathcal{HM}_n$, $\mathcal{X}_n$ are built. The interpretations witnessing $\mathcal{S}_n \equiv \mathcal{HM}_n$ are well known and will be presented for completeness, whereas the ones for $\mathcal{HM}_n \equiv \mathcal{X}_n$ will be provided for the first time.

Theorem 3.1.4 (synthesis of Theorems 3.1.1, 3.1.2, 3.1.3)**.** *For any $n > 1$,*

$$\mathcal{S}_n \equiv \mathcal{HM}_n \equiv \mathcal{X}_n.$$

Proof. In this proof, we will just define the interpretations and omit the verifications of the corresponding identities, since they can be carefully deduced by looking at the proofs of Theorems 3.1.1, 3.1.2 and 3.1.3 (let k_n be defined as in the statement of Theorem 3.1.3).

- $\mathcal{HM}_n \leq \mathcal{S}_n$: define the terms $p_1, \ldots, p_{n-1}$ by using $s_0, \ldots, s_n$, for $1 \leq i \leq n-1$ as follows

$$p_i(x, y, z) := s_i(\overbrace{x, \ldots, x}^{i \text{ times}}, y, \underbrace{z, \ldots, z}_{n-i \text{ times}}).$$

- $\mathcal{S}_n \leq \mathcal{HM}_n$: given $p_1, \ldots, p_{n-1}$, define $s_0, \ldots, s_n$ as

$$\begin{aligned} s_0(x_0, \ldots, x_n) &:= x_0; \\ s_i(x_0, \ldots, x_n) &:= p_i(x_{i-1}, x_i, x_{i+1}) \text{ for } 1 \leq i \leq n-1; \\ s_n(x_0, \ldots, x_n) &:= x_n. \end{aligned}$$

- $\mathcal{HM}_n \leq \mathcal{X}_n$: given $t_1, \ldots, t_{n-1}$, define the terms $p_1, \ldots, p_{n-1}$, as shown below

$$p_i(x, y, z) :=$$

$$= t_i(\overbrace{\underbrace{x, \ldots, x}_{i \text{ times}}, y, \ldots, y}^{n+1}, \overbrace{\underbrace{x, \ldots, x}_{i \text{ times}}, y, \ldots, y}^{n+1}, \overbrace{\underbrace{x, \ldots, x}_{i-1 \text{ times}}, y, \ldots, y}^{n}, \ldots, \overbrace{x, y, \ldots, y}^{n-i+2}, \overbrace{z, \ldots, z}^{\frac{n^2-(2i-3)n+i^2-3i}{2}}),$$

for $1 \leq i \leq n-1$ [notice $\frac{n^2-(2i-3)n+i^2-3i}{2} = (n-i+1) + (n-i) + \ldots + 2$].

- $\mathcal{X}_n \leq \mathcal{HM}_n$: define $t_1, \ldots, t_{n-1}$ by using $p_1, \ldots, p_{n-1}$, for $1 \leq i \leq n-1$ as follows

$$t_i(x_0, \ldots, x_{k_n-1}) := p_i(x_0, x_n, x_{k_n-1}).$$

□

It is also worth mentioning that there exists a Maltsev condition characterizing congruence n-permutable varieties which makes use of a single term of some suitable arity depending on n, instead of a sequence of terms of fixed arity, both functions of n, as it is the case for the three previously presented conditions. This result is due to M. Kozik, A. Krokhin, M. Valeriote and R. Willard, and can be consulted in [21]. We rephrase their result as follows

Theorem 3.1.5 ([21])**.** *Let $n \geq 1$ and $\mathcal{T}_n$ be the variety of type $\langle 3^n \rangle$ and with basic operation symbol t, axiomatized by the following 3^n equations:*

$$t(\underbrace{x,\ldots,x}_{i\ times}, x^{(i)}_{i+1},\ldots,x^{(i)}_{3^n}) \approx t(y^{(i)}_1,\ldots,y^{(i)}_{i-1}, \underbrace{y,\ldots,y}_{3^n-i+1\ times}),\ 1 \leq i \leq 3^n,$$

where $x^{(i)}_{i+1},\ldots,x^{(i)}_{3^n}, y^{(i)}_1,\ldots,y^{(i)}_{i-1} \in \{x,y\}$, for all $1 \leq i \leq 3^n$.

Let further $\mathcal{V}$ be any variety. Thus, if $\mathcal{V}$ is congruence $(n+1)$-permutable, then $\mathcal{T}_n \leq \mathcal{V}$; conversely, if $\mathcal{T}_n \leq \mathcal{V}$, then $\mathcal{V}$ is congruence $(2n-1)$-permutable.

We omit the proof of this theorem but we remark that, unlike the conditions $\mathcal{S}_n$, $\mathcal{HM}_n$ and $\mathcal{X}_n$, in the axiomatization of $\mathcal{T}_n$ the only parameter which depends on n is the arity of the term t, which, by itself, is able to capture congruence m-permutability.

Since the strong Maltsev classes induced by the varieties $\mathcal{S}_n$, $\mathcal{HM}_n$ and $\mathcal{X}_n$ coincide with the class of congruence n-permutable varieties, we can use a uniform name to denote these objects. Therefore, from now on, when dealing with congruence n-permutable varieties we will use Hagemann-Mitschke terms for convenience, and hence we will rename, for $n > 1$,

$$\mathcal{CP}_n := \mathcal{HM}_n.$$

Moreover, for fixed $n > 1$, the strong Maltsev class of congruence n-permutable varieties will be denoted by CP_n, i.e.

$$CP_n = \{\mathcal{V} : \mathcal{CP}_n \leq \mathcal{V}\},$$

whereas the Maltsev class of congruence n-permutable varieties, for some $n > 1$, is defined by

$$CP_\omega = \bigcup_{1<n<\omega} CP_n.$$

Throughout this thesis, we will also be dealing with the classes of idempotent varieties in CP_n or CP_ω, which we will denote by

$$CP^{id}_n := CP_n \cap L^{id},$$
$$CP^{id}_\omega := CP_\omega \cap L^{id}.$$

3.2 Failure algebras and Hagemann relations

In the previous section we mainly focused on three characterizations of congruence n-permutability for a variety $\mathcal{V}$, and we remarked several times that they come from two different points of view: Schmidt's Theorem 3.1.1 produces a Maltsev condition by considering one algebra (in fact the free algebra) in the variety $\mathcal{V}$, along with two n-permuting congruences of it; whereas Hagemann and Mitschke's Theorem 3.1.2 and Theorem 3.1.3 produce Maltsev conditions by considering some suitable reflexive relations satisfying $R^{-1} \subseteq R \circ^{n-1} R$. We will show in this section that there is a local connection between these objects, in a sense that we will explain soon.

Theorem 3.1.2 also implies that a variety $\mathcal{V}$ is not congruence n-permutable (fixed $n > 1$), if and only if $\mathcal{V}$ contains an algebra $\mathbf{A}$ carrying a reflexive $R \leq \mathbf{A} \times \mathbf{A}$ failing $R^{-1} \subseteq R \circ^{n-1} R$. This latter object is deservedly given a name

Definition 3.2.1. Let $n \geq 1$, $\mathbf{A}$ be an algebra and $R \leq \mathbf{A} \times \mathbf{A}$ a reflexive relation on A. We say that R is an *n-dimensional Hagemann relation*, or *Hagemann relation of dimension n*, if

$$R^{-1} \nsubseteq R \circ^n R.$$

Referring to the notation of Corollaries 3.1.2 and 3.1.3, R and R_n are $(n-1)$-dimensional Hagemann relations whenever the variety $\mathcal{V}$ is not congruence n-permutable.

Suppose now that $\mathbf{A}$ is an algebra failing to be congruence n-permutable, for $n > 1$. If we consider $\mathcal{V} = \mathbf{HSP}(\mathbf{A})$, which *a fortiori* is not congruence n-permutable, then Corollary 3.1.2 and Corollary 3.1.3 ensure that some Hagemann relations of dimension $n-1$ can be found as subuniverses of $\mathbf{F}_{\mathcal{V}}(\{x,y\}) \times \mathbf{F}_{\mathcal{V}}(\{x,y\})$, which in turn is a subuniverse of $\mathbf{A}^{A^2} \times \mathbf{A}^{A^2}$, and as subuniverses of $\mathbf{F}_{\mathcal{V}}(\{x_0,\dots,x_n\}) \times \mathbf{F}_{\mathcal{V}}(\{x_0,\dots,x_n\}) \leq \mathbf{A}^{A^{n+1}} \times \mathbf{A}^{A^{n+1}}$.

In other words, if $\mathbf{A}$ is a non-congruence n-permutable algebra, then an $(n-1)$-dimensional Hagemann relation can always be found as a subuniverse of $\mathbf{A}^k \times \mathbf{A}^k$, for some $k \geq |A|^2$. In fact, we are able to show that the value of k can be lowered to $k = n-1$. This is what we mean when claiming that the construction of an $(n-1)$-dimensional Hagemann relation can be localized to a failure of n-permutability. Indeed, there is no need to build any free algebra since the relation is definable via primitive positive formulas.

Before presenting the result, we need to define some tools that generalize the ones of Definition 1.1.5 and that will ease the notation and comprehension.

Definition 3.2.2. Let $n \geq 1$, A be any set and $R, S \subseteq A \times A$ binary relations on A. Define inductively for $n \geq 1$

$$R \otimes^1 S = R;$$
$$R \otimes^{n+1} S = R \otimes (S \otimes^n R);$$

and

$$R \star^0 S = A;$$
$$R \star^1 S = R;$$
$$R \star^{n+1} S = R \star (S \star^n R).$$

Notice that $R \otimes^2 S = R \otimes S$ and $R \star^2 S = R \star S$. Moreover, if $\mathbf{A}$ is an algebra and $R, S \leq \mathbf{A} \times \mathbf{A}$, then for all $n \geq 1$, $R \otimes^n S \leq \mathbf{A}^n \times \mathbf{A}^n$ and for all $n \geq 0$, $R \star^n S \leq \mathbf{A}^{n+1}$.

Theorem 3.2.1. *Let $n \geq 1$, $\mathbf{A}$ be an algebra and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.*

If α and β do not $(n+1)$-permute, with $\beta \circ^{n+1} \alpha \not\subseteq \alpha \circ^{n+1} \beta$, then the subalgebra $\mathbf{B}$ of $\mathbf{A}^n$, whose universe is

$$B = \beta \star^{n-1} \alpha$$

carries the relation $R \leq \mathbf{B} \times \mathbf{B}$, defined by

$$R = B^2 \cap ((\beta \circ \alpha) \otimes^n (\alpha \circ \beta)),$$

which is a Hagemann relation of dimension n.

Proof. Let $n \geq 1$, $\mathbf{A}$ an algebra with $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, $B \leq \mathbf{A}^n$ and $R \leq \mathbf{B} \times \mathbf{B}$ as defined in the statement. Let us first prove that R is reflexive.

For every $\vec{b} = (a_1, \dots, a_n) \in B$, we have that $(a_i, a_i) \in \beta \circ \alpha$ and $(a_i, a_i) \in \alpha \circ \beta$, for all $1 \leq i \leq n$, proving that $(\vec{b}, \vec{b}) \in R$.

Henceforth, we need to prove that $R^{-1} \not\subseteq R \circ^n R$. Since $\beta \circ^{n+1} \alpha \not\subseteq \alpha \circ^{n+1} \beta$, there exists a pair $(a_0, a_{n+1}) \in \beta \circ^{n+1} \alpha$, with $(a_0, a_{n+1}) \notin \alpha \circ^{n+1} \beta$. Call $a_1, \dots, a_n \in A$ the elements such that

$$(a_i, a_{i+1}) \in \beta \text{ for even } 0 \leq i \leq n;$$
$$(a_i, a_{i+1}) \in \alpha \text{ for odd } 0 \leq i \leq n.$$

By definition of B, we have that the n-tuples $\vec{b}_0 := (a_0, a_1, \dots, a_{n-1})$, $\vec{b}_n := (a_2, a_3, \dots, a_{n+1})$ are elements of B. Moreover,

$$\vec{b}_0 = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix} \begin{matrix} \beta \circ \alpha \\ \alpha \circ \beta \\ \vdots \\ \cdots \\ \cdots \end{matrix} \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \\ a_{n+1} \end{bmatrix} = \vec{b}_n,$$

which actually means $(\vec{b}_0, \vec{b}_n) \in R$.

Nevertheless, $(\vec{b}_n, \vec{b}_0) \notin R \circ^n R$. As a matter of fact, if we assume otherwise, there exist $\vec{b}_1, \ldots, \vec{b}_{n-1} \in B$, where $\vec{b}_i := (b_{i,1}, \ldots, b_{i,n})$, such that, for $1 \leq i \leq n$

$$(\vec{b}_i, \vec{b}_{i-1}) \in R.$$

Expanding this latter relationship, we get

$$\vec{b}_n = \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \\ a_{n+1} \end{bmatrix} R \begin{bmatrix} b_{n-1,1} \\ b_{n-1,2} \\ \vdots \\ b_{n-1,n-1} \\ b_{n-1,n} \end{bmatrix} R \cdots R \begin{bmatrix} b_{1,1} \\ b_{1,2} \\ \vdots \\ b_{1,n-1} \\ b_{1,n} \end{bmatrix} R \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix} = \vec{b}_0$$

By looking at the components of the n-tuples $\vec{b}_i$'s and how R is defined, we have that,

$$\begin{aligned}
&a_0 \ \alpha \circ \beta \ b_{1,1}; \\
&b_{i,i} \ \beta \ b_{i,i+1} \ \beta \circ \alpha \ b_{i+1,i+1} \text{ for odd } 1 \leq i \leq n-2; \\
&b_{i,i} \ \alpha \ b_{i,i+1} \ \alpha \circ \beta \ b_{i+1,i+1} \text{ for even } 1 \leq i \leq n-2; \\
&b_{n-1,n-1} \ \beta \ b_{n-1,n} \ \beta \circ \alpha \ a_{n+1} \text{ if } n \text{ is even;} \\
&b_{n-1,n-1} \ \alpha \ b_{n-1,n} \ \alpha \circ \beta \ a_{n+1} \text{ if } n \text{ is odd.}
\end{aligned}$$

Therefore, if n is odd, the previous list yields

$$a_0 \ \alpha \circ \beta \ b_{1,1} \beta \circ \alpha \ \cdots \ b_{n-1,n-1} \alpha \circ \beta \ a_{n+1};$$

while for n even,

$$a_0 \ \alpha \circ \beta \ b_{1,1} \beta \circ \alpha \ \cdots \ b_{n-1,n-1} \beta \circ \alpha \ a_{n+1}.$$

In either case, $(a_0, a_n) \in (\alpha \circ \beta) \circ^n (\beta \circ \alpha) = \alpha \circ^{n+1} \beta$, contradicting the initial assumption. $\square$

A direct corollary of this theorem is

Corollary 3.2.1. *If* $\mathbf{A}$ *is a non-congruence* $(n+1)$*-permutable algebra, then an* n*-dimensional Hagemann relation can be found on an algebra in* $\mathbf{SP}_{fin}(\mathbf{A})$*, precisely in* $\mathbf{S}(\mathbf{A}^n)$*.*

In order to clarify the construction of Theorem 3.2.1, we provide an example which deals with the case of congruence 5-permutability.

Example 3.2.1 ($n = 4$)**.** Suppose $\mathbf{A}$ is an algebra which fails congruence 5-permutability, and let $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ witness such a failure. Suppose $a_0 \ \beta \ a_1 \ \alpha \ a_2 \ \beta \ a_3 \ \alpha \ a_4 \ \beta \ a_5$, but $(a_0, a_5) \notin \alpha \circ^5 \beta$. Using the same notation as in Theorem 3.2.1, let $B = \beta \star \alpha \star \beta \leq \mathbf{A}^4$ and $R = B^2 \cap ((\beta \circ \alpha) \otimes^4 (\alpha \circ \beta))$. Moreover, let $\vec{b}_0 = (a_0, a_1, a_2, a_3)$ and $\vec{b}_4 = (a_2, a_3, a_4, a_5)$. Therefore, it is impossible that the following occurs

$$\vec{b}_4 = \begin{bmatrix} a_2 \\ \beta \\ a_3 \\ \alpha \\ a_4 \\ \beta \\ \underline{a_5} \end{bmatrix} \begin{matrix} \beta \circ \alpha \\ \\ \alpha \circ \beta \\ \\ \beta \circ \alpha \\ \\ \underline{\alpha \circ \beta} \end{matrix} \begin{bmatrix} b_{3,1} \\ \beta \\ b_{3,2} \\ \alpha \\ \underline{b_{3,3}} \\ \underline{\beta} \\ \underline{b_{3,4}} \end{bmatrix} \begin{matrix} \beta \circ \alpha \\ \\ \alpha \circ \beta \\ \\ \underline{\beta \circ \alpha} \\ \\ \alpha \circ \beta \end{matrix} \begin{bmatrix} b_{2,1} \\ \beta \\ \underline{b_{2,2}} \\ \underline{\alpha} \\ \underline{b_{2,3}} \\ \beta \\ b_{2,4} \end{bmatrix} \begin{matrix} \beta \circ \alpha \\ \\ \underline{\alpha \circ \beta} \\ \\ \beta \circ \alpha \\ \\ \alpha \circ \beta \end{matrix} \begin{bmatrix} \underline{b_{1,1}} \\ \underline{\beta} \\ \underline{b_{1,2}} \\ \alpha \\ b_{1,3} \\ \beta \\ b_{1,4} \end{bmatrix} \begin{matrix} \underline{\beta \circ \alpha} \\ \\ \alpha \circ \beta \\ \\ \beta \circ \alpha \\ \\ \alpha \circ \beta \end{matrix} \begin{bmatrix} \underline{a_0} \\ \beta \\ a_1 \\ \alpha \\ a_2 \\ \beta \\ a_3 \end{bmatrix} = \vec{b}_0$$

because otherwise we would get

$$a_0 \ \alpha \circ \beta \ b_{1,1} \ \beta \circ \alpha \ b_{2,2} \ \alpha \circ \beta \ b_{3,3} \ \beta \circ \alpha \ a_5,$$

contradicting the assumption of non-5-permutability of α and β.

On the other hand, suppose $\mathbf{A}$ is an algebra carrying an n-dimensional Hagemann relation $R \leq \mathbf{A} \times \mathbf{A}$ ($n \geq 1$): Theorem 3.1.2 guarantees that $\mathcal{V} = \mathbf{HSP}(\mathbf{A})$ is not congruence $(n+1)$-permutable, which in turn means $\mathcal{V}$ must contain an algebra having two congruences that do not $(n+1)$-permute. In general, the algebra $\mathbf{A}$ itself need not be a failure of congruence $(n+1)$-permutability, and a counterexample is trivially the following.

Example 3.2.2. Consider the variety of sets, $\mathcal{S}ets$, and let $\mathbf{2}$ be the (only, up to isomorphism) algebra with two elements, i.e. $\mathbf{2} = \langle \{0,1\}, \varnothing \rangle$. $\mathcal{S}ets$ is congruence n-permutable for no $n \geq 2$ (if it had been for some n, the whole lattice of interpretability types would have been the strong Maltsev class CP_n). It is obvious that $\mathbf{F}_{\mathcal{S}ets}(2) = \mathbf{2}$, hence by Corollary 3.1.2, $R = \{(0,0),(0,1),(1,1)\} \leq \mathbf{2} \times \mathbf{2}$ is a Hagemann relation of dimension $n-1$, for every $n \geq 2$. Nonetheless, $\mathbf{2}$ is not a failure of congruence n-permutability, for any $n \geq 2$; indeed it is congruence 2-permutable (every 2-element algebra is).

In the proof of Theorem 3.1.2, an explicit construction of a failure of congruence n-permutability is not provided given an $(n-1)$-dimensional Hagemann relation. We will show that, even for this case, it is possible to exhibit a "local" procedure which does not require building any free algebra, but that just depends on the Hagemann relation and the parameter n.

Theorem 3.2.2. *Let $n \geq 1$, $\mathbf{A}$ be an algebra and $R \leq \mathbf{A} \times \mathbf{A}$ be a Hagemann relation of dimension n. Let $\mathbf{S}$ be the subalgebra of $\mathbf{A}^{n+1}$ whose universe is*

$$S = R \star^n R.$$

If $E, O \subseteq n+1$ denote, respectively, the sets of even and odd numbers in $n+1$, and θ_E, θ_O are defined as

$$\theta_E := \bigwedge_{i \in E} \ker \pi_{i|S};$$

$$\theta_O := \bigwedge_{i \in O} \ker \pi_{i|S};$$

then $\mathbf{S}$ is a failure of congruence $(n+1)$-permutability, where $\theta_E \circ^{n+1} \theta_O \neq \theta_O \circ^{n+1} \theta_E$.

Proof. Let $n \geq 1$, $\mathbf{A}$, $R \leq \mathbf{A} \times \mathbf{A}$, $S \leq \mathbf{A}^{n+1}$, $\theta_E, \theta_O \in \operatorname{Con} \mathbf{S}$ as in the statement. Because of the exponent $n+1$ that appears, in this setting we are going to use the notation $(a_0, \dots, a_n)$ to denote an element of $\mathbf{A}^{n+1}$.

Since R is an n-dimensional Hagemann relation, there exist $a_0, a_n \in A$, with $(a_0, a_n) \in R$ but $(a_n, a_0) \notin R \circ^n R$. Moreover, we remark that $(a_0, a_0), (a_n, a_n) \in R$, by reflexivity.

Define the $(n+1)$-tuple $s_i \in A^{n+1}$, for $0 \leq i \leq n+1$:

$$\vec{s}_i = (\overbrace{a_0, \dots, a_0}^{n+1-i \text{ times}}, \underbrace{a_n, \dots, a_n}_{i \text{ times}}).$$

It is straightforward to notice that $\vec{s}_0, \dots, \vec{s}_{n+1}$ are elements of $R \star^n R = S$.

Furthermore, for every $0 \leq i \leq n$,

$$\text{if } n \text{ is odd, then } \begin{cases} \vec{s}_i \ \theta_E \ \vec{s}_{i+1} & \text{if } i \text{ is even;} \\ \vec{s}_i \ \theta_O \ \vec{s}_{i+1} & \text{if } i \text{ is odd;} \end{cases}$$

$$\text{if } n \text{ is even, then } \begin{cases} \vec{s}_i \ \theta_O \ \vec{s}_{i+1} & \text{if } i \text{ is even;} \\ \vec{s}_i \ \theta_E \ \vec{s}_{i+1} & \text{if } i \text{ is odd.} \end{cases}$$

Expanding these relationships, we get that, for n odd,

$$\vec{s}_0 \ \theta_E \ \vec{s}_1 \ \theta_O \ \cdots \ \theta_E \ \vec{s}_n \ \theta_O \ \vec{s}_{n+1},$$

with $n+1$ alternating occurrences of θ_O and θ_E, yielding $(\vec{s}_0, \vec{s}_{n+1}) \in \theta_E \circ^{n+1} \theta_O$.

On the other hand, for n even,

$$\vec{s}_0 \ \theta_O \ \vec{s}_1 \ \theta_E \ \cdots \ \theta_E \ \vec{s}_n \ \theta_O \ \vec{s}_{n+1},$$

again with $n+1$ alternating occurrences of θ_O and θ_E, giving as a result $(\vec{s}_0, \vec{s}_{n+1}) \in \theta_O \circ^{n+1} \theta_E$.

The next step is to show that θ_O and θ_E do not $(n+1)$-permute. Assume first n is odd. We have just proven $(\vec{s}_0, \vec{s}_{n+1}) \in \theta_E \circ^{n+1} \theta_O$, and hence it suffices to prove that $(\vec{s}_0, \vec{s}_{n+1}) \notin \theta_O \circ^{n+1} \theta_E$. Suppose otherwise, namely there exist $\vec{r}_1, \ldots, \vec{r}_n \in S$, such that

$$\begin{aligned}
&\vec{s}_0 \ \theta_O \ \vec{r}_1;\\
&\vec{r}_i \ \theta_E \ \vec{r}_{i+1} \text{ for odd } 1 \leq i \leq n-1;\\
&\vec{r}_i \ \theta_O \ \vec{r}_{i+1} \text{ for even } 1 \leq i \leq n-1;\\
&\vec{r}_n \ \theta_E \ \vec{s}_{n+1}.
\end{aligned}$$

For $1 \leq i \leq n$, say $\vec{r}_i = (r_{i,0}, \ldots, r_{i,n})$. The first of the above relationships yields that $r_{1,i} = a_0$, for $i \in O$, while from the fourth we deduce $r_{n,i} = a_n$, for $i \in E$. The other two, instead, mean

$$r_{i,j} = r_{i+1,j} \text{ for odd } 1 \leq i \leq n-2, \ j \in E;$$

$$r_{i,j} = r_{i+1,j} \text{ for even } 1 \leq i \leq n-2, \ j \in O.$$

Pictorially:

$$\vec{s}_0 = \begin{bmatrix} a_0 \\ a_0 \\ \vdots \\ a_0 \\ a_0 \end{bmatrix} \theta_O \begin{bmatrix} r_{1,0} \\ a_0 \\ \vdots \\ r_{1,n-1} \\ a_0 \end{bmatrix} \theta_E \begin{bmatrix} r_{1,0} \\ r_{2,1} \\ \vdots \\ r_{1,n-1} \\ r_{2,n} \end{bmatrix} \theta_O \cdots \theta_E \begin{bmatrix} r_{n-2,0} \\ r_{n-1,1} \\ \vdots \\ r_{n-2,n-1} \\ r_{n-1,n} \end{bmatrix} \theta_O \begin{bmatrix} a_n \\ r_{n-1,1} \\ \vdots \\ a_n \\ r_{n-1,n} \end{bmatrix} \theta_E \begin{bmatrix} a_n \\ a_n \\ \vdots \\ a_n \\ a_n \end{bmatrix} = \vec{s}_{n+1}.$$

As a result, by definition of S, we have that

$$\begin{aligned}
a_n \ R \ r_{n-1,1};&\\
r_{n-i,i} \ R \ r_{n-i-1,i+1} &\text{ for } 1 \leq i \leq n-2;\\
r_{1,n-1} \ R \ a_0.&
\end{aligned}$$

These imply $(a_n, a_0) \in R \circ^n R$, contradicting the initial hypothesis.

In an analogous way we proceed for the case of n even in order to get the same contradiction. □

The previous reasoning somehow generalizes the implicit argument presented in the last few lines of Example 3.8 of [22]. Again, Theorem 3.2.2 has a straightforward corollary which we just state.

Corollary 3.2.2. *If $n \geq 1$ and $\mathbf{A}$ is an algebra such that $R \leq \mathbf{A} \times \mathbf{A}$ is an n-dimensional Hagemann relation, then a failure of congruence $(n+1)$-permutability can be found in $\mathbf{SP}_{fin}(\mathbf{A})$, precisely in $\mathbf{S}(\mathbf{A}^{n+1})$.*

Since in the proof of Theorem 3.2.2 we did not discuss entirely the case n even, we owe the reader one supporting example dealing with such an instance, which may also help understand the general case.

Example 3.2.3 ($n = 4$)**.** Suppose R is a 4-dimensional Hagemann relation on an algebra $\mathbf{A}$. Using the same notation as in Theorem 3.2.2, let $(a_0, a_4) \in R$, with $(a_4, a_0) \notin R \circ^4 R$. Moreover, $E = \{0, 2, 4\}$, $O = \{1, 3\}$ and

$$\vec{s}_0 = (a_0, a_0, a_0, a_0, a_0);$$

$$\vec{s}_1 = (a_0, a_0, a_0, a_0, a_4);$$

$$\vec{s}_2 = (a_0, a_0, a_0, a_4, a_4);$$

$$\vec{s}_3 = (a_0, a_0, a_4, a_4, a_4);$$
$$\vec{s}_4 = (a_0, a_4, a_4, a_4, a_4);$$
$$\vec{s}_5 = (a_4, a_4, a_4, a_4, a_4).$$

Notice that $\vec{s}_0 \; \theta_O \; \vec{s}_1 \; \theta_E \; \vec{s}_2 \; \theta_O \; \vec{s}_3 \; \theta_E \; \vec{s}_4 \; \theta_O \; \vec{s}_5$, but $(\vec{s}_0, \vec{s}_n) \notin \theta_E \circ^5 \theta_O$, or else the following would hold

$$\vec{s}_0 = \begin{bmatrix} a_0 \\ R \\ a_0 \\ R \\ a_0 \\ R \\ a_0 \\ R \\ a_0 \end{bmatrix} \begin{matrix} = \\ \\ \\ \\ = \\ \\ \\ \\ = \end{matrix} \begin{bmatrix} a_0 \\ R \\ r_{1,1} \\ R \\ a_0 \\ R \\ \underline{r_{1,3}} \\ \underline{R} \\ \underline{a_0} \end{bmatrix} \begin{matrix} \\ \\ = \\ \\ \\ \\ = \\ \\ \\ \end{matrix} \begin{bmatrix} r_{2,0} \\ R \\ r_{1,1} \\ R \\ \underline{r_{2,2}} \\ \underline{R} \\ \underline{r_{1,3}} \\ R \\ r_{2,4} \end{bmatrix} \begin{matrix} = \\ \\ \\ \\ = \\ \\ \\ \\ = \end{matrix} \begin{bmatrix} r_{2,0} \\ R \\ \underline{r_{3,1}} \\ \underline{R} \\ \underline{r_{2,2}} \\ R \\ r_{3,3} \\ R \\ r_{2,4} \end{bmatrix} \begin{matrix} \\ \\ = \\ \\ \\ \\ = \\ \\ \\ \end{matrix} \begin{bmatrix} \underline{a_4} \\ \underline{R} \\ \underline{r_{3,1}} \\ R \\ a_4 \\ R \\ r_{3,3} \\ R \\ a_4 \end{bmatrix} \begin{matrix} = \\ \\ \\ \\ = \\ \\ \\ \\ = \end{matrix} \begin{bmatrix} a_4 \\ R \\ a_4 \\ R \\ a_4 \\ R \\ a_4 \\ R \\ a_4 \end{bmatrix} = \vec{s}_5,$$

implying

$$a_4 \; R \; r_{3,1} \; R \; r_{2,2} \; R \; r_{1,3} \; R \; a_0,$$

and hence contradicting $(a_4, a_0) \notin R \circ^4 R$.

In later chapters, we use the constructions of Theorem 3.2.1 and Theorem 3.2.2 for particular cases of congruence 2-permutability and 3-permutability in a way that allows us to produce special configurations of 1-dimensional and 2-dimensional Hagemann relations. In fact, such configurations will be involved in some primeness arguments as, respectively, 2-permutability and 3-permutability "blockers", as one can directly verify in Chapter 5.

Chapter 4

A family of prime Maltsev conditions

In the previous chapter, in order to find the Maltsev condition described in Theorem 3.1.3 characterizing congruence n-permutability, we have used a relation, that we have called R_n in Corollary 3.1.3, which has turned out to be an $(n-1)$-dimensional Hagemann relation when and only when $\mathcal{V}$ is not congruence n-permutable ($n > 1$).

We further recall that, in fact, $(x_n, x_0) \in R_n \circ^{n-1} R_n$ if and only if $\mathcal{V}$ is congruence n-permutable. Hence, when dealing with a non-congruence n-permutable variety $\mathcal{V}$, it does make sense to search within $\mathcal{V}$ for some suitable Hagemann relations having a special configuration, which is reminiscent of the shape of the directed graph $\langle X; V\rangle$ mentioned in the proof of Theorem 3.1.3. In the next sections we will formalize these concepts and present some results which relate to congruence n-permutability.

4.1 Special Hagemann relations and omission Maltsev classes

Let us begin with the following definition

Definition 4.1.1. Let $n \geq 1$ and $\mathbf{A}$ be an algebra with $R \leq \mathbf{A} \times \mathbf{A}$. We say that R is an *n-dimensional special Hagemann relation*, or a *special Hagemann relation of dimension* n (briefly SHR_n), if there exists a partition $\{A_i : 0 \leq i \leq n+1\}$ of A, such that

$$R = [(A_0 \cup A_1) \times A] \cup \bigcup_{i=2}^{n+1} \left[A_i \times \left(\bigcup_{j=i-1}^{n+1} A_j \right) \right].$$

In this setting, we also say that the partition $\{A_i : 0 \leq i \leq n+1\}$ *induces* R, or analogous expressions.

A Cartesian representation of a special Hagemann relation of dimension n (Figure 4.1) could help visualize the particular configuration of such an object.

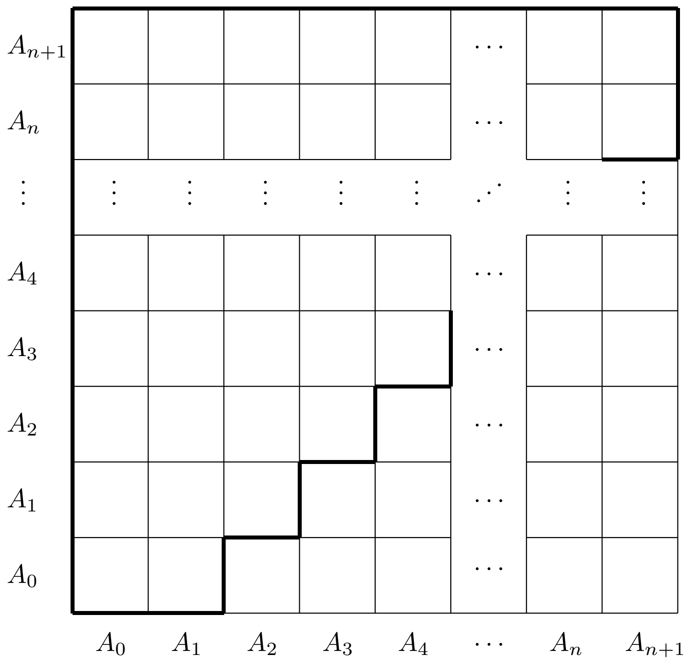

Figure 4.1: A Cartesian representation of an n-dimensional special Hagemann relation

It is straightforward to notice that Definition 4.1.1 defines indeed an n-dimensional Hagemann relation for every $n \geq 1$: as a matter of fact, since $A_i \times A_i \subseteq R$, for every $0 \leq i \leq n+1$, R is reflexive; moreover, $A_0 \times A_{n+1} \subseteq R$, but there is no way to $R \circ^n R$-connect any $a_{n+1} \in A_{n+1}$ to any $a_0 \in A_0$. However, it is always possible to $R \circ^{n+1} R$-connect any two elements of A, i.e. $(a, b) \in R \circ^{n+1} R$, for every $a, b \in A$ (equivalently $\langle A; R \circ^{n+1} R\rangle$ is a complete graph); this yields that no n-dimensional special Hagemann relation can be an $(n+1)$-dimensional special Hagemann relation, for any $n \geq 1$.

Let us now consider all those varieties in which it is not possible to find any algebra carrying a special Hagemann relation of dimension n. Hence, let us give the following definition

Definition 4.1.2. Let $n \geq 1$.

A variety $\mathcal{V}$ is said to *admit* SHR_n if there exist an algebra $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$, such that R is a special Hagemann relation of dimension n.

A variety $\mathcal{V}$ *omits* SHR_n if $\mathcal{V}$ does not admit SHR_n.

The class of varieties that omit SHR_n is called the *omission class of* SHR_n and is denoted by $\Omega(SHR_n)$.

The class $\Omega(SHR_n)$ is definitely related to the class CP_{n+1}, for $n \geq 1$: if a variety is congruence $(n+1)$-permutable, then by Theorem 3.1.2 it does not realize any n-dimensional Hagemann relation and, in particular, it must omit SHR_n. This means $CP_{n+1} \subseteq \Omega(SHR_n)$, for all $n \geq 1$. We know that in general the previous inclusion is strict and we will give a proof of this later on. What we are interested in at this point is to prove that the class $\Omega(SHR_n)$ defines a Maltsev condition for every $n \geq 1$, as stated below.

Before proceeding, we need to point out again that in the next theorem, every variety in $\Omega(SHR_n)$ is intended to be considered as its interpretability type so that the class $\Omega(SHR_n)$ can be thought as a subclass of the lattice of interpretability types.

Theorem 4.1.1. *For $n \geq 1$, $\Omega(SHR_n)$ is a Maltsev class.*

Proof. Fix any $n \geq 1$. In order to verify the statement, we could alternatively use Theorem 3.1 from [38], by giving a first order description of an n-dimensional special Hagemann relation; instead, we are going to make use of Theorem 2.2.1 to prove that $\Omega(SHR_n)$ is a Maltsev class, and verify every point of that theorem by following the same enumeration.

1. *Well definedness of* $\Omega(SHR_n)$: Suppose $\mathcal{V} \equiv \mathcal{W}$ are two equi-interpretable varieties and assume $\mathcal{V}$ omits SHR_n. If $\mathcal{W}$ admits SHR_n, then there exists an algebra $\mathbf{B} \in \mathcal{W}$ and $R \leq \mathbf{B} \times \mathbf{B}$ such that R is a special Hagemann relation of dimension n. Call ι the interpretation witnessing $\mathcal{V} \leq \mathcal{W}$. Then, by Definition 2.1.1, the algebra $\mathbf{B}^{(\iota)} \in \mathcal{V}$ and still $R \leq \mathbf{B}^{(\iota)} \times \mathbf{B}^{(\iota)}$ is an n-dimensional special Hagemann relation, contradicting the initial assumption on $\mathcal{V}$.

2. *Closure under subvarieties*: Let $\mathcal{V}$ be a variety in $\Omega(SHR_n)$ and $\mathcal{W} \subseteq \mathcal{V}$ a subvariety. Obviously, if $\mathcal{W}$ contained an algebra carrying a special Hagemann relation of dimension n, then the same algebra would lie in $\mathcal{V}$, yielding a contradiction.

3. *Closure under finite products*: It is sufficient to prove the validity for the product of two varieties.

   By contrapositive argument, assume $\mathcal{V} \otimes \mathcal{W} \notin \Omega(SHR_n)$, for two varieties $\mathcal{V}, \mathcal{W}$. By definition of $\Omega(SHR_n)$, there must exist an algebra $\mathbf{Q} \in \mathcal{V}\otimes\mathcal{W}$ and a subuniverse $R \leq \mathbf{Q}\times\mathbf{Q}$ which is a special Hagemann relation of dimension n. Let $\{Q_0, \dots, Q_{n+1}\}$ be the partition of Q inducing R.

   Therefore, by Theorem 2.1.3, there exist $\mathbf{U} \in \mathcal{V}$ and $\mathbf{V} \in \mathcal{W}$, such that
   $$\mathbf{Q} \cong \mathbf{U} \otimes \mathbf{V},$$
   and
   $$R \leq \mathbf{Q}^2 \cong (\mathbf{U} \otimes \mathbf{V})^2 \cong \mathbf{U}^2 \otimes \mathbf{V}^2;$$
   which in turns yields that there exist $R_1 \leq \mathbf{U} \times \mathbf{U}$ and $R_2 \leq \mathbf{V} \times \mathbf{V}$ with
   $$\mathbf{R} \cong \mathbf{R}_1 \otimes \mathbf{R}_2.$$
   Without loss of generality we will assume $Q = U \times V$ and $R = R_1 \otimes R_2$ (in the sense of Definition 3.2.2), allowing us to use the notation previously used several times, according to which:
   $$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \; R \; \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \text{ if and only if } (u_1, u_2) \in R_1 \text{ and } (v_1, v_2) \in R_2.$$
   CLAIM 4.1.1.1. Both R_1 and R_2 are reflexive relations.

   Proof. Let u be any element of U. Then, for every $v \in V$,
   $$\begin{bmatrix} u \\ v \end{bmatrix} \; R \; \begin{bmatrix} u \\ v \end{bmatrix}$$
   by reflexivity of R, meaning that $(u, u) \in R_1$, and hence that R_1 is reflexive.
   The same reasoning can be reproduced for R_2. □

   The next claim is the crucial part of the proof: the fact that $\mathbf{R} \cong \mathbf{R}_1 \otimes \mathbf{R}_2$ does force a fairly strong property for R_1 and R_2.
   CLAIM 4.1.1.2. Either $R_1 = U \times U$ or $R_2 = V \times V$.

   Proof. Suppose $R_2 \subsetneq V^2$ and let $(v, v') \in V^2$ with $(v, v') \notin R_2$. Furthermore, pick any $(u_1, u_2) \in U^2$ and consider the pair $\begin{bmatrix} u_2 \\ v \end{bmatrix}$.

   If we assume $\begin{bmatrix} u_2 \\ v \end{bmatrix} \; R \; \begin{bmatrix} u_2 \\ v' \end{bmatrix}$, then $(v, v') \in R_2$, which is a contradiction. This means that there exists at least one element of Q to which $\begin{bmatrix} u_2 \\ v \end{bmatrix}$ cannot be R-related: since $(Q_0 \cup Q_1) \times Q \subseteq R$, we deduce that $\begin{bmatrix} u_2 \\ v \end{bmatrix} \notin Q_0 \cup Q_1$. Therefore,
   $$\begin{bmatrix} u_2 \\ v \end{bmatrix} \in \bigcup_{i=2}^{n+1} Q_i.$$

Suppose then that $\begin{bmatrix} u_2 \\ v \end{bmatrix} \in Q_i$, for some $2 \leq i \leq n+1$, and consider the pair $\begin{bmatrix} u_1 \\ v' \end{bmatrix}$. If it is the case that

$$\begin{bmatrix} u_1 \\ v' \end{bmatrix} \in \bigcup_{j=i-1}^{n+1} Q_j,$$

then, by the fact that $Q_i \times (Q_{i-1} \cup \ldots \cup Q_{n+1}) \subseteq R$, we would have that $\begin{bmatrix} u_2 \\ v \end{bmatrix} \; R \; \begin{bmatrix} u_1 \\ v' \end{bmatrix}$, proving $(v, v') \in R_2$ and thus contradicting the initial assumption on this pair.
Hence, it follows that

$$\begin{bmatrix} u_1 \\ v' \end{bmatrix} \in \bigcup_{j=0}^{i-2} Q_j,$$

and because $Q_j \times Q_i \subseteq R$, for all $0 \leq j \leq i-2$, then

$$\begin{bmatrix} u_1 \\ v' \end{bmatrix} \; R \; \begin{bmatrix} u_2 \\ v \end{bmatrix},$$

proving that $(u_1, u_2) \in R_1$. Since this pair was arbitrary in U^2, this shows that $R_1 = U^2$, as desired.
A symmetric proof (which we omit) would also demonstrate that if $R_1 \subsetneq U^2$, then $R_2 = V^2$. $\square$

Henceforth, we can assume without any loss of generality that $R = R_1 \otimes V^2$ (the symmetric case $R = U^2 \otimes R_2$ would proceed analogously). If $\pi : Q \to U$ is the projection map onto U, then call $U_i := \pi(Q_i)$ for all $0 \leq i \leq n+1$.
CLAIM 4.1.1.3. For every $0 \leq i \leq n+1$, U_i is non-empty and

$$\bigcup_{i=0}^{n+1} U_i = U.$$

Proof. For any $0 \leq i \leq n+1$, since Q_i is non-empty, then so is U_i.
For the second statement, if $u \in U$, then for any $v \in V$ the pair $\begin{bmatrix} u \\ v \end{bmatrix}$ lies in Q, and hence in Q_i, for some $0 \leq i \leq n+1$, yielding $u = \pi\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \in \pi(Q_i) = U_i$. $\square$

Moreover,
CLAIM 4.1.1.4. For $0 \leq i, j \leq n+1$, $i \neq j$, $U_i \cap U_j = \varnothing$.

Proof. It is sufficient to prove that for any fixed $0 \leq i < n+1$ and $i < j \leq n+1$, $U_i \cap U_j = \varnothing$.
Let us begin with $0 \leq i \leq n-1$ and $i < j \leq n+1$: suppose $U_i \cap U_j \neq \varnothing$, and let $u \in U_i \cap U_j$.
Thus, by definition of the U_i's there exist $v, v' \in V$ such that $\begin{bmatrix} u \\ v \end{bmatrix} \in Q_i$ and $\begin{bmatrix} u \\ v' \end{bmatrix} \in Q_j$.
Because $j > i$, $Q_{i+2} \times Q_j \subseteq R$, which yields there exists $\begin{bmatrix} x \\ y \end{bmatrix} \in Q_{i+2}$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix} \; R \; \begin{bmatrix} u \\ v' \end{bmatrix}.$$

This proves $(x, u) \in R_1$, and since $R = R_1 \otimes V^2$, we deduce

$$\begin{bmatrix} x \\ y \end{bmatrix} \; R \; \begin{bmatrix} u \\ v \end{bmatrix},$$

contradicting the fact that $(Q_{i+2} \times Q_i) \cap R = \varnothing$. The contradiction derives from not assuming $U_i \cap U_j = \varnothing$, which instead holds.

The only remaining case is when $i = n$ and $j = n + 1$. Therefore, suppose $U_n \cap U_{n+1}$ contains an element u and, again, let v, v' be elements of V such that $\begin{bmatrix} u \\ v \end{bmatrix} \in Q_n$ and $\begin{bmatrix} u \\ v' \end{bmatrix} \in Q_{n+1}$. Since $Q_n \times Q_{n-1} \subseteq R$, let $\begin{bmatrix} x \\ y \end{bmatrix} \in Q_{n-1}$ with

$$\begin{bmatrix} u \\ v \end{bmatrix} \ R \ \begin{bmatrix} x \\ y \end{bmatrix},$$

which also means $(u, x) \in R_1$. Thus, since $R = R_1 \otimes V^2$, we also have

$$\begin{bmatrix} u \\ v' \end{bmatrix} \ R \ \begin{bmatrix} x \\ y \end{bmatrix},$$

contradicting the fact that $(Q_{n+1} \times Q_{n-1}) \cap R = \varnothing$. Again, the contradiction comes from not having considered $U_n \cap U_{n+1}$ empty, which indeed is. $\square$

Finally, we prove that R_1 has the desired configuration.

CLAIM 4.1.1.5.

$$R_1 = [(U_0 \cup U_1) \times U] \cup \bigcup_{i=2}^{n+1} [U_i \times (U_{i-1} \cup \ldots \cup U_{n+1})].$$

Proof. Call T the right-hand-side of the equality displayed above.

If $(u, u') \in R_1$, then there exist $v, v' \in V$ such that $\begin{bmatrix} u \\ v \end{bmatrix} \ R \ \begin{bmatrix} u' \\ v' \end{bmatrix}$, which means that $\begin{bmatrix} u \\ v \end{bmatrix} \in Q_0$, or $\begin{bmatrix} u \\ v \end{bmatrix} \in Q_i$ and $\begin{bmatrix} u' \\ v' \end{bmatrix} \in Q_j$, for some $1 \leq i \leq n+1$, $i - 1 \leq j \leq n + 1$. Hence, either $u \in U_0$, or $u \in U_i$ and $u' \in U_j$, proving that $(u, u') \in T$.

By inverting the above reasoning, we prove the other inclusion and hence the equality hold $\square$

These claims show that $\{U_i : 0 \leq i \leq n+1\}$ is a partition of U and such a partition induces R_1, which is an n-dimensional special Hagemann relation on the algebra $\mathbf{U}$. Since $\mathbf{U}$ is a member of $\mathcal{V}$, we have indeed proven that $\mathcal{V}$ admits SHR_n, as desired.

4. *Finite presentability*: Again, in order to prove this last statement we will argue by contrapositive.

   Suppose Σ is a set of equations of type σ such that for every finite $\Psi \subseteq \Sigma$, $\mathrm{Mod}(\Psi) \notin \Omega(SHR_n)$ [more precisely and formally, think of Σ as a set of universally quantified sentences of the form $\forall x_1 \ldots \forall x_k (p(x_1, \ldots, x_k) \approx q(x_1, \ldots, x_k))$]. We aim to prove that also $\mathrm{Mod}(\Sigma) \notin \Omega(SHR_n)$.

   Hence, call F the set of fundamental operation symbols of $\mathrm{Mod}(\Sigma)$ of type σ, and define the first order language $\mathcal{L}$ as follows:

$$\mathcal{L} = F \cup \{R^{(2)}\} \cup \{A_i^{(1)} : 0 \leq i \leq n+1\},$$

   where each $P^{(i)}$ denotes a predicate of arity i (we will omit the superscript (i) whenever it is clear from the context). Furthermore, let Γ be the set of first order sentences listed below:

$$\forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_k \left[\left(\bigwedge_{i=1}^{k} R(x_i, y_i) \right) \to R(f(x_1, \ldots, x_k), f(y_1, \ldots, y_k)) \right],$$

for all $f \in F$ of arity k (compatibility with the fundamental operations);

$$\forall x \left[\bigvee_{i=0}^{n+1} A_i(x) \right] \text{ (covering property);}$$

$$\exists x A_i(x) \text{ for } i \in \{0, \ldots, n+1\} \text{ (non-emptiness);}$$

$$\forall x [A_i(x) \to \neg A_j(x)] \text{ for } 0 \leq i, j \leq n+1, i \neq j \text{ (disjointness);}$$

$$\forall x \forall y \left[R(x,y) \leftrightarrow \left(A_0(x) \vee A_1(x) \vee \bigvee_{i=2}^{n+1} \left(A_i(x) \to \bigwedge_{j=0}^{i-2} \neg A_j(y) \right) \right) \right].$$

At this point, let Δ be any finite subset of $\Sigma \cup \Gamma$.

CLAIM 4.1.1.6. Δ has a model.

Proof. Let Ψ be any finite subset of Σ: by assumption $\mathrm{Mod}(\Psi)$ admits SHR_n, i.e. there exists an algebra $\mathbf{A} = \langle A; \{f^{\mathbf{A}} : f \in F\} \rangle \in \mathrm{Mod}(\Psi)$ such that $R' \leq \mathbf{A} \times \mathbf{A}$ is a special Hagemann relation of dimension n. Let $\{A'_0, \ldots, A'_{n+1}\}$ be the partition inducing R'. If we interpret the predicates of $\mathcal{L}$ in the natural way, namely, for all $x, y \in A$, $0 \leq i \leq n+1$:

$$R^{\mathfrak{A}}(x,y) \Longleftrightarrow (x,y) \in R';$$

$$A_i^{\mathfrak{A}}(x) \Longleftrightarrow x \in A'_i;$$

then the structure $\mathfrak{A} = \langle \mathbf{A}; R^{\mathfrak{A}}, A_0^{\mathfrak{A}}, \ldots, A_{n+1}^{\mathfrak{A}} \rangle$ is a model of Δ by construction. $\square$

Therefore, by the compactness theorem, we deduce that also $\Sigma \cup \Gamma$ has a model: if we consider only the universe of such a model along with the function symbols of F, we get an algebra $\mathbf{A}$ carrying a special Hagemann relation of dimension n, which is a model of Σ, proving that $\mathrm{Mod}(\Sigma) \notin \Omega(SHR_n)$, as we wanted to prove.

$\square$

Unfortunately, we do not know whether $\Omega(SHR_n)$ is a strong Maltsev class or not, even though we do know that it is so in some particular cases that we are going to treat later. Also, we are not able to provide an explicit description of the finitely presentable varieties that define the Maltsev condition: as a matter of fact, the proof of Theorem 4.1.1 invokes the compactness theorem at some point, and the fact that the compactness argument is not constructive, somehow prevents one from accessing (or at least from accessing with no big effort) the terms that characterize the Maltsev condition.

On the other hand, the tight shape of a special Hagemann relation naturally yields a primeness argument for those varieties omitting it. Yet, before proving this property, we need the following lemma:

Lemma 4.1.1. *For $n \geq 1$, and an algebra $\mathbf{A}$, if $R \leq \mathbf{A} \times \mathbf{A}$ is a special Hagemann relation of dimension n, then every power of $\mathbf{A}$ carries one, i.e. for every cardinal $\kappa > 0$, there exists $R_\kappa \leq \mathbf{A}^\kappa \times \mathbf{A}^\kappa$ which is a special Hagemann relation of dimension n.*

Proof. Let $n \geq 1$ and $\mathbf{A}$ and R as in the statement. For every cardinal $\kappa > 0$ define

$$R_0 := R \leq \mathbf{A} \times \mathbf{A};$$

$$R_\kappa := R \otimes (A^\kappa)^2 \leq \mathbf{A}^{1+\kappa} \times \mathbf{A}^{1+\kappa}.$$

If $\{A_0, \ldots, A_{n+1}\}$ is the partition of A inducing R, then it is a straightforward verification that $\{A_i \times A^\kappa : 0 \leq i \leq n+1\}$ is a partition of $A^{1+\kappa}$ which induces the n-dimensional special Hagemann relation R_κ, for all $\kappa > 0$. $\square$

Since we will be using them in the proof of the next theorem, call $A_i^{(1+\kappa)} := A_i \times A^\kappa$ for $\kappa > 0$, $0 \leq i \leq n+1$, which are the members of the partition used in the previous proof. Then, we have that

Theorem 4.1.2. *For each $n \geq 1$, the Maltsev filter $\Omega(SHR_n)$ is prime in the lattice of interpretability types. In other words, omitting SHR_n is a prime Maltsev condition.*

Proof. Fix $n \geq 1$. We need to prove that for any two varieties $\mathcal{V}, \mathcal{W}$, if $\mathcal{V} \vee \mathcal{W} \in \Omega(SHR_n)$, then $\mathcal{V} \in \Omega(SHR_n)$ or $\mathcal{W} \in \Omega(SHR_n)$.

By contrapositive, suppose neither $\mathcal{V}$, nor $\mathcal{W}$ lie in $\Omega(SHR_n)$. This means that there exist $\mathbf{A} \in \mathcal{V}$, $\mathbf{B} \in \mathcal{W}$ with $R \leq \mathbf{A} \times \mathbf{A}$ and $S \leq \mathbf{B} \times \mathbf{B}$ which are n-dimensional special Hagemann relations. Call $\{A_i : 0 \leq i \leq n+1\}$ and $\{B_i : 0 \leq i \leq n+1\}$ the partitions of, respectively, A and B, inducing R and S. We want to use these objects to build an algebra in $\mathcal{V} \vee \mathcal{W}$ carrying a special Hagemann relation of dimension n, too.

By Lemma 4.1.1, we can consider powers of $\mathbf{A}$ and $\mathbf{B}$ that still carry special Hagemann relations of dimension n. In fact, we need to consider a large enough cardinal $\lambda > 0$, such that for every $0 \leq i \leq n+1$, $|A_i^{(\kappa)}| = |B_i^{(\kappa)}|$, where $\kappa = 1 + \lambda$ (notice that we require the same κ for all the i's). Call s_i any arbitrarily chosen bijection from $A_i^{(\kappa)}$ into $B_i^{(\kappa)}$, for $0 \leq i \leq n+1$. Since $\{A_0^{(\kappa)}, \ldots, A_{n+1}^{(\kappa)}\}$ and $\{B_0^{(\kappa)}, \ldots, B_{n+1}^{(\kappa)}\}$ are partitions, we can unify the bijections s_i's by defining $s : A^\kappa \to B^\kappa$, for $a \in A^\kappa$, as

$$s(a) = s_i(a) \text{ if } a \in A_i^{(\kappa)},$$

for some $i \in \{0, \ldots, n+1\}$. Such an s is obviously a bijection which allows us to consider the algebra $\mathbf{A}^\kappa \amalg_s \mathbf{B}^\kappa \in \mathcal{V} \amalg \mathcal{W} = \mathcal{V} \vee \mathcal{W}$.

Moreover, since $\{A_0^{(\kappa)}, \ldots, A_{n+1}^{(\kappa)}\}$ and $\{B_0^{(\kappa)}, \ldots, B_{n+1}^{(\kappa)}\}$ respectively induce R_λ and S_λ, we have that $(s \times s)(R_\lambda) = S_\lambda$. By Theorem 2.1.4, we deduce

$$R_\lambda \leq (\mathbf{A}^\kappa)^2 \amalg_{s \times s} (\mathbf{B}^\kappa)^2 = (\mathbf{A}^\kappa \amalg_s \mathbf{B}^\kappa)^2,$$

which is an n-dimensional special Hagemann relation on $\mathbf{A}^\kappa \amalg_s \mathbf{B}^\kappa$, showing that $\mathcal{V} \vee \mathcal{W} \notin \Omega(SHR_n)$, as desired. □

To close this section, it makes sense to ask what relationship occurs between any two classes $\Omega(SHR_n)$ and $\Omega(SHR_m)$ for $m, n \geq 1$ with $m \neq n$. It would be optimal to have, for instance, $\Omega(SHR_n) \subseteq \Omega(SHR_{n+1})$, but we do not know whether this is the case in general (although we do when restricting to idempotent varieties, as we will show further on). However, we are still able to argue that the following holds.

Theorem 4.1.3. *Let $n \geq 1$ be an integer and $\mathbf{A}$ an algebra carrying a special Hagemann relation of dimension n $R \leq \mathbf{A} \times \mathbf{A}$ induced by the partition $\{A_i : 0 \leq i \leq n+1\}$. Then the relation $R \circ^n R \leq \mathbf{A} \times \mathbf{A}$ is a 1-dimensional special Hagemann relation on $\mathbf{A}$, induced by the partition $\{A_0, \bigcup_{i=1}^n A_i, A_{n+1}\}$.*

Proof. Fix $n \geq 1$ and let $R \leq \mathbf{A} \times \mathbf{A}$ be a special Hagemann relation of dimension n on the algebra $\mathbf{A}$. Since R is a subuniverse, then so is $R \circ^n R$. Thus define,

$$B_0 := A_0;$$

$$B_1 := \bigcup_{i=1}^{n} A_i;$$

$$B_2 := A_{n+1}.$$

The set $\{B_0, B_1, B_2\}$ is obviously a partition of A; hence we need to prove that

$$R \circ^n R = [(B_0 \cup B_1) \times A] \cup [B_2 \times (B_1 \cup B_2)].$$

Easily, $B_0 \times A = A_0 \times A \subseteq R \subseteq R \circ^n R$.

For the others, choose any $a_i \in A_i$, for all $0 \le i \le n+1$. Then, for all $0 \le j \le i \le n$

$$a_{i+1}\ R\ a_i\ R\ \cdots\ R\ a_j$$

with $i-j+1$ occurrences of R, proving that $A_{i+1} \times A_j \subseteq R \circ^{i-j+1} R$, which means

$$B_2 \times B_1 = A_{n+1} \times \bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} R \circ^{n-i+1} R = R \circ^n R,$$

and

$$B_1 \times A = \left(\bigcup_{i=1}^{n} A_i\right) \times A \subseteq \bigcup_{0\le j\le i\le n} R \circ^{i-j+1} R = R \circ^n R.$$

Also $B_2 \times B_2 \subseteq R \subseteq R \circ^n R$, proving that the inclusion "$\supseteq$" holds.

Conversely, if a pair is not in $[(B_0 \cup B_1) \times A] \cup [B_2 \times (B_1 \cup B_2)]$, then it is of the form $(a_{n+1}, a_0) \in A_{n+1} \times A_0$. If such a pair lay in $R \circ^n R$, then R would not be a n-dimensional Hagemann relation, showing that "$\subseteq$" must hold as well. □

Therefore, the next corollary follows straightforwardly

Corollary 4.1.1. *For every $n \ge 1$, $\Omega(SHR_1) \subseteq \Omega(SHR_n)$.*

Proof. For $n \ge 1$, if $\mathcal{V}$ is a variety that admits SHR_n, then the same algebra of $\mathcal{V}$ carrying an n-dimensional special Hagemann relation, by Theorem 4.1.3, also carries a 1-dimensional special Hagemann relation, meaning that $\mathcal{V}$ also admits SHR_1. This reasoning shows that $\Omega(SHR_1) \subseteq \Omega(SHR_n)$, as desired. □

In the next section, we are going to restrict attention to the case of idempotent varieties omitting SHR_n.

4.2 Idempotent varieties omitting SHR_n

In this section, we will focus on the analysis of the idempotent varieties in the classes $\Omega(SHR_n)$, for $n \ge 1$. We will see that with respect to idempotent varieties it is possible to state stronger versions of Theorem 4.1.3 and Corollary 4.1.1 and observe more significant connections with congruence n-permutable varieties.

First, let us define the following class

Definition 4.2.1. For $n \ge 1$,

$$\Omega^{id}(SHR_n) = \Omega(SHR_n) \cap L^{id}.$$

If an algebra $\mathbf{A}$ is idempotent and carries a special Hagemann relation R of dimension n ($n \ge 1$), then some subsets of A included in the partition inducing R, or suitable union of them, turn out to be subuniverses of $\mathbf{A}$. This crucial fact yields a stronger property than the one expressed in Theorem 4.1.3, which we present below.

Theorem 4.2.1. *Let $n \ge 1$ and $\mathbf{A}$ be an idempotent algebra such that $R \le \mathbf{A} \times \mathbf{A}$ is a special Hagemann relation of dimension $n+1$ induced by the partition $\{A_i : 0 \le i \le n+2\}$. If we call*

$$B := \bigcup_{i=1}^{n+2} A_i,$$

$$S := R \cap (B \times B),$$

then B is a subuniverse of $\mathbf{A}$ and $S \le \mathbf{B} \times \mathbf{B}$ is a special Hagemann relation of dimension n induced by the partition $\{A_i : 1 \le i \le n+2\}$.

Proof. Fix $n \geq 1$ and let $\mathbf{A}$, R, B and S as defined in the statement.

Fix any $a_{n+2} \in A_{n+2}$, and notice that for every $a_i \in A_i$, $0 \leq i \leq n+1$, and $a \in A_{n+2}$,

$$a_{n+2}\ R\ a_{n+1}\ R\ \cdots\ R\ a_{i+1}\ R\ a_i,$$

$$a_{n+2}\ R\ a,$$

meaning that $(a_{n+2}, a_i) \in R \circ^{n+2-i} R$. In particular, for $1 \leq i \leq n+1$, $(a_{n+2}, a_i) \in R \circ^{n+2-i} R \subseteq R \circ^{n+1} R$, which yields that, for each $b \in B$, $(a_{n+2}, b) \in R \circ^{n+1} R$.

On the other hand, if $b \in a_{n+2}/R \circ^{n+1} R$, then $b \notin A_0$, otherwise contradicting the fact that R is an $(n+1)$-dimensional Hagemann relation. Hence b must lie in B.

The previous reasoning shows that

$$B = a_{n+2}/R \circ^{n+1} R,$$

which, by idempotence, means B is a subuniverse of $\mathbf{A}$. Therefore, S is also a subuniverse of $\mathbf{B}^2$ and B is partitioned by the sets $A_1, \ldots, A_{n+2}$. Moreover, the following set theoretical calculation shows that S is an n-dimensional special Hagemann relation on $\mathbf{B}$ induced by $\{A_1, \ldots, A_{n+2}\}$:

$$S = R \cap B^2 = \left[(A_0 \times A) \cup \bigcup_{i=1}^{n+2} [A_i \times (A_{i-1} \cup \ldots \cup A_{n+2})] \right] \cap B^2 =$$

$$= [(A_0 \times A) \cap B^2] \cup \left[\bigcup_{i=1}^{n+2} [A_i \times (A_{i-1} \cup \ldots \cup A_{n+2})] \right] \cap B^2 =$$

$$= [\varnothing \times B] \cup [A_1 \times B] \cup \bigcup_{i=2}^{n+2} [A_i \times (A_{i-1} \cup \ldots \cup A_{n+2})] =$$

$$= [(A_1 \cup A_2) \times B] \cup \bigcup_{i=3}^{n+2} [A_i \times (A_{i-1} \cup \ldots \cup A_{n+2})].$$

□

This theorem has a direct consequence which is stated in the next corollary.

Corollary 4.2.1. *For every $n \geq 1$, $\Omega^{id}(SHR_n) \subseteq \Omega^{id}(SHR_{n+1})$.*

Proof. By contrapositive, suppose $\mathcal{V}$ is an idempotent variety admitting SHR_{n+1}, for fixed $n \geq 1$. This is to say there exists $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$ which is a special Hagemann relation of dimension $n+1$. By Theorem 4.2.1, there exists a subalgebra $\mathbf{B}$ of $\mathbf{A}$ carrying $S = R \cap B^2$, which is a special Hagemann relation of dimension n. Because $\mathbf{B} \in \mathcal{V}$, then $\mathcal{V}$ also admits SHR_n, completing the proof. □

Corollary 4.2.1 ensures that, with respect to idempotent varieties, the omission Maltsev classes of SHR_n form an ascending chain as subclasses of L^{id}

$$\Omega^{id}(SHR_1) \subseteq \Omega^{id}(SHR_2) \subseteq \ldots \subseteq \Omega^{id}(SHR_n) \subseteq \Omega^{id}(SHR_{n+1}) \subseteq \ldots$$

Let us define the following class:

Definition 4.2.2. The class of idempotent varieties omitting SHR_n, for some $n \geq 1$, is denoted by $\Omega^{id}(SHR_\omega)$ and corresponds to

$$\Omega^{id}(SHR_\omega) = \bigcup_{1 \leq n < \omega} \Omega^{id}(SHR_n).$$

Within $\mathbf{L}^{id}$, being a nested union of Maltsev filters , $\Omega^{id}(SHR_\omega)$ is in turn a Maltsev filter. Thus, we can reasonably ask whether there is a connection with other well known Maltsev conditions. The next arguments will prove that such a connection exists indeed.

In [19] (Theorem 9.14) the authors prove, using the techniques of Tame Congruence Theory, that a locally finite idempotent variety is congruence n-permutable for some $n \geq 2$ if and only if it is not interpretable in the variety of distributive lattices. Years later, M. Valeriote and R. Willard in [42] generalized that result by removing the local finiteness hypothesis and making use of an ultraproduct construction. By completeness, we rephrase this mentioned result as follows

Theorem 4.2.2 ([42])**.** *Let $\mathcal{V}$ be an idempotent variety and let $\mathcal{D}$ denote the variety of distributive lattices. Then*

$$\mathcal{V} \in CP_\omega \text{ if and only if } \mathcal{V} \not\leq \mathcal{D}$$

In other words, recalling first that CP_ω^{id} denotes the class of (interpretability types of) idempotent varieties in CP_ω, we have

$$CP_\omega^{id} = \{\mathcal{V} : \mathcal{V} \text{ is idempotent}, \mathcal{V} \not\leq \mathcal{D}\}.$$

We will use this characterization to prove the next theorem.

Theorem 4.2.3. *Let $\mathcal{V}$ be an idempotent variety. Then, $\mathcal{V}$ is congruence n-permutable for some $n \geq 2$ if and only if $\mathcal{V}$ omits SHR_m for some $m \geq 1$. In other words,*

$$CP_\omega^{id} = \Omega^{id}(SHR_\omega).$$

Proof. For the entire proof, let $\mathcal{V}$ be an idempotent variety.

Suppose that $\mathcal{V} \notin \Omega^{id}(SHR_\omega)$. This is to say $\mathcal{V}$ admits SHR_n, for every $n \geq 1$: in particular, this also means that for every $n \geq 1$, we can find in $\mathcal{V}$ an algebra carrying an n-dimensional Hagemann relation, yielding, by Theorem 3.1.2, that $\mathcal{V}$ cannot be congruence $(n+1)$-permutable, for any $n \geq 1$.

Conversely, assume $\mathcal{V}$ is not congruence n-permutable for any $n \geq 2$. By Theorem 4.2.2, $\mathcal{V} \leq \mathcal{D}$, the variety of distributive lattices. Call ι such an interpretation.

The variety $\mathcal{D}$ contains (up to isomorphism) an n-chain, for every $n \geq 1$, i.e. the lattice whose universe is $n = \{0, \dots, n-1\}$ and the operations of $\vee$ and $\wedge$ are defined as $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Denote the n-chain by $\mathbf{C}_n$, $n \geq 1$; it will suffice to prove that $\mathcal{D}$ admits SHR_n for all $n \geq 1$, say $R_n \leq \mathbf{A}_n \times \mathbf{A}_n$, for some $\mathbf{A}_n \in \mathcal{D}$: if so, then the algebra $\mathbf{A}_n^{(\iota)}$ in $\mathcal{V}$ will carry the same n-dimensional special Hagemann relation R_n, proving that $\mathcal{V}$ admits SHR_n, as well.

Therefore, let us fix $n \geq 3$; we are going to show that the n-chain $\mathbf{C}_n$ itself carries a special Hagemann relation of dimension $n-2$. Hence, define on n the following relation

$$S_n := (\{0\} \times n) \cup \bigcup_{i=1}^{n-1} (\{i\} \times \{i-1, \dots, n-1\}).$$

We need to verify that S_n is a subuniverse of $\mathbf{C}_n \times \mathbf{C}_n$ or, in other words, that S_n is compatible with $\wedge$ and $\vee$. For convenience, call $X := \{0\} \times n$ and $Y := \bigcup_{i=1}^{n-1}(\{i\} \times \{i-1, \dots, n-1\})$, so that $S_n = X \cup Y$. If $(0, k), (0, h) \in X$, then

$$(0, k) \wedge (0, h) = (0, k \wedge h),$$

which is an element of X and then of S_n. If $(0, k) \in X$ and $(i, j) \in Y$, then

$$(0, k) \wedge (i, j) = (0 \wedge i, k \wedge j) = (0, k \wedge j) \in X,$$

showing the closure also for this case. Finally, if $(i, j), (h, k) \in Y$, then necessarily $i - 1 \leq j < n$ and $h - 1 \leq k < n$, implying that $\min\{j, k\} \geq \min\{i - 1, h - 1\} = \min\{i, h\} - 1$ and hence showing that

$$(i, j) \wedge (h, k) = (i \wedge h, j \wedge k) = (\min\{i, h\}, \min\{j, k\}) \in Y.$$

For the closure under $\vee$, we can dualize the previous reasoning by noting that S_n may also be viewed as

$$S_n = (n \times \{n-1\}) \cup \bigcup_{i=0}^{n-2} (\{0, \ldots, i+1\} \times \{i\}).$$

Likewise for the previous argument, call $U := n \times \{n-1\}$ and $V := \bigcup_{i=0}^{n-2}(\{0, \ldots, i+1\} \times \{i\})$ so that $S_n = U \cup V$; then, pick any $(i, n-1), (j, n-1) \in U$ and notice that

$$(i, n-1) \vee (j, n-1) = (i \vee j, n-1) \in U.$$

For the case $(i, n-1) \in U$ and $(h, k) \in V$, then their join turns out to be

$$(i, n-1) \vee (h, k) = (i \vee h, n-1 \vee k) = (i \vee h, n-1) \in U.$$

The last case is that of $(i, j), (h, k) \in V$, yielding $0 \leq i \leq j+1$ and $0 \leq h \leq k+1$. Hence, we can observe that $0 \leq \max\{i, h\} \leq \max\{j+1, k+1\} = \max\{j, k\} + 1$, which actually shows that

$$(i, j) \vee (h, k) = (i \vee h, j \vee k) = (\max\{i, h\}, \max\{j, k\}) \in V.$$

Therefore, $S_n \leq \mathbf{C}_n \times \mathbf{C}_n$ and, by definition, it is clearly a special Hagemann relation of dimension $n-2$ induced by the partition $\{\{i\} : i \in n\}$ of n.

Thus, by the arbitrariness of $n \geq 3$, we have that $\mathcal{V}$ admits SHR_m, for every $m \geq 1$, where $m = n-2$. This means that $\mathcal{V}$ lies outside of $\bigcup_{1 \leq m < \omega} \Omega^{id}(SHR_m) = \Omega^{id}(SHR_\omega)$, as we aimed to prove. $\square$

The previous argument is essentially based on proving that the variety of distributive lattices admits SHR_n, for every $n \geq 1$; alternatively, we can provide another construction, which is a specialization of the one presented in Theorem 3.2.1, of a special Hagemann relation of dimension $n \geq 1$ and show how it can be built out of an $(n+2)$-chain, regardless of the language of the idempotent variety $\mathcal{V} \leq \mathcal{D}$ (in other words, such a special Hagemann relation is obtained by a primitive positive construction that is valid for every language, provided that the variety is idempotent). Even though we are not going to present the details of the proof, we want to provide the definition of such a relation.

With the same notation as the one used in the proof of Theorem 4.2.3, consider the interpreted n-chain $\mathbf{C}_n^{(\iota)} \in \mathcal{V}$, for $n \geq 3$. Without loss of generality, we may assume that $\mathbf{C}_n \in \mathcal{V}$. Then, consider the non-$(n-1)$-permuting congruences α and β of $\mathbf{C}_n$ such that $(0, n-1) \in (\beta \circ^{n-1} \alpha) - (\alpha \circ^{n-1} \beta)$. If we call $B = \beta \star^{n-3} \alpha$, then by Theorem 3.2.1, we have that the algebra $\mathbf{B}$ carries the $(n-2)$-dimensional Hagemann relation $R = B^2 \cap ((\beta \circ \alpha) \otimes^{n-2} (\alpha \circ \beta)) \leq \mathbf{B} \times \mathbf{B}$. Let us now define $\vec{x} = (x_0, \ldots, x_{n-3}), \vec{y} = (y_0, \ldots, y_{n-3}) \in B$ as the tuples having $x_i = i$ and $y_i = i+2$, for $0 \leq i \leq n-3$. By using these, define

$$E = \vec{x}/R \cap \vec{y}/R^{-1} \leq \mathbf{B}.$$

We could prove by a rather long sequence of computations (which will be omitted) that $R_{|E} \leq \mathbf{E} \times \mathbf{E}$ is a special Hagemann relation of dimension $n-2$ induced by $\{\{\vec{e}\} : \vec{e} \in E\}$.

Theorem 4.2.2 *per se* already provides a proof of the fact that "congruence n-permutability for some $n \geq 2$" is a prime Maltsev condition with respect to idempotent varieties, or equivalently CP_ω^{id} is a prime Maltsev filter. As a matter of fact, if $\mathcal{V}$ and $\mathcal{W}$ are two idempotent varieties which are not congruence n-permutable for any $n \geq 2$, then $\mathcal{V} \leq \mathcal{D}$ and $\mathcal{W} \leq \mathcal{D}$, which implies $\mathcal{V} \vee \mathcal{W} \leq \mathcal{D}$ (the join is the least upper bound), namely $\mathcal{V} \vee \mathcal{W}$ is not congruence n-permutable for any $n \geq 2$.

On the other hand, Theorem 4.2.3, which also represents a primeness argument for CP_ω^{id}, emphasizes the aspect of not being congruence n-permutable for every $n \geq 2$, which is equivalent to realizing a special Hagemann relation of dimension n, for every $n \geq 1$. We present this straightforward result as a corollary of Theorem 4.2.3.

Corollary 4.2.2. *CP_ω^{id} is a prime Maltsev filter in $\mathbf{L}^{id}$.*

Proof. We already know by Theorem 4.1.2 that $\Omega(SHR_n)$ is a prime Maltsev class, for every $n \geq 1$; in particular, the filter $\Omega^{id}(SHR_n) \leq \mathbf{L}^{id}$ satisfies the primeness property as well. If $\mathcal{V}, \mathcal{W}$ are two idempotent varieties failing to be congruence n-permutable for each $n \geq 2$, then by Theorem 4.2.3 they also admit SHR_m, for every $m \geq 1$. By Theorem 4.1.2, $\mathcal{V} \vee \mathcal{W} \notin \Omega(SHR_m)$, for every $m \geq 1$, meaning $\mathcal{V} \vee \mathcal{W} \notin \Omega^{id}(SHR_\omega) = CP_\omega^{id}$, validating the primeness property. □

This concluding corollary somewhat anticipates what the argument of the next chapter is going to be. In particular, we will be concentrating on some primeness arguments for specific cases of congruence n-permutability for some values of n.

Chapter 5

Some primeness arguments

In the previous chapter, we have dealt with special Hagemann relations which have allowed us to prove that congruence n-permutability for some $n \geq 2$ is a prime Maltsev condition when restricting to idempotent varieties. Also, we have considered singularly $\Omega(SHR_n)$, for $n \geq 1$, and proven that is a prime Maltsev condition as well. A plausible expectation would be that (at least in the idempotent case where the property expressed in Corollary 4.2.1 holds certainly) congruence n-permutability can be characterized by the omission of SHR_{n-1}, for $n \geq 2$: this would imply that CP_n (or CP_n^{id} within $\mathbf{L}^{id}$) is a prime strong Maltsev condition, for every $n \geq 2$.

For specific values of n or in some restricted contexts, we know that the property of being prime holds for CP_n. Historically, the first result in this sense was proven by S. Tschantz in his unpublished work [41], where he answered affirmatively the question asked in [15] about whether congruence 2-permutability is a prime condition.

Theorem 5.0.1 ([41])**.** *Congruence* 2*-permutability is a prime strong Maltsev condition.*

Tschantz's proof contains computationally heavy techniques arising from a syntactic approach which does not seem to be easily generalizable not even to the case $n = 3$. The power of this theorem is that such a result holds for a generic variety satisfying no particular hypothesis. Other attempts have been made in order to find a different and clearer proof, but unfortunately without succeeding for the general case, not even using semantic approaches. However, some partial results have been discovered recently by J. Opršal in [31] and by K. Kearnes and A. Szendrei in [23]. In both articles, the authors prove that congruence 2-permutability is a prime condition with respect to idempotent varieties, and they do so by looking at 2-permutability as a particular case of another Maltsev condition which is referred to as "having an n-cube term": indeed, a 2-cube term is a Maltsev term (or equivalently a Hagemann-Mitschke term for 2-permutability). Their proofs can be considered semantic in the sense that they manipulate suitable algebras so as to obtain some so called *cube term blockers* (see also [26]) or *compatible crosses*.

Furthermore, again J. Opršal in [31] proves another partial result which this time involves congruence n-permutability for fixed n: he shows that for given $n \geq 2$, being congruence n-permutable is a prime strong Maltsev condition with respect to linear varieties. In order to prove this, the author uses the technique of "colorability by a relational structure" presented in [4], which is a refinement of another technique defined by L. Sequeira in [36] and [37], called "compatibility with the projections", through which he provides, among other things, a measure (in a precise sense that we do not specify explicitly) of how complex it could be to prove syntactically that 2-permutability and 3-permutability are prime conditions.

To summarize, there exist semantic proofs of the primeness of CP_2^{id} within $\mathbf{L}^{id}$ and of CP_n, for fixed $n \geq 2$, with respect to linear varieties. We rephrase these claims in the following theorems.

Theorem 5.0.2 ([31],[23])**.** *CP_2^{id} is a prime filter in $\mathbf{L}^{id}$.*

Theorem 5.0.3 ([31]). *For fixed $n \geq 2$, if $\mathcal{V}, \mathcal{W}$ are linear varieties failing to be congruence n-permutable, then $\mathcal{V} \vee \mathcal{W}$ is not congruence n-permutable either.*

In other words, CP_n is a prime strong Maltsev filter in $\mathbf{L}$ when restricted to (the interpretability types of) linear varieties.

In the next sections, we will provide another proof of Theorem 5.0.2 by using a 1-dimensional special Hagemann relation instead of a 2-cube term blocker. We will also deal with the case of congruence 3-permutable varieties being idempotent and locally finite, and prove that, in such a case, a 2-dimensional special Hagemann relation is no longer sufficient for a characterization, for which we will need a slightly modified version of that.

5.1 The case of idempotent congruence 2-permutability

We have already mentioned that congruence 2-permutability has been known to be a prime strong Maltsev condition since Tschantz's Theorem 5.0.1 was proven, although never published.

The approaches taken in [31] and [23] consider congruence 2-permutability as a demonstration of the strong Maltsev condition "having a 2-cube term". More generally, the strong Maltsev condition certifying the existence of an n-cube term was first defined in [6]; precisely, given a variety $\mathcal{V}$, for $n \geq 2$, an *n-cube term* for $\mathcal{V}$ is a $(2^n - 1)$-ary term $c \in F_{\mathcal{V}}(\{x_1, \dots, x_{2^n-1}\})$ such that, if $\{x, y\}^n - \{y\}^n = \{\vec{v}_1, \dots, \vec{v}_{2^n-1}\}$, then

$$c^{\mathbf{F}^n_{\mathcal{V}}(x,y)}(\vec{v}_1, \dots, \vec{v}_{2^n-1}) \in \{y\}^n.$$

Therefore a variety $\mathcal{V}$ has a 2-cube term if and only if there exists a ternary term c of $\mathcal{V}$ satisfying

$$c^{\mathbf{F}^2_{\mathcal{V}}(x,y)}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix},$$

implying that the equations $c(x, x, y) \approx y$ and $c(x, y, x) \approx y$ hold in $\mathcal{V}$. By flipping the first two variables in c, we get the term $p_1(x, y, z) := c(y, x, z)$ satisfying

$$y \approx p_1(y, x, x) \text{ and } p_1(x, x, y) \approx y.$$

Hence p_1 is a Hagemann-Mitschke term for congruence 2-permutability (i.e. a Maltsev term).

Our perspective, instead, will produce a proof for idempotent congruence 2-permutability by characterizing it via the omission of special Hagemann relations of dimension 1, or equivalently, as we will show, via the omission of algebras with a particular configuration. The idea of using 1-dimensional special Hagemann relations for this purpose comes from Lemma 2 of [8], where the authors essentially build a special Hagemann relation of dimension 1 out of a finite idempotent non-congruence 2-permutable algebra. Indeed, we generalize this fact by dropping the assumption of finiteness.

Let us then define what could be considered the simplest configuration of a non-congruence 2-permutable algebra.

Definition 5.1.1. Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be similar algebras such that $\mathbf{A} \leq_{sd} \mathbf{B} \times \mathbf{C}$. We say that $\mathbf{A}$ is a *special failure of congruence 2-permutability*, briefly SF_2, if there exist proper subsets $P \subsetneq B$ and $Q \subsetneq C$ such that

$$A = (P \times C) \cup (B \times Q).$$

It is clear by Definition 5.1.1 that a special failure of congruence 2-permutability $\mathbf{A}$ is a non-congruence 2-permutable algebra. As a matter of fact, if $\mathbf{A} \leq_{sd} \mathbf{B} \times \mathbf{C}$ as in the definition, since $P \subsetneq B$ and $Q \subsetneq C$, there have to exist $b \in B - P$ and $c \in C - Q$. Call α and β the kernels of the projection maps onto, respectively, B and C, with domain restricted to A, or in other words

$$\alpha = (0_B \otimes 1_C) \cap A^2,$$

$$\beta = (1_B \otimes 0_C) \cap A^2$$

Also, let $p \in P$ and $q \in Q$ be any two elements (which exist because P and Q are non-empty). Then we have $(p, c) \in P \times C \subseteq A$, $(p, q) \in P \times Q \subseteq A$, $(b, q) \in B \times Q \subseteq A$ and hence

$$\begin{bmatrix} p \\ c \end{bmatrix} \alpha \begin{bmatrix} p \\ q \end{bmatrix} \beta \begin{bmatrix} b \\ q \end{bmatrix}.$$

Yet, $(p, c) \not\mathrel{\beta \circ \alpha} (b, q)$, because otherwise the following would hold

$$\begin{bmatrix} p \\ c \end{bmatrix} \beta \begin{bmatrix} b \\ c \end{bmatrix} \alpha \begin{bmatrix} b \\ q \end{bmatrix},$$

implying $(b, c) \in A$, which is to say $b \in P$ or $c \in Q$, contradicting the initial assumptions on them.

A special failure of congruence 2-permutability is indeed in a deep connection with 1-dimensional special Hagemann relations, not only in the idempotent case, but in the general one, as we will see in the next theorem. Before that, we need the following definition

Definition 5.1.2. We say that a variety $\mathcal{V}$ *admits* SF_2, if $\mathcal{V}$ contains a special failure of congruence 2-permutability. We say that $\mathcal{V}$ *omits* SF_2 otherwise.

The class of varieties omitting SF_2 is denoted by $\Omega(SF_2)$, and the subclass of $\Omega(SF_2)$ containing idempotent varieties is denoted by $\Omega^{id}(SF_2)$.

With this notation, we have

Theorem 5.1.1. *For any variety $\mathcal{V}$, $\mathcal{V}$ admits SHR_1 if and only if it admits SF_2.*

In other words,

$$\Omega(SHR_1) = \Omega(SF_2).$$

Proof. Let $\mathcal{V}$ be any variety.

First, suppose $\mathcal{V}$ admits SHR_1, i.e there exists $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$, such that R is a 1-dimensional special Hagemann relation. Let $\{A_0, A_1, A_2\}$ be the partition of A inducing R, which is to say

$$R = [(A_0 \cup A_1) \times A] \cup [A_2 \times (A_1 \cup A_2)].$$

Since R is reflexive, in fact $R \leq_{sd} \mathbf{A} \times \mathbf{A}$. Call $P := A_0 \cup A_1$ and $Q := A_1 \cup A_2$ and notice that $\varnothing \neq P, Q \subsetneq A$, by the fact that $\{A_0, A_1, A_2\}$ is a partition. Moreover:

$$R = [(A_0 \cup A_1) \times A] \cup [A \times (A_1 \cup A_2)] = [P \times A] \cup [A \times Q],$$

proving that $\mathbf{R}$ itself is a special failure of congruence 2-permutability.

Conversely, let $\mathbf{A} \leq_{sd} \mathbf{B} \times \mathbf{C}$ be a special failure of congruence 2-permutability, for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$, and let P and Q be the proper subsets of, respectively, B and C with

$$A = (P \times C) \cup (B \times Q).$$

Define the following three subsets of A:

$$A_0 := P \times (C - Q);$$

$$A_1 := P \times Q;$$

$$A_2 := (B - P) \times Q.$$

CLAIM 5.1.1.1. $\{A_0, A_1, A_2\}$ is a partition of A.

Proof. Since P and Q are proper subsets of B and C respectively, then in particular P, Q, $B - P$ and $C - Q$ are non-empty, and so are A_0, A_1 and A_2. Moreover,

$$\begin{aligned} A_0 \cup A_1 \cup A_2 &= [P \times (C - Q)] \cup [P \times Q] \cup [(B - P) \times Q] = \\ &= ([P \times (C - Q)] \cup [P \times Q]) \cup ([P \times Q] \cup [(B - P) \times Q]) = \\ &= (P \times C) \cup (B \times Q) = A. \end{aligned}$$

Finally, since $P \cap (B - P) = \varnothing$ and $Q \cap (C - Q) = \varnothing$, then A_0, A_1 and A_2 are mutually disjoint. $\square$

If π_B and π_C denote the projection maps from $B \times C$ onto, respectively, B and C, then call

$$\alpha := \ker \pi_{B|A} = (0_B \otimes 1_C) \cap A^2,$$

$$\beta := \ker \pi_{C|A} = (1_B \otimes 0_C) \cap A^2.$$

Furthermore, define

$$R := \alpha \circ \beta \leq \mathbf{A} \times \mathbf{A}.$$

CLAIM 5.1.1.2. R is a special Hagemann relation of dimension 1 induced by $\{A_0, A_1, A_2\}$.

Proof. Let $(p, c) \in A_0 \cup A_1$ and $(x, y) \in A$. In such a case, $p \in P$ and hence $(p, y) \in P \times C \subseteq A$, implying

$$\begin{bmatrix} p \\ c \end{bmatrix} \ \alpha \ \begin{bmatrix} p \\ y \end{bmatrix} \ \beta \ \begin{bmatrix} x \\ y \end{bmatrix},$$

namely $(p, c)\ R\ (x, y)$. This proves $(A_0 \cup A_1) \times A \subseteq R$.

If $(b, q) \in A_2$ and $(x, y) \in A_1 \cup A_2$, then $q, y \in Q$ which yields $(b, y) \in B \times Q \subseteq A$.

Therefore,

$$\begin{bmatrix} b \\ q \end{bmatrix} \ \alpha \ \begin{bmatrix} b \\ y \end{bmatrix} \ \beta \ \begin{bmatrix} x \\ y \end{bmatrix},$$

meaning $(b, q)\ R\ (x, y)$ and proving $A_2 \times (A_1 \cup A_2) \subseteq R$.

Conversely, assume there exist $(b, q) \in A_2$ and $(p, c) \in A_0$ such that $(b, q)\ R\ (p, c)$. This is to say there exists $(u, v) \in A$ such that

$$\begin{bmatrix} b \\ q \end{bmatrix} \ \alpha \ \begin{bmatrix} u \\ v \end{bmatrix} \ \beta \ \begin{bmatrix} p \\ c \end{bmatrix}.$$

By definition of α and β, we have that $(u, v) = (b, c)$ which also yields that $(b, c) \in A$. Yet, this is a contradiction because $(b, q) \in A_2$ implies $b \in B - P$, whereas $(p, c) \in A_0$ implies $c \in C - Q$, i.e. $(b, c) \notin A$.

Therefore, we have that $R = [(A_0 \cup A_1) \times A] \cup [A_2 \times (A_1 \cup A_2)]$, as desired. □

The previous claim shows that $\mathbf{A} \in \mathcal{V}$ carries the special Hagemann relation of dimension 1 R. The equality $\Omega(SHR_1) = \Omega(SF_2)$ is a direct consequence of the reasonings above. □

As a result, we deduce that $\Omega(SF_2)$ is also a Maltsev class. Moreover, notice that a congruence 2-permutable variety $\mathcal{V}$ always omits SF_2, or equivalently SHR_1, showing that $CP_2 \subseteq \Omega(SF_2) = \Omega(SHR_1)$. We do not know whether in general this former inclusion is strict or, in fact, the equality holds[1]: if so, we would have a new proof of Theorem 5.0.1. When a variety $\mathcal{V}$ fails to be congruence 2-permutable, a failure of it could look pretty wild, in the sense that it could be very far from having the nice shape of a special failure. However, when the variety $\mathcal{V}$ is idempotent, then we can certainly find within $\mathcal{V}$ a failure which is special. This result is essentially contained in Lemma 2.8 of [24]: K. Kearnes and S. Tschantz build a special failure of congruence 2-permutability as a subdirect power of the squared of the 2-generated free algebra in any idempotent non-congruence 2-permutable variety. We will include a slightly modified and more detailed proof of Lemma 2.8[24] in the proof of the next theorem.

Theorem 5.1.2. *Let $\mathcal{V}$ be an idempotent variety. Then, the following statements are equivalent.*

1. *$\mathcal{V}$ is congruence 2-permutable;*
2. *$\mathcal{V}$ omits SF_2;*
3. *$\mathcal{V}$ omits SHR_1.*

[1]See Appendix B, added after the external reviewer's comments.

Proof. We already know that (2) and (3) are equivalent due to Theorem 5.1.1. In addition, as already noted previously, a congruence 2-permutable variety obviously contains no failures of congruence 2-permutability, in particular it omits SF_2. This proves that (1) implies (2). Therefore, the only missing part is the implication (2)$\Rightarrow$(1).

(2)$\Rightarrow$(1)[modification of the proof of Lemma 2.8[24]]: Let us argue by contrapositive, namely let us assume $\mathcal{V}$ is an idempotent variety which is not congruence 2-permutable. Denote by $\mathbf{F}$ the free algebra of $\mathcal{V}$ generated by $\{x, y\}$. By Corollary 3.1.2, we have that $R^{-1} \not\subseteq R$, because $(y, x) \notin R$, where R is the subuniverse of $\mathbf{F} \times \mathbf{F}$ generated by $\{(x, x), (x, y), (y, y)\}$, i.e.

$$R = \mathrm{Sg}^{\mathbf{F}^2}(\{(x, x), (x, y), (y, y)\}) = \mathrm{Sg}^{\mathbf{F}^2}([\{x\} \times \{x\}] \cup \{(x, y), (y, y)\}).$$

There is a reason why we have written R in two equivalent ways, which is going to appear clearer soon.

Let us first notice that, for every $t \in F$, $(x, t) \in R$ and $(t, y) \in R$. This holds because, by idempotence,

$$x = t^{\mathbf{F}}(x, x) \; R \; t^{\mathbf{F}}(x, y) = t,$$
$$t = t^{\mathbf{F}}(x, y) \; R \; t^{\mathbf{F}}(y, y) = y.$$

Let us define a family of subuniverses of $\mathbf{F} \times \mathbf{F}$. For any $P \leq \mathbf{F}$ call

$$R_P := \mathrm{Sg}^{\mathbf{F}^2}([P \times \{x\}] \cup \{(x, y), (y, y)\}).$$

With this notation, $R = R_{\{x\}}$ (note $\{x\} \leq \mathbf{F}$ by idempotence). Furthermore, if $x \in P$, then $R \subseteq R_P$, and the following also holds: for every $p, q \in F$,

$$\text{if } (p, x) \in R_P \text{ then } (p, q) \in R_P,$$

$$\text{if } (y, q) \in R_P \text{ then } (p, q) \in R_P.$$

As a matter of fact,

$$p = q^{\mathbf{F}}(p, p) \; R_P \; q^{\mathbf{F}}(x, y) = q,$$
$$p = p^{\mathbf{F}}(x, y) \; R_P \; q^{\mathbf{F}}(q, q) = q.$$

At this point, define the following set of subuniverses of $\mathbf{F}$, call it H:

$$H := \{P \leq \mathbf{F} : x \in P \text{ and } (y, x) \notin R_P\}.$$

H is definitely non-empty because it contains $\{x\}$. Moreover, if we consider the poset $\langle H; \subseteq \rangle$, then every totally ordered subset of H has a maximal element. Indeed, if $\{P_i : i < \kappa\} \subseteq H$ forms a chain, then the set

$$P := \bigcup_{i<\kappa} P_i$$

is a subuniverse, contains x and satisfies $(y, x) \notin R_P$, because otherwise we could deduce that $(y, x) \in R_{P_i}$, for some $i < \kappa$, which is a contradiction.

By Zorn's Lemma, we can find a $\subseteq$-maximal element in H, which we call P. For such a maximal P, the relation R_P inherits a special configuration.

CLAIM 5.1.2.1. $R_P = (P \times F) \cup (F \times y/R_P)$.

Proof. Let us first prove the inclusion $\supseteq$: if $p \in P$, then $(p, x) \in P \times \{x\} \subseteq R_P$, which implies $(p, q) \in R_P$, for each $q \in F$, as we observed above. This means $P \times F \subseteq R_P$.

Analogously, if $q \in y/R_P$, i.e. $(y, q) \in R_P$, then for each $p \in F$, $(p, q) \in R_P$, proving that also $F \times y/R_P \subseteq R_P$.

The crucial part of the whole proof is what we present next. We show that if a pair lies in R_P, then it must belong to $(P \times F) \cup (F \times y/R_P)$. Hence, consider any $(a, b) \in R_P$ and assume that $a \notin P$.

Define

$$P_a := \mathrm{Sg}^{\mathbf{F}}(P \cup \{a\}).$$

Before proceeding further, we need to point out that R_{P_a} can be described easily as

$$R_{P_a} = \mathrm{Sg}^{\mathbf{F}^2}([(P \cup \{a\}) \times \{x\}] \cup \{(x, y), (y, y)\}).$$

To prove this, it is clear that the generators of the subuniverse on the right hand side are contained in R_{P_a}, implying that the right-to-left inclusion holds.

Conversely, any pair (c, d) in R_{P_a} is of the form

$$\begin{bmatrix} c \\ d \end{bmatrix} = t^{\mathbf{F}^2}\left(\begin{bmatrix} t_1^{\mathbf{F}}(p_{1,1}, \ldots, p_{1,k_1}, a) \\ x \end{bmatrix}, \ldots, \begin{bmatrix} t_m^{\mathbf{F}}(p_{m,1}, \ldots, p_{m,k_m}, a) \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right)$$

for suitable terms $t, t_1, \ldots, t_m$ of $\mathcal{V}$ and $p_{i,j}$'s from P. By idempotency, though, we can rewrite the previous expression as

$$\begin{bmatrix} c \\ d \end{bmatrix} = t^{\mathbf{F}^2}\left(t_1^{\mathbf{F}^2}\left(\begin{bmatrix} p_{1,1} \\ x \end{bmatrix}, \ldots, \begin{bmatrix} p_{1,k_1} \\ x \end{bmatrix}, \begin{bmatrix} a \\ x \end{bmatrix}\right), \ldots, t_m^{\mathbf{F}^2}\left(\begin{bmatrix} p_{m,1} \\ x \end{bmatrix}, \ldots, \begin{bmatrix} p_{m,k_m} \\ x \end{bmatrix}, \begin{bmatrix} a \\ x \end{bmatrix}\right), \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right).$$

Therefore, if we consider the term $r(\vec{u}_1, \ldots, \vec{u}_m, x, y, z) := t(t_1(\vec{u}_1, x), \ldots, t_m(\vec{u}_m, x), y, z)$, we have that $(c, d) \in r^{\mathbf{F}^2}([(P \cup \{a\}) \times \{x\}] \cup \{(x, y), (y, y)\})$, as expected.

We are then ready to argue the final part of this proof. Because $a \notin P$, obviously $P \subsetneq P_a$. Moreover, since $x \in P_a$, by maximality of P, $P_a \notin H$, yielding that $(y, x) \in R_{P_a}$. Due to the previous observation, this means there exists a term t of $\mathcal{V}$, and $p_1, \ldots, p_k \in P$, for some $k \geq 1$, such that

$$\begin{bmatrix} y \\ x \end{bmatrix} = t^{\mathbf{F}^2}\left(\begin{bmatrix} p_1 \\ x \end{bmatrix}, \ldots, \begin{bmatrix} p_k \\ x \end{bmatrix}, \begin{bmatrix} a \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right).$$

This is to say, $y = t^{\mathbf{F}}(p_1, \ldots, p_k, a, x, y)$ and $x = t^{\mathbf{F}}(x, \ldots, x, x, y, y)$. Since these equalities hold in the free algebra, then they hold in the whole variety, in particular the second identity

$$x \approx t(\underbrace{x, \ldots, x, x}_{k+1 \text{ times}}, y, y).$$

If we substitute b at x in this latter equation, we get that $b = t^{\mathbf{F}}(b, \ldots, b, b, y, y)$, and hence

$$\begin{bmatrix} y \\ b \end{bmatrix} = t^{\mathbf{F}^2}\left(\begin{bmatrix} p_1 \\ b \end{bmatrix}, \ldots, \begin{bmatrix} p_k \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right) \in R_P,$$

because $(a, b) \in R_P$ by assumption and $P \times F \subseteq R_P$, $(x, y), (y, y) \in R_P$. This proves that $b \in y/R_P$, i.e. $(a, b) \in F \times y/R_P$. □

Therefore, the algebra $\mathbf{R}_P \leq_{sd} \mathbf{F} \times \mathbf{F}$, and if we call $Q := y/R_P$, the previous claim shows that $R_P = (P \times F) \cup (F \times Q)$. Moreover, the fact that $(y, x) \notin R_P$ yields $y \notin P$ and $x \notin Q$, meaning $P, Q \subsetneq F$ and showing that $\mathbf{R}_P$ is a special failure of congruence 2-permutability in $\mathcal{V}$, concluding the proof. □

This theorem has a direct corollary which has been indeed the main aim of this section.

Corollary 5.1.1. *In the lattice of idempotent interpretability types* $\mathbf{L}^{id}$*,*

$$CP_2^{id} = \Omega^{id}(SHR_1) = \Omega^{id}(SF_2).$$

As a result, CP_2^{id} *is a prime filter in* $\mathbf{L}^{id}$*.*

Proof. Theorem 5.1.2 essentially yields the equalities of the classes displayed, and since $\Omega(SHR_1)$ is a prime Maltsev filter by Theorem 4.1.2, then so is $\Omega^{id}(SHR_1) = CP_2^{id}$. □

Also, this is an indirect proof of the fact that $\Omega(SHR_1)$ is a strong Maltsev class with respect to idempotent varieties.

We close this section with another observation on congruence 2-permutable idempotent varieties which is a consequence of the characterization given in Theorem 5.1.2.

In [22], K. Kearnes presents a characterization of congruence n-permutable varieties in terms of tolerance identities satisfied by the tolerance lattices of the members of the varieties themselves. In particular, for congruence 2-permutability, the characterization claims that a variety $\mathcal{V}$ is in CP_2 if and only if $T \circ T \subseteq T$, for every tolerance T of any algebra in $\mathcal{V}$. Theorem 5.1.2 indeed yields that an idempotent variety failing congruence 2-permutability realizes an algebra having a tolerance T for which the property $T \circ T \supsetneq T$ is witnessed in a precise way.

Corollary 5.1.2. *Let $\mathcal{V}$ be an idempotent variety.*

$\mathcal{V}$ is not congruence 2-permutable if and only if there exist $\mathbf{A} \in \mathcal{V}$ and a partition $\{A_0, A_1, A_2\}$ of $\mathbf{A}$ such that $T = (A_0 \cup A_1)^2 \cup (A_1 \cup A_2)^2 \in \mathrm{Tol}(\mathbf{A})$.

Proof. Suppose that $\mathcal{V}$ is a non-congruence 2-permutable idempotent variety. By Theorem 5.1.2, $\mathcal{V}$ admits SHR_1, i.e. there exist $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$ such that R is a 1-dimensional special Hagemann relation. Let also $\{A_0, A_1, A_2\}$ be the partition of A inducing R.

Define the relation T as

$$T = R \cap R^{-1}.$$

By definition T is a subuniverse of $\mathbf{A}^2$, it is reflexive because R and R^{-1} are and it is obviously symmetric, namely T is a tolerance on $\mathbf{A}$. Moreover

$$T = R \cap R^{-1} = [[(A_0 \cup A_1) \times A] \cup [A_2 \times (A_1 \cup A_2)]] \cap [[A \times (A_0 \cup A_1)] \cup [(A_1 \cup A_2) \times A_2]] =$$

$$= (A_0 \cup A_1)^2 \cup (A_1 \times A_2) \cup (A_2 \times A_1) \cup A_2^2 = (A_0 \cup A_1)^2 \cup (A_1 \cup A_2)^2.$$

Conversely, if $\mathbf{A} \in \mathcal{V}$ has a tolerance T satisfying the condition in the statement, then $T \circ T = A^2 \supsetneq T$, since $A_0 \times A_2 \subseteq A^2 - T$, proving that $\mathcal{V}$ is not congruence 2-permutable. $\square$

Unfortunately, the last results of this section have been found by making a heavy use of idempotency and hence they look pretty far from being able to be extended also to the non-idempotent case.

In the next section, we will deal with the case of congruence 3-permutability and we will see that the situation becomes unexpectedly more complicated than 2-permutability, even only restricting to locally finite idempotent varieties.

5.2 The case of locally finite idempotent congruence 3-permutability

In this section we will analyze the strong Maltsev condition of congruence 3-permutability and for the first time we will provide a primeness argument that is valid for idempotent and locally finite varieties.

Although one could expect that such a primeness argument is likely an almost trivial generalization of what is shown in 5.1.2, it is not even the case that 3-permutability can be captured by the omission of 2-dimensional special Hagemann relations, not even in the locally finite idempotent restriction, as we will justify later on in this current section.

More generally, congruence 3-permutability has been thoroughly studied in the past decades and many interesting results have been found about it. Among these, one particularly interesting theorem was proven by R. Wille in [43] which characterizes congruence 3-permutable varieties in the following way.

Theorem 5.2.1 ([43])**.** *Let $\mathcal{V}$ be a variety. Then $\mathcal{V}$ is congruence 3-permutable if and only if for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and for each surjective homomorphism $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$, $\alpha \in \mathrm{Con}\,\mathbf{A}$ implies also $(h \times h)(\alpha) \in \mathrm{Con}\,\mathbf{B}$.*

As a matter of fact, if **A** and **B** are two similar algebras and h is an onto homomorphism from **A** to **B**, then the relation

$$(h \times h)(\alpha) = \{(h(u), h(v)) : (u, v) \in \alpha\}$$

need not be a congruence of **B** whenever α is a congruence of **A**, because it might fail transitivity. Yet, congruence 3-permutability for the variety $\mathcal{V}$ does force what the above theorem states. This very strong and peculiar property isolates in some sense all congruence 3-permutable (and in particular congruence 2-permutable) varieties.

This is not the only property distinguishing congruence 2 and 3-permutability from congruence n-permutability for every $n \geq 4$. Perhaps, one of the most characteristic features of congruence 3-permutability for a variety is that of implying congruence modularity.

Theorem 5.2.2 ([20])**.** *For any variety $\mathcal{V}$, if $\mathcal{V}$ is congruence 3-permutable, then it is congruence modular.*

On the other hand, there exist congruence 4-permutable varieties which are not congruence modular (see, for example, [32]), preventing Theorem 5.2.2 from being generalized to further levels of n-permutability.

Indeed, Theorem 5.2.2 and other facts that we will expose later, have led us to making reasonable considerations and conjectures regarding the primeness problem for congruence n-permutability for $n \geq 4$: we will discuss such topics in the next section.

After the previous brief observations about 3-permutability, let us concentrate on showing the main result of this section which is a proof of the fact that congruence 3-permutability is a prime strong Maltsev condition with respect to locally finite idempotent varieties. In order to get to that point, we need to present some preliminary definitions and results that will eventually end up establishing a primeness argument.

Likewise for the case of 2-permutability, it is worth considering some particular failures of congruence 3-permutability which are to play the same role as the ones defined in Definition 5.1.1.

Definition 5.2.1. Let $\mathbf{S}, \mathbf{P}$ and $\mathbf{Q}$ be similar algebras such that **S** is a subdirect product of $\mathbf{P} \times \mathbf{Q}$. We say that **S** is a *special failure of congruence 3-permutability* if there exist

- a partition $\{X, Y, Z\}$ of P;
- a partition $\{U, W\}$ of $Q - V$, for some (potentially empty) $V \subsetneq Q$;

such that

$$S = (X \times U) \cup (Y \times Q) \cup (Z \times W).$$

Pictorially:

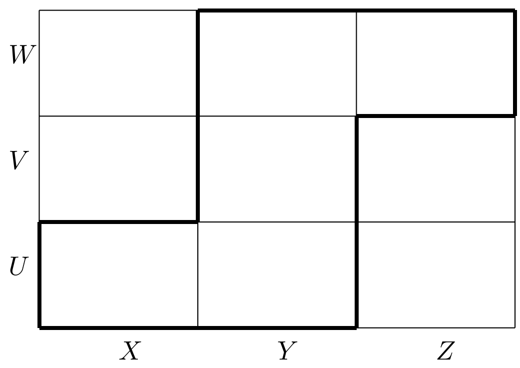

Figure 5.1: A special failure of congruence 3-permutability

If $V = \varnothing$, we say that **S** is a *special failure of congruence 3-permutability without middle portion*; otherwise, **S** is a *special failure of congruence 3-permutability with middle portion* and, in such a case, V is said to be the *middle portion*.

Notice that, in accordance with the name, the existence of a special failure of congruence 3-permutability implies the failure of congruence 3-permutability, that is the existence of an algebra carrying two congruences that do not 3-permute.

In order to see this, let $\mathbf{S}$ be a special failure of congruence 3-permutability as defined in Definition 5.2.1, and call

$$\alpha := (0_P \otimes 1_Q) \cap S^2,$$
$$\beta := (1_P \otimes 0_Q) \cap S^2,$$

i.e. the restrictions to S of the kernels of the projection maps from $P \times Q$ onto, respectively, P and Q.

Furthermore, notice that for every $(x, u) \in X \times U$ and $(z, w) \in Z \times W$, there exists $y \in Y$ such that

$$\begin{bmatrix} x \\ u \end{bmatrix} \;\beta\; \begin{bmatrix} y \\ u \end{bmatrix} \;\alpha\; \begin{bmatrix} y \\ w \end{bmatrix} \;\beta\; \begin{bmatrix} z \\ w \end{bmatrix},$$

but $(x, u) \; \cancel{\alpha \circ \beta \circ \alpha} \; (z, w)$, because otherwise the following would hold

$$\begin{bmatrix} x \\ u \end{bmatrix} \;\alpha\; \begin{bmatrix} x \\ q \end{bmatrix} \;\beta\; \begin{bmatrix} z \\ q \end{bmatrix} \;\alpha\; \begin{bmatrix} z \\ w \end{bmatrix},$$

for some $q \in Q$ such that $(x, q), (z, q) \in S$. This is to say $q \in U \cap W$, which contradicts the fact that $\{U, W\}$ is a partition of $Q - V$ (notice that this reasoning holds regardless of V being empty or not). Hence α and β cannot 3-permute, making $\mathbf{S}$ a failure of congruence 3-permutability.

Unlike the case of congruence 2-permutability, here we need to consider two similar but formally different special failures of congruence 3-permutability: this is the first complication arising from the passage from $n = 2$ to $n = 3$ in the current analysis of n-permutability.

The notion of a special failure of congruence 3-permutability is related (in a sense that we will show later) to the notion of a special Hagemann relation of dimension 2: by Definition 4.1.1, we already know what such an object is, but in this context we will also need to deal with a generalized version of that. Therefore, we are going to define the notion of a generalized 2-dimensional special Hagemann relation in a way that includes Definition 4.1.1($n = 2$) as a particular case.

Definition 5.2.2. Let $\mathbf{A}$ be an algebra and $R \leq \mathbf{A} \times \mathbf{A}$. We say that R is a *generalized* 2-*dimensional special Hagemann relation* (or *generalized special Hagemann relation of dimension* 2) if there exists a partition $\{A_0, A_1, A_2, A_3\}$ of $A - M$, for some (potentially empty) $M \subsetneq A$, such that

$$R = [(A_0 \cup A_1) \times A] \cup [(M \cup A_2) \times (A - A_0)] \cup [A_3 \times (A_2 \cup A_3)].$$

Pictorially:

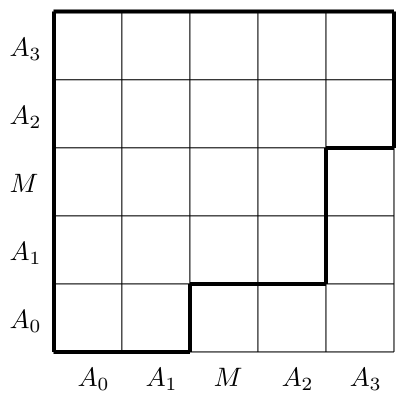

Figure 5.2: A generalized special Hagemann relation of dimension 2

Whenever M is empty, we say that R is a *generalized* 2-*dimensional special Hagemann relation without middle part*; otherwise, R is a *generalized* 2-*dimensional special Hagemann relation with middle part*, where M is the *middle part*.

It is straightforward to observe that a generalized 2-dimensional special Hagemann relation without middle part is exactly a 2-dimensional special Hagemann relation as presented in Definition 4.1.1. On the other hand, if R is a generalized special Hagemann relation of dimension 2 with middle part, then notice that $\{A_0, A_1, M, A_2, A_3\}$ is a partition of A.

To be precise, referring to Definition 5.2.2, let us define

$$\rho_R^{\pm} := \begin{cases} \rho_R^{+} = \{A_0, A_1, M, A_2, A_3\} & \text{if } M \neq \varnothing; \\ \rho_R^{-} = \{A_0, A_1, A_2, A_3\} & \text{if } M = \varnothing. \end{cases}$$

In either case, we say that the partition $\rho_R^{\pm}$ *induces* R.

Using the same notation as in previous chapters, let us define the following concepts and omission classes

Definition 5.2.3. We say that a variety $\mathcal{V}$ *admits* $M^{-}SF_3$ (resp. $M^{+}SF_3$), if there exists $\mathbf{S} \in \mathcal{V}$ which is a special failure of congruence 3-permutability without (resp. with) middle portion.

Similarly, a variety $\mathcal{V}$ *admits* $M^{-}SHR_2$ (resp. $M^{+}SHR_2$), if there is $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$, such that R is a generalized special Hagemann relation of dimension 2 without (resp. with) middle part.

Furthermore, a variety $\mathcal{V}$ *omits* $\bigstar \in \{M^{-}SF_3, M^{+}SF_3, M^{-}SHR_2, M^{+}SHR_2\}$ if $\mathcal{V}$ does not admit $\bigstar$, and the class of varieties omitting $\bigstar$ is denoted by $\Omega(\bigstar)$.

Finally, we denote by $\Omega^{id}(\bigstar)$ the class of idempotent varieties omitting $\bigstar$.

As already observed, $\Omega(M^{-}SHR_2) = \Omega(SHR_2)$, which has been formerly proven in Theorem 4.1.1 to be a Maltsev class. An analogous feature can be verified for the omission class of $M^{+}SHR_2$, as the next theorem states. Nonetheless, we do not know how the two classes $\Omega(M^{-}SHR_2)$ and $\Omega(M^{+}SHR_2)$ interact (nor do we know for $\Omega(M^{-}SF_3)$ and $\Omega(M^{+}SF_3)$), although we can describe such an interaction as far as $\Omega^{id}(M^{+}SHR_2)$ and $\Omega^{id}(M^{-}SHR_2)$ are concerned. As a matter of fact, for these we have been able to argue that one class is properly contained in the other (precisely, the omission of $M^{+}SHR_2$ implies the omission of $M^{-}SHR_2$, for idempotent varieties), as will be shown further on. Let us first focus on proving that omitting $M^{+}SHR_2$ has a Maltsev characterization.

Theorem 5.2.3. *$\Omega(M^{+}SHR_2)$ is a Maltsev class. In other words, for a variety, omitting a generalized 2-dimensional special Hagemann relation with middle part is equivalent to satisfying a Maltsev condition.*

Proof. The proof of this fact is in most parts the same as the one of Theorem 4.1.1; hence, we will not specify all the details but only those points which differ from that proof.

Regarding the closure under equi-interpretable varieties and subvarieties, it is still obvious that $\Omega(M^{+}SHR_2)$ meets such requirements. Furthermore, we can still show the validity of the closure under finite products via the same strategy as in the proof of Theorem 4.1.1: we can first verify that, given any two varieties $\mathcal{V}, \mathcal{W}$, if $R \leq \mathbf{Q}^2$ is a generalized special Hagemann relation of dimension 2 with middle part for some $\mathbf{Q} \cong \mathbf{U} \otimes \mathbf{V} \in \mathcal{V} \otimes \mathcal{W}$ (which is to say $\mathbf{U} \in \mathcal{V}$ and $\mathbf{V} \in \mathcal{W}$), then $\mathbf{R} \cong \mathbf{R}_1 \otimes \mathbf{R}_2$, for some $R_1 \leq \mathbf{U} \times \mathbf{U}$ and $R_2 \leq \mathbf{V} \times \mathbf{V}$. Eventually, it turns out that either $R_1 = U^2$ or $R_2 = V^2$, implying that either $\mathbf{R} \cong \mathbf{R}_1 \otimes \mathbf{V}^2$ or $\mathbf{R} \cong \mathbf{U}^2 \otimes \mathbf{R}_2$.

In order to do so, we assume without loss of generality that $\mathbf{Q} = \mathbf{U} \otimes \mathbf{V}$ and that R is induced by the partition $\rho_R^{+} = \{Q_0, Q_1, M, Q_2, Q_3\}$ (M is the middle part) of Q. We further suppose that $R_2 \neq V^2$ and we aim to show that necessarily $R_1 = U^2$. Thus, pick $(v, v') \in V^2 - R_2$ and any $(u_1, u_2) \in U^2$: if we consider the pair $\begin{bmatrix} u_2 \\ v \end{bmatrix}$, to avoid the contradiction that $(v, v') \in R_2$, it must be the case that

$$\begin{bmatrix} u_2 \\ v \end{bmatrix} \in M \cup Q_2 \cup Q_3.$$

Therefore, if we take the pair $\begin{bmatrix} u_1 \\ v' \end{bmatrix}$, in order to avoid the contradiction $(v, v') \in R_2$, then we need deduce that $\begin{bmatrix} u_1 \\ v' \end{bmatrix} \in Q_0 \cup Q_1 \cup M$. Since $(Q_0 \cup Q_1 \cup M) \times M \cup Q_2 \cup Q_3 \subseteq R$, we have that

$$\begin{bmatrix} u_1 \\ v' \end{bmatrix} \; R \; \begin{bmatrix} u_2 \\ v \end{bmatrix},$$

which implies $(u_1, u_2) \in R_1$, and hence $R_1 = U^2$.

An analogous reasoning leads, from assuming $R_1 \neq U^2$, to proving $R_2 = V^2$.

From now on, the proof almost coincides with the corresponding part of the proof of Theorem 4.1.1: for instance, by assuming that $R = R_1 \otimes V^2$, we get that $R_1 \leq \mathbf{U} \times \mathbf{U}$ is indeed a generalized 2-dimensional special Hagemann relation with middle part, proving that $\mathcal{V}$ admits M^+SHR_2. Likewise for the symmetric circumstance.

Finally, as far as the "finite presentability" is concerned, a compactness argument (which we omit) yields the thesis: the difference from Theorem 4.1.1 is in the fact that we need to specify all the properties of the middle part by using the following language

$$\mathcal{L} = F \cup \{R^{(2)}\} \cup \{A_0^{(1)}, A_1^{(1)}, M^{(1)}, A_2^{(1)}, A_3^{(1)}\},$$

and the following first order sentences

$$\forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_k \left[\left(\bigwedge_{i=1}^{k} R(x_i, y_i) \right) \to R(f(x_1, \ldots, x_k), f(y_1, \ldots, y_k)) \right],$$

for all $f \in F$ of arity k;

$$\forall x \left[M(x) \vee \bigvee_{i=0}^{3} A_i(x) \right];$$

$$\exists x A_i(x) \text{ for } i \in \{0, 1, 2, 3\};$$

$$\exists x M(x);$$

$$\forall x [A_i(x) \to \neg A_j(x)] \text{ for } 0 \leq i, j \leq 3, i \neq j;$$

$$\forall x [A_i(x) \to \neg M(x)] \text{ for } 0 \leq i \leq 3;$$

$$\forall x \forall y \big[R(x, y) \leftrightarrow$$

$$\leftrightarrow [A_0(x) \vee A_1(x) \vee [(M(x) \vee A_2(x)) \wedge (A_1(y) \vee M(y) \vee A_2(y) \vee A_3(y))] \vee (A_3(x) \wedge (A_2(y) \vee A_3(y)))]\big].$$

This completes the proof. $\square$

Likewise for Theorem 4.1.1, we can naturally deduce the following result.

Theorem 5.2.4. *The Maltsev filter $\Omega(M^+SHR_2)$ is prime in $\mathbf{L}$. In other words, omitting a generalized 2-dimensional special Hagemann relation with middle part, for a variety, is a prime Maltsev condition.*

Proof. By an analogous argument to the one expressed in Lemma 4.1.1, we can claim that, if an algebra carries a generalized special Hagemann relation with middle part, then so does each of its powers. Therefore, similarly to the reasoning in the proof of Theorem 4.1.2, we can prove that, given two varieties $\mathcal{V}$ and $\mathcal{W}$, whenever $R \leq \mathbf{A} \times \mathbf{A}$ and $S \leq \mathbf{B} \times \mathbf{B}$ are generalized special Hagemann relations of dimension 2 with middle part, for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$, then we can find a suitable power of these algebras, say $\mathbf{A}^\kappa$ and $\mathbf{B}^\kappa$ ($\kappa > 0$), also carrying generalized 2-dimensional special Hagemann relations with middle part, say

$$R_\kappa \leq \mathbf{A}^\kappa \times \mathbf{A}^\kappa,$$

$$S_\kappa \leq \mathbf{B}^\kappa \times \mathbf{B}^\kappa,$$

so that the corresponding members of the respective partitions inducing them have the same cardinalities. If we fix a bijection $s : A^\kappa \to B^\kappa$ matching the corresponding members of those partitions, we can prove that $(s \times s)(R_\kappa) = S_\kappa$ and

$$R_\kappa \leq (\mathbf{A}^\kappa \amalg_s \mathbf{B}^\kappa)^2$$

is a generalized 2-dimensional special Hagemann relation with middle part, showing that $\mathcal{V} \vee \mathcal{W}$ admits M^+SHR_2, validating the primeness of $\Omega(M^+SHR_2)$. ☐

At this point, one may reasonably expect a connection between $\Omega(M^-SHR_2)$ (respectively $\Omega(M^+SHR_2)$) and $\Omega(M^-SF_3)$ (respectively $\Omega(M^+SF_3)$). As far as the case of congruence 2-permutability is concerned, Theorem 5.1.1 guarantees the coincidence of the two corresponding classes; unfortunately, we are not able to prove whether it is the same case in this context. However, if we restrict to idempotent varieties, we can establish a result which is analogous to Theorem 5.1.1. Before presenting that, though, we prove a lemma which states a crucial fact that will be used afterwards multiple times. Such a lemma somewhat provides a more refined procedure than the one presented in Theorem 3.2.1($n = 2$) for building a generalized special Hagemann relation of dimension 2 out of a failure of congruence 3-permutability.

Lemma 5.2.1. *Suppose $\mathbf{P}$ and $\mathbf{Q}$ are similar algebras, let $X, Y, Z \subsetneq P$ be non-empty such that $X \cap Y = Z \cap Y = \varnothing$, $U, W \subsetneq Q$ be non-empty and disjoint, and $V \subsetneq Q$ (potentially empty), with $V \cap U = V \cap W = \varnothing$, such that*

$$[X \times U] \cup [Y \times (U \cup V \cup W)] \leq \mathbf{P} \times \mathbf{Q},$$

$$[Y \times (U \cup V \cup W)] \cup [Z \times W] \leq \mathbf{P} \times \mathbf{Q}.$$

Thus, if $V = \varnothing$, then $\mathbf{HSP}(\mathbf{P}\times\mathbf{Q})$ admits M^-SHR_2; otherwise, $\mathbf{HSP}(\mathbf{P}\times\mathbf{Q})$ admits M^+SHR_2.

Proof. Let $\mathbf{P}$ and $\mathbf{Q}$ be similar algebras as presented in the statement, and denote by $\mathcal{V}$ the variety generated by $\mathbf{P}$ and $\mathbf{Q}$, i.e. $\mathcal{V} = \mathbf{HSP}(\mathbf{P}, \mathbf{Q}) = \mathbf{HSP}(\mathbf{P} \times \mathbf{Q})$.

Call I and J respectively the subuniverses of $\mathbf{P} \times \mathbf{Q}$ displayed above, namely

$$I = [X \times U] \cup [Y \times (U \cup V \cup W)] \leq \mathbf{P} \times \mathbf{Q},$$

$$J = [Y \times (U \cup V \cup W)] \cup [Z \times W] \leq \mathbf{P} \times \mathbf{Q}.$$

Clearly, $I \cap J = Y \times (U \cup V \cup W) \leq \mathbf{P} \times \mathbf{Q}$ and $\mathbf{I}, \mathbf{J}, \mathbf{I} \wedge \mathbf{J}, \mathbf{I} \times \mathbf{J} \in \mathcal{V}$.

As usual, call α and β the kernels of the projection maps from $P \times Q$ onto, respectively, P and Q, that is

$$\alpha = \ker \pi_P = 0_P \otimes 1_Q,$$

$$\beta = \ker \pi_Q = 1_P \otimes 0_Q.$$

Furthermore, define the algebra $\mathbf{A} \in \mathcal{V}$, whose universe is

$$A = (I \times J) \cap \beta \leq (\mathbf{P} \times \mathbf{Q})^2,$$

and call R the subuniverse of $\mathbf{A} \times \mathbf{A}$ defined by

$$R = [(\beta_{|I} \circ \alpha_{|I}) \otimes (\alpha_{|J} \circ \beta_{|J})] \cap [A \times A].$$

For the remainder of this proof, we will show that R is a generalized 2-dimensional special Hagemann relation induced by the partition $\rho_R^\pm$ of A, where

$$A_0 = [(X \times U) \times (Y \times U)] \cap \beta,$$

$$A_1 = [(Y \times U) \times (Y \times U)] \cap \beta,$$

$$M = [(Y \times V) \times (Y \times V)] \cap \beta,$$

$$A_2 = [(Y \times W) \times (Y \times W)] \cap \beta,$$

$$A_3 = [(Y \times W) \times (Z \times W)] \cap \beta.$$

Note that A_0, A_1, M (when non-empty) A_2 and A_3 are non-empty mutually disjoint sets due to how X, Y, Z and U, V, W are defined, and

$$A_0 \cup A_1 \cup M \cup A_2 \cup A_3 =$$

$$= \big[[(X \times U) \times (Y \times U)] \cup [(Y \times U) \times (Y \times U)] \cup [(Y \times V) \times (Y \times V)] \cup$$

$$\cup [(Y \times W) \times (Y \times W)] \cup [(Y \times W) \times (Z \times W)]\big] \cap \beta =$$

$$= \big[[[X \times U] \cup [Y \times (U \cup V \cup W)]] \times [[Y \times (U \cup V \cup W)] \cup [Z \times W]]\big] \cap \beta =$$

$$= [I \times J] \cap \beta = A,$$

showing that $\rho_R^{\pm}$ is a partition of A indeed. Therefore, we just need to prove that $\rho_R^{\pm}$ induces R, and for this purpose, we are going to analyze the R-relationship case by case.

We will denote any element of A by

$$\begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix}$$

where $(p,q) \in I$, $(r,q) \in J$ and their second components coincide because they have to be β-related by definition of A.

- $(A_0 \cup A_1) \times A \subseteq R$: let $\begin{bmatrix} (s,u) \\ (y,u) \end{bmatrix} \in A_0 \cup A_1$ and $\begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix} \in A$. We get that

  $$\begin{bmatrix} (s,u) \\ (y,u) \end{bmatrix} \begin{matrix} \beta \\ \alpha \end{matrix} \begin{bmatrix} (p,u) \\ (y,q) \end{bmatrix} \begin{matrix} \alpha \\ \beta \end{matrix} \begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix}.$$

  Notice that $(p,u) \in (X \cup Y) \times U \subseteq I$ and $(y,q) \in Y \times (U \cup V \cup W) \subseteq J$, meaning that, in fact

  $$\begin{bmatrix} (s,u) \\ (y,u) \end{bmatrix} R \begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix}.$$

- $(M \cup A_2) \times (A - A_0) \subseteq R$: pick any $\begin{bmatrix} (y,t) \\ (y',t) \end{bmatrix} \in M \cup A_2$ and $\begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix} \in A - A_0$. The following holds:

  $$\begin{bmatrix} (y,t) \\ (y',t) \end{bmatrix} \begin{matrix} \beta \\ \alpha \end{matrix} \begin{bmatrix} (p,t) \\ (y',q) \end{bmatrix} \begin{matrix} \alpha \\ \beta \end{matrix} \begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix}.$$

  Indeed, because $p \notin X$ as long as $(p,q) \notin A_0$, then $(p,t) \in Y \times (V \cup W) \subseteq I$ and $(y',q) \in Y \times (U \cup V \cup W)$, which implies

  $$\begin{bmatrix} (y,t) \\ (y',t) \end{bmatrix} R \begin{bmatrix} (p,q) \\ (r,q) \end{bmatrix}.$$

- $A_3 \times (A_2 \cup A_3) \subseteq R$: let $\begin{bmatrix} (y,w) \\ (z,w) \end{bmatrix} \in A_3$ and $\begin{bmatrix} (y',w') \\ (r,w') \end{bmatrix} \in A_2 \cup A_3$. Hence

  $$\begin{bmatrix} (y,w) \\ (z,w) \end{bmatrix} \begin{matrix} \beta \\ \alpha \end{matrix} \begin{bmatrix} (y',w) \\ (z,w') \end{bmatrix} \begin{matrix} \alpha \\ \beta \end{matrix} \begin{bmatrix} (y',w') \\ (r,w') \end{bmatrix}.$$

  Since $(y',w) \in Y \times W \subseteq I$ and $(z,w') \in Z \times W \subseteq J$, then the above yields

  $$\begin{bmatrix} (y,w) \\ (z,w) \end{bmatrix} R \begin{bmatrix} (y',w') \\ (r,w') \end{bmatrix}.$$

- $(M \cup A_2 \cup A_3) \times A_0 \subseteq A^2 - R$: Suppose there exist $\begin{bmatrix}(y,q)\\(s,q)\end{bmatrix} \in M \cup A_2 \cup A_3$ and $\begin{bmatrix}(x,u)\\(r,u)\end{bmatrix} \in A_0$ such that
$$\begin{bmatrix}(y,q)\\(s,q)\end{bmatrix} \; R \; \begin{bmatrix}(x,u)\\(r,u)\end{bmatrix}.$$
This is to say
$$\begin{bmatrix}(y,q)\\(s,q)\end{bmatrix} \begin{matrix}\beta_{|I}\\ \alpha_{|J}\end{matrix} \begin{bmatrix}(x,q)\\(s,u)\end{bmatrix} \begin{matrix}\alpha_{|I}\\ \beta_{|J}\end{matrix} \begin{bmatrix}(x,u)\\(r,u)\end{bmatrix},$$
which, in particular, yields $(x,q) \in I$. Yet, this is impossible, because $(x,q) \in X \times (V \cup W) \subseteq (P \times Q) - I$. Therefore, the situation initially assumed cannot occur, proving the claim.

- $A_3 \times (A_1 \cup M) \subseteq A^2 - R$: Assume there exist $\begin{bmatrix}(y,w)\\(z,w)\end{bmatrix} \in A_3$ and $\begin{bmatrix}(p,q)\\(r,q)\end{bmatrix} \in A_1 \cup M$ such that
$$\begin{bmatrix}(y,w)\\(z,w)\end{bmatrix} \; R \; \begin{bmatrix}(p,q)\\(r,q)\end{bmatrix}.$$
In other words,
$$\begin{bmatrix}(y,w)\\(z,w)\end{bmatrix} \begin{matrix}\beta_{|I}\\ \alpha_{|J}\end{matrix} \begin{bmatrix}(p,w)\\(z,q)\end{bmatrix} \begin{matrix}\alpha_{|I}\\ \beta_{|J}\end{matrix} \begin{bmatrix}(p,q)\\(r,q)\end{bmatrix},$$
which in particular implies $(z,q) \in J$, contradicting the fact that $(z,q) \in Z \times (U \times V) \subseteq (P \times Q) - J$.

This sequence of inclusions proves that R is a generalized special Hagemann relation of dimension 2 induced by $\rho_R^{\pm}$. Moreover, M is built in such a way that M is empty whenever V is, showing that R has a middle part if and only if V is not the empty set, concluding the proof. ☐

We also want to point out that in the hypotheses of Lemma 5.2.1, the idempotence of $\mathbf{P}$ and $\mathbf{Q}$ is not required. However, the subsets I and J can always be found as subuniverses whenever we assume the idempotence of a variety admitting M^+SF_3 or M^-SF_3, as better exposed in the theorem below.

Theorem 5.2.5. *If $\mathcal{V}$ is any idempotent variety, $\mathcal{V}$ omits M^-SF_3 (resp. M^+SF_3) if and only if $\mathcal{V}$ omits M^-SHR_2 (resp. M^+SHR_2), i.e.*
$$\Omega^{id}(M^-SF_3) = \Omega^{id}(M^-SHR_2),$$
$$\Omega^{id}(M^+SF_3) = \Omega^{id}(M^+SHR_2).$$

Proof. For the rest of this proof, let $\mathcal{V}$ be an idempotent variety.

Equivalently to the statement, we will prove that $\mathcal{V}$ admits M^-SF_3 (resp. M^+SF_3) if and only if it admits M^-SHR_2 (resp. M^-SHR_2). In order to deal with generalized 2-dimensional special Hagemann relations with or without middle parts, as well as with special failures of congruence 3-permutability with or without middle portions in parallel, we will adopt the notation $M^{\pm}SHR_2$ and $M^{\pm}SF_3$ and we will remark explicitly when the middle part or middle portion are needed.

First, assume $\mathcal{V} \notin \Omega(M^{\pm}SHR_2)$, which is to say there exist an algebra $\mathbf{A} \in \mathcal{V}$ and $R \leq \mathbf{A} \times \mathbf{A}$ which is a generalized special Hagemann relation of dimension 2 with/without middle part, induced by the partition $\rho_R^{\pm}$ of A.

Call Q the subset of A defined by
$$Q := A_1 \cup M \cup A_2.$$
Notice that this definition makes sense even when M is empty. In fact, the following claim holds.

CLAIM 5.2.5.1. Q is a subuniverse of $\mathbf{A}$. In particular
$$Q = \bigcap_{a \in A} a/(R \circ R) \cap a/(R \circ R)^{-1}.$$

Proof. By idempotence, for every $a \in A$, the sets $a/R \circ R$ and $a/(R \circ R)^{-1}$ are subuniverses of $\mathbf{A}$, and hence the intersections of all of them is a subuniverse of $\mathbf{A}$. Therefore, the claim is proven if we verify the equality displayed in the statement.

Recall that

$$R = [(A_0 \cup A_1) \times A] \cup [(M \cup A_2) \times (A_1 \cup M \cup A_2 \cup A_3)] \cup [A_3 \times (A_2 \cup A_3)].$$

Therefore, fix $a \in A$ and $q \in Q$. If $a \in A_0 \cup A_1 \cup M \cup A_2$, then $(a, q) \in R \circ R$, since $(A_0 \cup A_1 \cup M \cup A_2) \times Q \subseteq R \subseteq R \circ R$. Instead, If $a \in A_3$, then for all $a_2 \in A_2$, $a\ R\ a_2\ R\ q$, proving that $(a, q) \in R \circ R$.

On the other hand, whenever $a \in A_1 \cup M \cup A_2 \cup A_3$, since $Q \times (A_1 \cup M \cup A_2 \cup A_3) \subseteq R$, then $(q, a) \in R \subseteq R \circ R$. Moreover, if $a \in A_0$, then for each $a_1 \in A_1$, $q\ R\ a_1\ R\ a$, proving that $(a, q) \in (R \circ R)^{-1}$. So far, we have shown that $Q \subseteq a/(R \circ R) \cap a/(R \circ R)^{-1}$, and given the arbitrariness of $a \in A$, Q is contained in the intersection of all of them.

Conversely, notice that for all $a \in A_0$ and $a' \in A_3$, $(a, a') \notin R \circ R$, otherwise contradicting that R is a 2-dimensional Hagemann relation and proving both $a' \notin a/(R \circ R)$ and $a \notin a'/(R \circ R)^{-1}$. This shows that $A_0 \cup A_3$ is contained in the complement of $\bigcap_{a \in A} a/(R \circ R) \cap a/(R \circ R)^{-1}$, which verifies the equality. □

Therefore, the algebra $\mathbf{Q} \in \mathcal{V}$ and so $\mathbf{A} \times \mathbf{Q} \in \mathcal{V}$.

At this point, define

$$S := (R \cap R^{-1}) \cap (A \times Q).$$

Clearly $\mathbf{S}$ is a subalgebra of $\mathbf{A} \times \mathbf{Q}$. A set theoretical calculation easily shows that

$$R \cap R^{-1} = [A_0 \times (A_0 \cup A_1)] \cup [A_1 \times (A_0 \cup Q)] \cup [M \times Q] \cup [A_2 \times (Q \cup A_3)] \cup [A_3 \times (A_2 \cup A_3)],$$

and hence, if we expand the above identity defining S, we get

$$S = (R \cap R^{-1}) \cap (A \times Q) =$$

$$= [A_0 \times A_1] \cup [A_1 \times Q] \cup [M \times Q] \cup [A_2 \times Q] \cup [A_3 \times A_2] =$$
$$= [A_0 \times A_1] \cup [Q \times Q] \cup [A_3 \times A_2].$$

Using the same notation as in Definition 5.2.1, if we call

$$X := A_0,$$

$$Y := Q,$$

$$Z := A_3,$$

as subsets of $P := A$, and

$$U := A_1,$$

$$V := M,$$

$$W := A_2,$$

as subsets of Q, then we have that $S = [X \times U] \cup [Y \times Q] \cup [Z \times W]$, where $\{X, Y, Z\}$ is a partition of P, $\{U, W\}$ is a partition of $Q - V$ (notice that V is empty if and only if M is), proving that $\mathbf{S}$ is a special failure of congruence 3-permutability with/without middle portion. Because $\mathbf{S} \in \mathcal{V}$, we get $\mathcal{V} \notin \Omega^{id}(M^{\pm}SF_3)$, as desired.

For the other direction, suppose $\mathcal{V}$ admits $M^{\pm}SF_3$, which is to say there exist $\mathbf{S}, \mathbf{P}, \mathbf{Q} \in \mathcal{V}$, such that $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$, and partitions $\{X, Y, Z\}$ and $\{U, W\}$ of, respectively, P and $Q - V$, for $V \subsetneq Q$, such that $S = (X \times U) \cup (Y \times Q) \cup (Z \times W)$. We have already observed that $\mathbf{S}$ has two congruences α and β which do not 3-permute, namely

$$\alpha := \ker \pi_P \cap S^2 = (0_P \otimes 1_Q) \cap S^2,$$

$$\beta := \ker \pi_Q \cap S^2 = (1_P \otimes 0_Q) \cap S^2.$$

Let us define the following two subsets of S:

$$I := (X \times U) \cup (Y \times Q),$$

$$J := (Y \times Q) \cup (Z \times W).$$

Again, idempotency of $\mathcal{V}$ yields the following

CLAIM 5.2.5.2. *I and J are subuniverses of* $\mathbf{S}$*. Indeed, for every $x \in X$, $u \in U$, $z \in Z$, $w \in W$*

$$I = (x, u)/\beta \circ \alpha,$$

$$J = (z, w)/\beta \circ \alpha.$$

Proof. Let us prove that $I = (x, u)/\beta \circ \alpha$, for any $(x, u) \in X \times U$. The equality involving J can be proven similarly, hence we omit that proof.

For every $(a, b) \in I$, we have that

$$\begin{bmatrix} x \\ u \end{bmatrix} \ \beta \ \begin{bmatrix} a \\ u \end{bmatrix} \ \alpha \ \begin{bmatrix} a \\ b \end{bmatrix},$$

because $(a, b) \in I$ implies $a \in X \cup Y$, and hence $(a, u) \in (X \cup Y) \times U \subseteq I \subseteq S$.

Conversely, if $(a, b) \in (x, u)/\beta \circ \alpha$, then it must be the case that

$$\begin{bmatrix} x \\ u \end{bmatrix} \ \beta \ \begin{bmatrix} a \\ u \end{bmatrix} \ \alpha \ \begin{bmatrix} a \\ b \end{bmatrix}.$$

This implies in particular that $(a, u), (a, b) \in S$. Moreover, since $u \in U$, then $a \in X \cup Y$, which forces $(a, b) \in I$, completing the proof of the claim. □

Therefore, since $Q = U \cup V \cup W$, the subuniverses I and J of $\mathbf{P} \times \mathbf{Q}$ satisfy the assumptions of Lemma 5.2.1, implying that $\mathcal{V}$ admits $M^{-}SHR_2$ whenever V is empty, or $\mathcal{V}$ admits $M^{+}SHR_2$, otherwise. This means $\mathcal{V} \notin \Omega^{id}(M^{\pm}SHR_2)$, finishing the proof. □

A natural question to ask at this point could be about the relationship that occurs, at least in the idempotent setting, between generalized special Hagemann relations of dimension 2 without middle part and those with middle part (or between special failures of congruence 3-permutability with and without middle portion, this being an equivalent approach as Theorem 5.2.5 guarantees). The result contained in the next theorem answers the above question and, besides, it represents another crucial point in the proof of the primeness of locally finite idempotent congruence 3-permutability.

Theorem 5.2.6. *Let $\mathcal{V}$ be an idempotent variety. If $\mathcal{V}$ admits $M^{-}SF_3$, then it also admits $M^{+}SF_3$.*

In other words,

$$\Omega^{id}(M^{+}SHR_2) = \Omega^{id}(M^{+}SF_3) \subseteq \Omega^{id}(M^{-}SF_3) = \Omega^{id}(M^{-}SHR_2).$$

Proof. The equalities in the displayed expression are the result of Theorem 5.2.5 and the inclusion follows from the statement above, which is the only part to be proven.

Let $\mathcal{V}$ be an idempotent variety admitting $M^{-}SF_3$, and hence, let $\mathbf{S} \in \mathcal{V}$ be a special failure of congruence 3-permutability without middle portion. Call $\mathbf{P}, \mathbf{Q} \in \mathcal{V}$ the two algebras such that $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$ and let $\{X, Y, Z\}$ and $\{U, W\}$ be partitions of, respectively, P and Q such that $S = (X \times U) \cup (Y \times Q) \cup (Z \times W)$.

From this point on, starting from $\mathbf{S}$ we are going to build an algebra in $\mathcal{V}$ which is a special failure of congruence 3-permutability with middle portion.

Let us begin by fixing two elements in Q, say $0 \in U$ and $1 \in W$. Then, define the set $S' \subseteq S \times S \times (P \times Q)$ as

$$S' := \left\{ \begin{bmatrix} (s, 0) \\ (r, 1) \\ (p, q) \end{bmatrix} : (s, 0), (r, 1), (s, q), (r, q) \in S, \exists i \in Q[(s, i), (r, i), (p, i) \in S] \right\}.$$

Notice that, since S' is defined via a primitive positive sentence involving the subuniverse S, we deduce that $S' \leq \mathbf{S} \times \mathbf{S} \times (\mathbf{P} \times \mathbf{Q})$ by idempotence, which yields $\mathbf{S}' \in \mathcal{V}$. Our first goal is to prove that

$$\begin{gathered} S' = \big[(X \times \{0\}) \times (Y \times \{1\}) \times [(X \cup Y) \times U]\big] \cup \\ \cup \big[(Y \times \{0\}) \times (Y \times \{1\}) \times (P \times Q)\big] \cup \\ \cup \big[(Y \times \{0\}) \times (Z \times \{1\}) \times [(Y \cup Z) \times W]\big]. \end{gathered}$$

Let us start with "$\supseteq$".

- $(X \times \{0\}) \times (Y \times \{1\}) \times [(X \cup Y) \times U] \subseteq S'$: pick any $(x, 0) \in X \times \{0\}$, $(y, 1) \in Y \times \{1\}$ and $(p, u) \in (X \cup Y) \times U$. Notice that $(x, 0) \in X \times U \subseteq S$, $(y, 1) \in Y \times W \subseteq S$ and for the element $u \in U \subseteq Q$,

  $$(x, u) \in X \times U \subseteq S, \ (y, u) \in Y \times U \subseteq S, \ (p, u) \in (X \cup Y) \times U \subseteq S,$$

  showing that

  $$\begin{bmatrix} (x,0) \\ (y,1) \\ (p,u) \end{bmatrix} \in S'.$$

- $(Y \times \{0\}) \times (Y \times \{1\}) \times (P \times Q) \subseteq S'$: let $(y, 0) \in Y \times \{0\}$, $(y', 1) \in Y \times \{1\}$ and $(p, q) \in P \times Q$. We can immediately claim that $(y, 0) \in Y \times U \subseteq S$ and $(y', 1) \in Y \times W \subseteq S$. Moreover, because $Y \times Q \subseteq S$, we get $(y, q), (y', q) \in S$. Also, we need to distinguish two different cases for $p \in P$.

  If $p \in X \cup Y$, then for $i = 0 \in U \subseteq Q$ we have

  $$(y, 0) \in S, \ (y', 0) \in Y \times U \subseteq S, \ (p, 0) \in (X \cup Y) \times U \subseteq S.$$

  Instead, for $p \in Z$, by setting $i = 1 \in W$, we get

  $$(y, 1) \in Y \times W \subseteq S, \ (y', 1) \in S, \ (p, 1) \in Z \times W \subseteq S.$$

  In either case, we have proven

  $$\begin{bmatrix} (y,0) \\ (y',1) \\ (p,q) \end{bmatrix} \in S'.$$

- $(Y \times \{0\}) \times (Z \times \{1\}) \times [(Y \cup Z) \times W] \subseteq S'$: consider any $(y, 0) \in Y \times \{0\}$, $(z, 1) \in Z \times \{1\}$ and $(p, w) \in (Y \cup Z) \times W$. Since $(Y \cup Z) \times W \subseteq S$, we deduce $(z, 1) \in S$, $(y, w) \in S$, $(z, w) \in S$ and $(p, w) \in S$. Also, $(y, 0) \in Y \times U \subseteq S$. Furthermore, we have also shown that there exists $i = w \in W \subseteq Q$ with $y, z, p \ S \ i$, proving that even for this case

  $$\begin{bmatrix} (y,0) \\ (z,1) \\ (p,w) \end{bmatrix} \in S'.$$

For the other inclusion, pick any triple in S', say

$$\begin{bmatrix} (s,0) \\ (r,1) \\ (p,q) \end{bmatrix} \in S'.$$

The fact that $(s, 0), (r, 1) \in S$, along with $0 \in U$ and $1 \in W$, imply that $s \in X \cup Y$ and $r \in Y \cup Z$.

Suppose first $s \in X$: because $(s, q) \in S$, then q is forced to belong to U, which in turn forces $p \in X \cup Y$ and $r \in Y$ (since $Z \times U \subseteq (P \times Q) - S$). Therefore, we deduce that

$$\begin{bmatrix} (s,0) \\ (r,1) \\ (p,q) \end{bmatrix} \in (X \times \{0\}) \times (Y \times \{1\}) \times [(X \cup Y) \times U].$$

Likewise, if $r \in Z$, then because $(r, q) \in S$, we get $q \in W$. This yields $p \in Y \cup Z$ and $s \in Y$ (since $X \times W \subseteq (P \times Q) - S$). All these things together imply

$$\begin{bmatrix} (s,0) \\ (r,1) \\ (p,q) \end{bmatrix} \in (Y \times \{0\}) \times (Z \times \{1\}) \times [(Y \cup Z) \times W].$$

The only remaining case is $s, r \in Y$: directly by assumption, we have that $(s, 0) \in Y \times \{0\}$ and $(r, 1) \in Y \times \{1\}$ and no constraint is required for $(p, q) \in P \times Q$, showing that

$$\begin{bmatrix} (s,0) \\ (r,1) \\ (p,q) \end{bmatrix} \in (Y \times \{0\}) \times (Y \times \{1\}) \times (P \times Q).$$

Now that the equality has been verified, the next step is to define two algebras in $\mathcal{V}$ whose direct product contains $\mathbf{S}'$ subdirectly. Let then P' be the image of the projection map from S onto the first two coordinates and $Q' = P \times Q$.

Clearly $\mathbf{P}', \mathbf{Q}' \in \mathcal{V}$ and $\mathbf{S}' \leq_{sd} \mathbf{P}' \times \mathbf{Q}'$. Furthermore, if we call

$$X' := (X \times \{0\}) \times (Y \times \{1\}) \subseteq P',$$

$$Y' := (Y \times \{0\}) \times (Y \times \{1\}) \subseteq P',$$

$$Z' := (Y \times \{0\}) \times (Z \times \{1\}) \subseteq P',$$

and

$$U' := (X \cup Y) \times U \subseteq Q',$$

$$V' := (X \times W) \cup (Z \times U) \subseteq Q',$$

$$W' := (Y \cup Z) \times W \subseteq Q',$$

then it is straightforward to check that $\{X', Y', Z'\}$ is a partition of P' and $\{U', V', W'\}$ is a partition of Q' (because so are $\{X, Y, Z\}$ of P and $\{U, W\}$ of Q). In addition,

$$\begin{aligned} S' = \big[[(X \times \{0\}) \times (Y \times \{1\})] \times [(X \cup Y) \times U]\big] \cup \\ \cup \big[[(Y \times \{0\}) \times (Y \times \{1\})] \times (P \times Q)\big] \cup \\ \cup \big[[(Y \times \{0\}) \times (Z \times \{1\})] \times [(Y \cup Z) \times W]\big] = \\ = [X' \times U'] \cup [Y' \times Q'] \cup [Z' \times W']. \end{aligned}$$

We wish to emphasize that V' cannot be empty in this construction, proving that, in fact, $\mathbf{S}'$ is a special failure of congruence 3-permutability with middle portion (which is V'), and hence showing that $\mathcal{V}$ admits $M^{+}SF_3$, as desired. □

So far, we have been exposing some results which do not deal directly with congruence 3-permutable varieties, although it is clear that any congruence 3-permutable variety must omit $M^{\pm}SHR_2$ and $M^{\pm}SF_3$: in particular $CP_3^{id} \subseteq \Omega^{id}(M^{+}SF_3) = \Omega^{id}(M^{+}SHR_2)$ $(\subseteq \Omega^{id}(M^{-}SF_3) = \Omega^{id}(M^{-}SHR_2))$. Indeed, the next theorem will allow us to invert this property for locally finite idempotent varieties. We will show that given any finite idempotent algebra that is not congruence 3-permutable, there is a procedure that, via a sequence of reductions applied to it, is able to produce a pair of subalgebras satisfying the assumptions of Lemma 5.2.1. Let us see this in detail.

Theorem 5.2.7. *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{A}$ *is not congruence* 3*-permutable, then* $\mathbf{HSP}(\mathbf{A})$ *admits* M^+SHR_2*.*

Proof. Let $\mathbf{A}$ be a finite idempotent algebra having two congruences α and β failing to 3-permute and let $\mathcal{V}$ denote the variety generated by $\mathbf{A}$. Without loss of generality, we may assume that $\alpha \wedge \beta = 0_A$: if not, we replace our initial algebra $\mathbf{A}$ by its quotient $\mathbf{A}/\alpha \wedge \beta$, which still carries two congruences which do not 3-permute, namely $\alpha/\alpha \wedge \beta$ and $\beta/\alpha \wedge \beta$.

Also, we can assume that $\beta \circ \alpha \circ \beta \not\subseteq \alpha \circ \beta \circ \alpha$ and we will choose a pair (x, y) in $\beta \circ \alpha \circ \beta$ such that $(x, y) \notin \alpha \circ \beta \circ \alpha$. Let $u, v \in A$ such that $x \ \beta \ u \ \alpha \ v \ \beta \ y$. From this point on, we will provide a sequence of reductions which we will classify, for convenience, in *horizontal* and *vertical reductions*. In order to help the reader follow the current proof, we will sometimes provide some pictures describing potential scenarios. The following figure, for instance, represents a possible configuration of the algebra $\mathbf{A}$, along with a possible placement of the elements x, u, v, y, considering that two elements are β-related if they lie on the same horizontal line, while they are α-related when lying on the same vertical line.

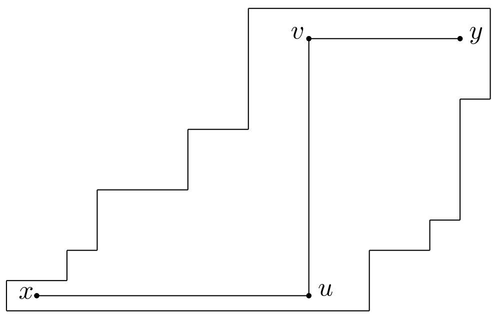

Figure 5.3: A potential failure of congruence 3-permutability

Horizontal reductions: Define, for $H \subseteq K \leq \mathbf{A}$, the following set

$$Q[H, K] = \bigcap_{h \in H} h/\alpha_{|K} \circ \beta_{|K} \subseteq K.$$

By idempotence, we can observe that $Q[H, K]$ is a subuniverse of $\mathbf{K}$, and hence the algebra $\mathbf{Q}[H, K]$ is a member of $\mathcal{V}$ whenever its universe is non-empty. In the case that $Q[H, K] \neq \varnothing$, it is not hard to notice that $Q[H, K]$ is a union of $\beta_{|K}$-classes of elements $k \in K$ such that $(h, k) \in \alpha_{|K} \circ \beta_{|K}$, for all $h \in H$.

Furthermore, define the following set for any $a \in B$, where $B \leq \mathbf{A}$

$$P[a, B] = \{X \subseteq a/\beta_{|B} : \exists b \in Q[X, B][(a, b) \notin \alpha_{|Q[X,B]} \circ \beta_{|Q[X,B]} \circ \alpha_{|Q[X,B]}]\}.$$

Any element b in the above definition will be referred to as an $(a, Q[X, B])$*-witness*, whenever $P[a, B]$ is not empty. In such a case, immediately note the following three facts:

- $a \in Q[X, B]$, for every $X \in P[a, B]$, because $a \ \beta_{|B} \ z \ \alpha_{|B} \ z$, for every $z \in X$;
- $a \notin X$, for every $X \in P[a, B]$, since otherwise, we would get that $(a, b) \in \alpha_{|Q[X,B]} \circ \beta_{|Q[X,B]}$, for any $(a, Q[X, B])$-witness $b \in Q[X, B]$, contradicting the definition of this latter element;
- for every $c \in Q[X, B]$, $a \ \beta_{|Q[X,B]} \ z \ \alpha_{|Q[X,B]} \circ \beta_{|Q[X,B]} \ c$, for all $z \in X$.

In order to further clarify what a non-empty $P[a, B]$ looks like, notice that it contains all those sets X for which $\mathbf{Q}[X, B]$ is an algebra that fails to be congruence 3-permutable, since it contains two elements a, b such that $(a, b) \in \beta_{|Q[X,B]} \circ \alpha_{|Q[X,B]} \circ \beta_{|Q[X,B]}$, but $(a, b) \notin \alpha_{|Q[X,B]} \circ \beta_{|Q[X,B]} \circ \alpha_{|Q[X,B]}$. Indeed, the horizontal reductions will consist of eventually reducing down to some minimal subalgebra of $\mathbf{A}$ of the form $\mathbf{Q}[X, B]$, for some suitable X and B.

For the case $a = x$ (recall x is the element chosen initially), $P[x, A]$ is non-empty because $\{u\} \in P[x, A]$. As a matter of fact, $Q[\{u\}, A] = u/\alpha \circ \beta$ contains y, which is an $(x, Q[\{u\}, A])$-witness. The following two figures distinguish two opposite cases for the demonstrating example initially considered in Figure 5.3, showing both when $X \in P[x, A]$ and when $X \notin P[x, A]$; the region surrounded by the thickest line denotes $Q[X, A]$.

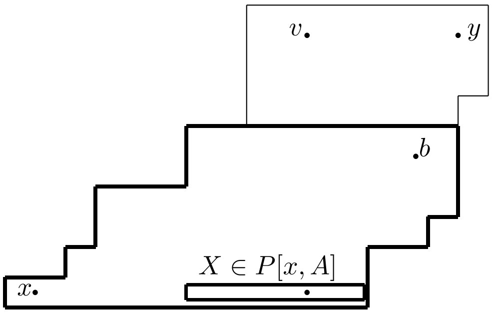

Figure 5.4: Example of $Q[X, A]$, for X lying in $P[x, A]$: in such a case, $\mathbf{Q}[X, A]$ fails 3-permutability, as witnessed by the pair (x, b)

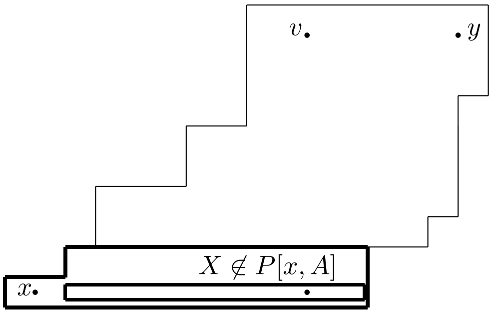

Figure 5.5: Example of $Q[X, A]$, for X not lying in $P[x, A]$: in such a case, $\alpha_{|Q[X,A]}$ and $\beta_{|Q[X,A]}$ 3-permute

In order to make the coming arguments more understandable and to support them with a more uniform notation, let us define the following objects. Call $x^{(0)} := x$, and define inductively

$$Q^{(0)} := A,$$

$$Q^{(n+1)} := Q[M_{x^{(n)}}, Q^{(n)}],$$

where $x^{(n+1)}$ is an $(x^{(n)}, Q^{(n+1)})$-witness, $M_{x^{(0)}}$ is a $\subseteq$-maximal extension of $\{u\}$ in $P[x^{(0)}, Q^{(0)}]$ and $M_{x^{(n+1)}}$ is a $\subseteq$-maximal element in $P[x^{(n+1)}, Q^{(n+1)}]$ (the existence of a maximal element is guaranteed since A is finite and it is needed, informally speaking, to eventually obtain the smallest possible non-congruence 3-permutable $\mathbf{Q}[X, Q^{(n)}]$: in fact, at every step, it is enough to add an extra element q so that $\mathbf{Q}[M_{x^{(n)}} \cup \{q\}, Q^{(n)}]$ is no longer a failure of congruence 3-permutability, as we will remark further on) such that $M_{x^{(n+1)}}$ extends $s_n(M_{x^{(n)}})$, where s_n is described in the following claim.

CLAIM 5.2.7.1. For every $n \geq 0$, there exists an injective map $s_n : M_{x^{(n)}} \to x^{(n+1)}/\beta_{|Q^{(n+1)}}$ such that, for all $p \in M_{x^{(n)}}$, $s_n(p)$ is the only element in $x^{(n+1)}/\beta_{|Q^{(n+1)}}$ such that $(p, s_n(p)) \in \alpha_{|Q^{(n+1)}}$.

Proof. Fix $n \geq 0$. The stated property is implicit in the definition of $Q^{(n+1)}$. Indeed, $x^{(n+1)} \in Q^{(n+1)}$, which means that, for all $p \in M_{x^{(n)}}$, there is $q \in Q^{(n)}$ satisfying

$$p \ \alpha_{|Q^{(n)}} \ q \ \beta_{|Q^{(n)}} \ x^{(n+1)}.$$

Notice that, for every $p' \in M_{x^{(n)}}$, $(p', x^{(n+1)}) \in \alpha_{|Q^{(n)}} \circ \beta_{|Q^{(n)}}$ and hence $(p', q) \in \alpha_{|Q^{(n)}} \circ \beta_{|Q^{(n)}}$. Therefore, $q \in Q^{(n+1)}$, implying that, in fact, $q \in x^{(n+1)}/\beta_{|Q^{(n+1)}}$. Moreover, such a q is unique because we have assumed $\alpha \wedge \beta = 0_A$. If we call $s_n(p) := q$, then we are actually defining the function described in the statement.

For a proof of injectivity, let $p_1, p_2 \in M_{x^{(n)}}$ such that $s_n(p_1) = s_n(p_2)$. Then, we have that

$$p_1 \ \alpha \ s_n(p_1) = s_n(p_2) \ \alpha \ p_2,$$

and also

$$p_1 \ \beta \ x^{(n)} \ \beta \ p_2.$$

Since α and β intersect trivially, $p_1 = p_2$, as desired. $\square$

Also notice that, at the $(n+1)^{st}$ step, there is at least $x^{(n)}$ as an $(x^{(n+1)}, Q[s_n(M_{x^{(n)}}), Q^{(n+1)}])$-witness; indeed, by definition, $x^{(n+1)}$ is an element of $Q^{(n+1)}$ satisfying

$$(x^{(n)}, x^{(n+1)}) \notin \alpha_{|Q^{(n+1)}} \circ \beta_{|Q^{(n+1)}} \circ \alpha_{|Q^{(n+1)}}.$$

In addition, $Q^{(n+1)} = Q[s_n(M_{x^{(n)}}), Q^{(n+1)}]$: while the inclusion " $\supseteq$ " is obvious, the other is proven by taking any $q \in Q^{(n+1)}$ and showing that $q \ \beta_{|Q^{(n+1)}} \circ \alpha_{|Q^{(n+1)}} \ w \ \alpha_{|Q^{(n+1)}} \ s_n(w)$, for all $w \in M_{x^{(n)}}$. Therefore, by the symmetry of $\alpha_{|Q^{(n+1)}} \circ \beta_{|Q^{(n+1)}} \circ \alpha_{|Q^{(n+1)}}$, we observe that $x^{(n)}$ is an $(x^{(n+1)}, Q[s_n(M_{x^{(n)}}), Q^{(n+1)}])$-witness. In particular, the previous reasoning yields that $\mathbf{Q}^{(k)}$ is a failure of congruence 3-permutability for every $k \geq 0$, witnessed by the non-3-permuting congruences $\alpha_{|Q^{(k)}}$ and $\beta_{|Q^{(k)}}$.

At this point, we invoke again the finiteness of A to determine the existence of a certain $m \geq 0$ such that $Q^{(m)} = Q^{(k)}$, for all $k \geq m$, which is to say the descending chain $Q^{(0)} \supseteq Q^{(1)} \supseteq \cdots \supseteq Q^{(i)} \supseteq \cdots$ must stabilize eventually.

Given that $Q^{(m+1)} = Q^{(m)}$, without loss of generality we can assume that $x^{(m+2)} = x^{(m)}$ and $M_{x^{(m+2)}} = M_{x^{(m)}}$, so as to have the maps s_m and s_{m+1} satisfy $s_m : M_{x^{(m)}} \to M_{x^{(m+1)}} \subseteq x^{(m+1)}/\beta_{|Q^{(m)}}$ and $s_{m+1} : M_{x^{(m+1)}} \to M_{x^{(m)}} \subseteq x^{(m)}/\beta_{|Q^{(m)}}$. In fact, we get even more, i.e.

CLAIM 5.2.7.2. The functions s_m and s_{m+1} map bijectively $M_{x^{(m)}}$ onto $M_{x^{(m+1)}}$, and viceversa.

Proof. We already know that s_m is one-to-one, so we just need prove that it is onto. Suppose it is not: then there exists $q \in M_{x^{(m+1)}}$ such that $s_m(p) \neq q$, for all $p \in M_{x^{(m)}}$. This means $(p, q) \notin \alpha_{|Q^{(m)}}$, for all $p \in M_{x^{(m)}}$, which also yields $(q, x^{(m)}) \notin \alpha_{|Q^{(m)}} \circ \beta_{|Q^{(m)}}$. As a matter of fact, if there is some $p' \in Q^{(m)}$ such that $q \ \alpha_{|Q^{(m)}} \ p' \ \beta_{|Q^{(m)}} \ x^{(m)}$, then for the previous observation $p' \notin M_{x^{(m)}}$, which contradicts the maximality of $M_{x^{(m+1)}}$, being that $x^{(m+1)}$ is still a $(x^{(m)}, Q[M_{x^{(m)}} \cup \{p'\}, Q^{(m)}])$-witness.

Nonetheless, the fact that $(q, x^{(m)}) \notin \alpha_{|Q^{(m)}} \circ \beta_{|Q^{(m)}}$ is also a contradiction, because it implies that $x^{(m)} \notin Q^{(m+2)} = Q^{(m)}$. Hence, s_m has to be surjective. An analogous argument holds for s_{m+1}, proving the claim. $\square$

Henceforth, call $Q := Q^{(m)} \leq \mathbf{A}$, and in order not to load on the notation, call $x := x^{(m)}$ and $y := x^{(m+1)}$, and denote by α and β, respectively, $\alpha_{|Q}$ and $\beta_{|Q}$ (which need not be the initial x, y and α, β we considered). Finally, call $s := s_m$, $S := M_{x^{(m)}} \subseteq x/\beta$ and $T := M_{x^{(m+1)}} = s(S) \subseteq y/\beta$. Also note that for any $q \in S$, $q/\alpha = s(q)/\alpha$, and hence, since s is a bijection, $S \wr \alpha = T \wr \alpha$[2]. We also remark that $\mathbf{Q} \in \mathcal{V}$ and α, β do not 3-permute. Indeed,

CLAIM 5.2.7.3. If $x' \in x/\beta - S$ and $y' \in y/\beta - T$, then $(x', y') \in \beta \circ \alpha \circ \beta$, but $(x', y') \notin \alpha \circ \beta \circ \alpha$.

Proof. First, note that $x' \ \beta \ x \ \beta \circ \alpha \circ \beta \ y \ \beta \ y'$. Moreover, for some $p \in S$,

$$x' \ \beta \ x \ \beta \ p \ \alpha \ s(p) \ \beta \ y \ \beta \ y'.$$

If $(x', y') = (x, y)$, then the claim is obvious due to the role x and y play in the definition of S and Q. Else, suppose $(x', y') \neq (x, y)$, and assume that $x' \neq x$. Suppose further that

$$x' \ \alpha \ q' \ \beta \ p' \ \alpha \ y',$$

[2] To recall the adopted notation, see Section 1.1.

for some $p', q' \in Q$. If $(x, p') \notin \alpha \circ \beta \circ \alpha$, then, because $x' \notin S$, we have that $S' := \{x'\} \cup S \supsetneq S$ and hence p' is a $(x, Q[S', Q])$-witness, contradicting the maximality of S.

If $(x, p') \in \alpha \circ \beta \circ \alpha$, then $(x, y') \in \alpha \circ \beta \circ \alpha$, since $(p', y') \in \alpha$. This yields $y' \neq y$. Say that

$$x \ \alpha \ u' \ \beta \ v' \ \alpha \ y'.$$

If $(y, u') \notin \alpha \circ \beta \circ \alpha$, then again we have that u' is a $(y, Q[\{y'\} \cup T, Q])$-witness, contradicting the maximality of T.

Instead, if $(y, u') \in \alpha \circ \beta \circ \alpha$, then $(y, x) \in \alpha \circ \beta \circ \alpha$, since $(u', x) \in \alpha$. This is against the definition of x and y. Therefore, the only possibility is that $(x', y') \notin \alpha \circ \beta \circ \alpha$, as desired. □

This is the end of the first sequence of reductions. The picture below shows how the algebra **Q** looks like in the example that we have been using throughout this proof: the two rectangles represent S and T and the arrows stand for the map $s : S \to T$.

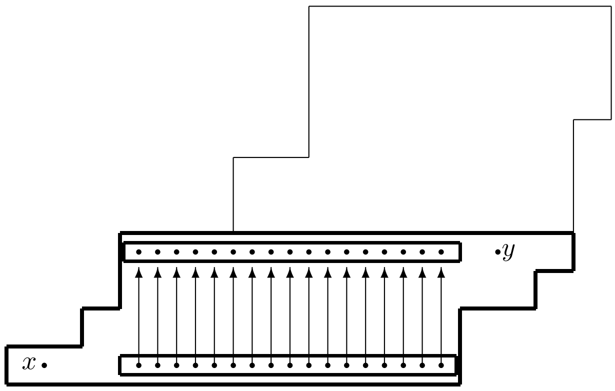

Figure 5.6: Horizontally reduced algebra with a representation of the bijection $s : S \to T$

First vertical reduction: We begin with the definition of the following maps. For every $p \in x/\beta$, $q \in S$, define the map $b_1[p, q] : p/\alpha \to q/\alpha$ satisfying

$$(p', b_1[p, q](p')) \in \beta,$$

for all $p' \in p/\alpha$.

CLAIM 5.2.7.4. For every $p \in x/\beta$ and $q \in S$, $b_1[p, q]$ is well defined and is injective. Moreover, $p \in S$ if and only if $b_1[p, q]$ is a bijection, whose inverse is $b_1[q, p]$.

Proof. Let us first prove the well-definedness of $b_1[p, q]$, for $q \in S$, $p \in x/\beta$ and $p' \in p/\alpha$. By definition $Q = \bigcap_{q \in S} q/\alpha \circ \beta$, and since $p' \in Q$, we have that $p' \ \beta \ r \ \alpha \ q$, for some $r \in Q$. Set $b_1[p, q] := r$, which is the only element (due to $\alpha \wedge \beta = 0_Q$) of q/α such that $p' \ \beta \ r = b_1[p, q](p')$.

Moreover, if $p', p'' \in p/\alpha$ are such that $b_1[p, q](p') = b_1[p, q](p'')$, then

$$p' \ \alpha \ p''$$

and

$$p' \ \beta \ b_1[p, q](p') = b_1[p, q](p'') \ \beta \ p'',$$

forcing $p' = p''$ and showing injectivity.

In addition, if $p \in S$, then for every $q' \in q/\alpha$, $(q', b_1[q, p](q')) \in \beta$ by definition of $b_1[q, p]$, and hence $b_1[p, q](b_1[q, p](q')) = q'$.

Conversely, if $p \in x/\beta$ and $q \in S$ are such that $b_1[p, q]$ is a bijection, then in particular there exists $p' \in p/\alpha$ such that $b_1[p, q](p') = s(q) \in q/\alpha$. Since also $s(q) \in T \subseteq y/\beta$, we have that

$$p \ \alpha \ p' \ \beta \ b_1[p, q](p') = s(q) \ \beta \ y.$$

In order not to contradict the maximality of S, we deduce that $p \in S$, proving that the rest of the claim also holds. □

Notice that the previous claim ensures that the α-classes of elements in S have the same (finite) cardinality. With this in mind, fix any $r \in S$ and let $\widehat{x} \in x/\beta - S$ be one element such that $b_1[\widehat{x}, r](\widehat{x}/\alpha) \subseteq r/\alpha$ is $\subseteq$-maximal. Again, the existence of such an $\widehat{x}$ is certified by the finiteness of Q. Hence, define the following subset of Q:

$$Q_1 := \bigcap_{q \in \widehat{x}/\alpha} q/\beta \circ \alpha.$$

Notice that $Q_1 \leq \mathbf{Q}$ by idempotence, which also yields $\mathbf{Q}_1 \in \mathcal{V}$. If we define

$$E := \{e \in x/\beta - S : b_1[e, r](e/\alpha) = b_1[\widehat{x}, r](\widehat{x}/\alpha)\},$$

we can claim that

CLAIM 5.2.7.5.

$$Q_1 = \left(\bigcup_{e \in E} e/\alpha \right) \cup \left(\bigcup_{p \in S} p/\alpha \right).$$

Proof. Let us first prove "$\supseteq$": if $w \in e/\alpha$, for some $e \in E$, then for some $e' \in e/\alpha$,

$$q \ \beta \ b_1[\widehat{x}, r](q) = b_1[e, r](e') \ \beta \ e' \ \alpha \ e \ w$$

for each $q \in \widehat{x}/\alpha$, showing that $w \in Q_1$.

On the other hand, if $w \in p/\alpha$, for some $p \in S$, then for every $q \in \widehat{x}/\alpha$, we have

$$q \ \beta \ b_1[\widehat{x}, p](q) \ \alpha \ p \ \alpha \ w,$$

again proving that $w \in Q_1$.

Conversely, let w be any element in Q_1, and suppose $w \notin p/\alpha$, for any $p \in S$. Because $w \in Q_1$, then

$$w \ \alpha \ a \ \beta \ \widehat{x},$$

for some $a \in \widehat{x}/\beta - S$.

Let $g \in b_1[\widehat{x}, r](\widehat{x}/\alpha)$, which is to say $g = b_1[\widehat{x}, r](q)$, for some $q \in \widehat{x}/\alpha$. Since $w \in Q_1$, there must be $d \in a/\alpha$ such that

$$g \ \beta \ d \ \alpha \ w.$$

Thus, we get

$$g = b_1[\widehat{x}, r](q) \ \beta \ d,$$

which, by definition of $b_1[a, r]$, yields $b_1[a, r](d) = g$. This proves that $b_1[\widehat{x}, r](\widehat{x}/\alpha) \subseteq b_1[a, r](a/\alpha)$. Yet, by maximality of $b_1[\widehat{x}, r](\widehat{x}/\alpha)$, the equality must hold in fact, i.e.

$$b_1[\widehat{x}, r](\widehat{x}/\alpha) = b_1[a, r](a/\alpha),$$

meaning, by definition of E, that $a \in E$. Since $(w, a) \in \alpha$, we have that

$$w \in \bigcup_{e \in E} e/\alpha,$$

completing the proof of the claim.

$\square$

Second vertical reduction: Likewise for the previous vertical reduction, for every $p \in y/\beta$ and $q \in T$, we define the map $b_2[p, q] : p/\alpha \to q/\alpha$ so that for all $p' \in p/\alpha$

$$(p', b_2[p, q](p')) \in \beta.$$

We have then the analogous result to Claim 5.2.7.4.

CLAIM 5.2.7.6. For every $p \in y/\beta$ and $q \in T$, $b_2[p, q]$ is well defined and is injective. Moreover, $p \in T$ if and only if $b_2[p, q]$ is a bijection, whose inverse is $b_2[q, p]$.

Proof. The proof of this claim proceeds exactly as the one of Claim 5.2.7.4, hence we omit it. □

Again, we note that $|q/\alpha| = |q'/\alpha|$ for every $q, q' \in T$. Then, let $t \in T$ be fixed and take one element $\widehat{y} \in y/\beta - T$ such that $b_2[\widehat{y}, t](\widehat{y}/\alpha) \subseteq t/\alpha$ is $\subseteq$-maximal. Furthermore, by using such a $\widehat{y}$, define

$$Q_2 := \bigcap_{q \in \widehat{y}/\alpha} q/\beta \circ \alpha,$$

which turns out to be a subuniverse of $\mathbf{Q}$ and hence $\mathbf{Q}_2 \in \mathcal{V}$. By defining

$$F := \{f \in y/\beta - T : b_2[f,t](f/\alpha) = b_2[\widehat{y}, t](\widehat{y}/\alpha)\},$$

we can infer that

CLAIM 5.2.7.7.

$$Q_2 = \left(\bigcup_{p \in T} p/\alpha \right) \cup \left(\bigcup_{f \in F} f/\alpha \right).$$

Proof. The proof, which is omitted, is basically the same as the one of Claim 5.2.7.5. □

The horizontal and vertical reductions have produced the algebras $\mathbf{Q}, \mathbf{Q}_1$ and $\mathbf{Q}_2$, all contained in the variety $\mathcal{V}$, and such that $\mathbf{Q}_i \leq \mathbf{Q}$, for $i = 1, 2$. Also notice that, due to $s(S) = T$, we get

$$\bigcup_{p \in S} p/\alpha = \bigcup_{s(p) \in T} s(p)/\alpha = \bigcup_{w \in T} w/\alpha.$$

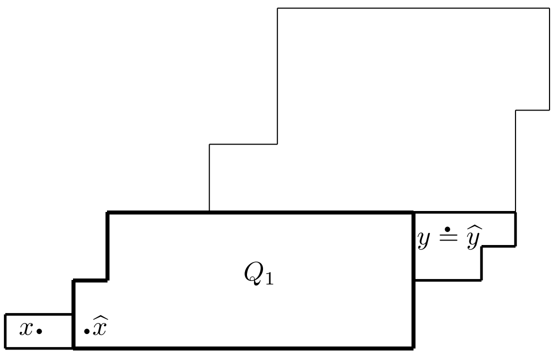

Figure 5.7: The subuniverse $Q_1 \leq \mathbf{Q}$

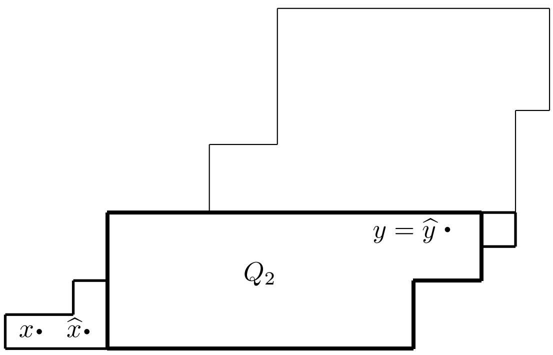

Figure 5.8: The subuniverse $Q_2 \leq \mathbf{Q}$

Since $\mathbf{Q}$ has two congruences α and β such that $\alpha \wedge \beta = 0_Q$, then we can look at $\mathbf{Q}$ as a subdirect product of the direct product of its quotients $\mathbf{Q}/\alpha$ and $\mathbf{Q}/\beta$, namely

$$\mathbf{Q} \leq_{sd} \mathbf{Q}/\alpha \times \mathbf{Q}/\beta.$$

Therefore, without loss of generality, we can consider Q, Q_1 and Q_2 as sets of pairs and the congruences α and β as the kernels of the projection maps restricted to Q, i.e.

$$\alpha = (0 \otimes 1) \cap Q^2,$$

$$\beta = (1 \otimes 0) \cap Q^2.$$

With this notation, if $\widehat{x} = (a, c)$ and $\widehat{y} = (z, d)$, then we have that $S = Y \times \{c\}$, for some $Y \subseteq Q/\alpha$. Moreover, note that $(b, c)/\alpha = \{b\} \times Q/\beta$, for all $b \in Y$. One inclusion is obvious, whereas for the other inclusion it suffices to observe that, for every $q \in Q/\beta$, there exists $p \in Q/\alpha$ such that $(p, q) \in Q$ and for every $b \in Y$, $(b, c)\ \alpha \circ \beta\ (p, q)$ (by definition of Q), showing that $(b, q)\ \alpha\ (b, c)$, as claimed. Therefore,

$$\bigcup_{p \in S} p/\alpha = \bigcup_{b \in Y} (b, c)/\alpha = \bigcup_{b \in Y} (\{b\} \times Q/\beta) = Y \times Q/\beta.$$

Also, $E = X \times \{c\}$ and $F = Z \times \{d\}$, for some $X, Z \subseteq Q/\alpha$, yielding

$$\bigcup_{e \in E} e/\alpha = \bigcup_{h \in X} (h, c)/\alpha = \bigcup_{h \in X} (\{h\} \times U) = X \times U,$$

$$\bigcup_{f \in F} f/\alpha = \bigcup_{k \in Z} (k, d)/\alpha = \bigcup_{k \in Z} (\{k\} \times W) = Z \times W,$$

for some $U, W \subseteq Q/\beta$, satisfying $U = \{u : (a, u) \in Q_1\}$ and $W = \{w : (z, w) \in Q_2\}$ and $|U| = |e/\alpha|$, $|W| = |f/\alpha|$, for every $e \in E$, $f \in F$, as results of Claim 5.2.7.5 and Claim 5.2.7.7.

It is obvious that X, Y, Z are non-empty and disjoint, because so are E and S, and F and T. For U and W, if $U \cap W \neq \varnothing$, then there exists $v \in Q/\beta$ such that

$$\widehat{x} = (a, c)\ \alpha\ (a, u)\ \beta\ (z, u)\ \alpha\ (z, d) = \widehat{y},$$

contradicting Claim 5.2.7.3.

If we call $V := Q/\beta - (U \cup W)$ (potentially empty), we translate Claim 5.2.7.5 and 5.2.7.7 in

$$Q_1 = (X \times U) \cup [Y \times (U \cup V \cup W)] \leq \mathbf{Q},$$

$$Q_2 = [Y \times (U \cup V \cup W)] \cup (Z \times W) \leq \mathbf{Q}.$$

By Lemma 5.2.1, we get that $\mathcal{V}$ admits M^-SHR_2 or M^+SHR_2. If it admits M^-SHR_2, then by Theorem 5.2.6, it also admits M^+SHR_2. In any case, the variety $\mathcal{V}$ admits M^+SHR_2, as we wanted to prove. ☐

Theorem 5.2.7 allows us to state the main result of this section and its direct corollary.

Theorem 5.2.8. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following are equivalent:*

1. *$\mathcal{V}$ is congruence 3-permutable;*
2. *$\mathcal{V}$ omits M^+SHR_2;*
3. *$\mathcal{V}$ omits M^+SF_3.*

Proof. The equivalence of (2) and (3) is clear by Theorem 5.2.5, which in particular holds for locally finite idempotent varieties.

If $\mathcal{V}$ is any variety which is congruence 3-permutable, then it has to contain neither 2-dimensional Hagemann relations nor failures of congruence 3-permutability; in particular it has to omit M^+SHR_2 and M^+SF_3, proving that (1) implies (2) and (3).

Finally, let $\mathcal{V}$ be a locally finite idempotent variety which fails to be congruence 3-permutable. By Corollary 3.1.1, the free algebra in $\mathcal{V}$ over 4 generators $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(\{x, y, z, w\})$ is not congruence 3-permutable. Moreover, $\mathbf{F}$ is finite and idempotent because $\mathcal{V}$ is locally finite and idempotent: by Theorem 5.2.7, $\mathbf{HSP}(\mathbf{F}) \subseteq \mathcal{V}$ admits M^+SHR_2, showing that (2) implies (1) and concluding the proof. ☐

Corollary 5.2.1. *If $\mathcal{V}$ and $\mathcal{W}$ are two locally finite idempotent varieties which are not congruence 3-permutable, then $\mathcal{V} \vee \mathcal{W}$ is not congruence 3-permutable either.*

In other words, congruence 3-permutability is a prime strong Maltsev condition with respect to locally finite idempotent varieties.

Proof. If $\mathcal{V}$ and $\mathcal{W}$ are locally finite idempotent varieties failing congruence 3-permutability, then $\mathcal{V}, \mathcal{W} \notin \Omega(M^+SHR_2)$ by Theorem 5.2.8. By Theorem 5.2.4, $\mathcal{V} \vee \mathcal{W} \notin \Omega(M^+SHR_2)$, which implies that $\mathcal{V} \vee \mathcal{W}$ is not congruence 3-permutable (Theorem 5.2.8), as we wished to prove. □

Likewise for the case of idempotent congruence 2-permutability, we can provide a characterization of congruence 3-permutability for locally finite idempotent varieties in terms of tolerances, as a consequence of Theorem 5.2.8.

Corollary 5.2.2. *Let $\mathcal{V}$ be a locally finite idempotent variety. The following statements are equivalent:*

1. *$\mathcal{V}$ is not congruence 3-permutable;*

2. *There exist $\mathbf{A} \in \mathcal{V}$ and a partition $\{A_0, A_1, M, A_2, A_3\}$ of A such that*

$$T := (A_0 \cup A_1)^2 \cup (A_1 \cup M \cup A_2)^2 \cup (A_2 \cup A_3) \in \operatorname{Tol} \mathbf{A}.$$

Proof. For the entire proof, let $\mathcal{V}$ be a locally finite idempotent variety.

If $\mathcal{V}$ is not congruence 3-permutable, then it admits M^+SHR_2 by Theorem 5.2.8, that is there exists $\mathbf{A} \in \mathcal{V}$ carrying a generalized special Hagemann relation of dimension 2 with middle part $R \leq \mathbf{A} \times \mathbf{A}$, induced by the partition $\rho_R^+ = \{A_0, A_1, M, A_2, A_3\}$, where M is the middle part.

Define $T := R \cap R^{-1} \subseteq A^2$. Because R and R^{-1} are subuniverses of $\mathbf{A} \times \mathbf{A}$, then so is T. Moreover, T is symmetric and reflexive, hence it is a tolerance on $\mathbf{A}$. A set theoretical calculation similar to the one computed in Corollary 5.1.2 shows that

$$T = (A_0 \cup A_1)^2 \cup (A_1 \cup M \cup A_2)^2 \cup (A_2 \cup A_3)^2,$$

proving that (1) implies (2).

Conversely, if $\mathcal{V}$ satisfies (2), then $\mathbf{T} \leq \mathbf{A} \times \mathbf{A}$ is a failure of congruence 3-permutability. To prove this, let

$$\alpha = (0_A \otimes 1_A) \cap T^2,$$
$$\beta = (1_A \otimes 0_A) \cap T^2.$$

For any $a_i \in A_i$, $i \in \{0, 1, 2, 3\}$, it is not hard to show, involving techniques already used several times, that

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \ \beta \ \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} \ \alpha \ \begin{bmatrix} a_2 \\ a_2 \end{bmatrix} \ \beta \ \begin{bmatrix} a_3 \\ a_2 \end{bmatrix},$$

although

$$\left(\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \begin{bmatrix} a_3 \\ a_2 \end{bmatrix} \right) \notin \alpha \circ \beta \circ \alpha,$$

showing that also (2) implies (1). □

Going back to Theorem 5.2.7, the power of such a result is that it provides an argument for the construction of a generalized 2-dimensional special Hagemann relation with middle part, which is completely semantic, namely without the appeal to any syntactic manipulation of terms or free algebras (as, for instance, we have seen in Theorem 5.1.2 for idempotent congruence 2-permutability). Moreover, the procedure described in the proof of the theorem (which we will refer to as the HV-procedure, standing for *Horizontal and Vertical reductions procedure*) can be applied to any finite idempotent non-3-permutable algebra: this fact can be rephrased by saying that we are always able to build *locally* a generalized 2-dimensional special Hagemann relation with middle part out of a given finite idempotent algebra that fails congruence 3-permutability.

Also, still looking at the proof of Theorem 5.2.7, we wish to point out that the finiteness of the algebra is invoked several times, namely when we need to maximize some sets, or when we need to prevent the chain of subuniverses $Q^{(n)}$'s from descending forever. This latter action need not be possible within an infinite algebra, for which a chain of subuniverses may have no lower bound at all.

Next, we are going to give an example of a countably infinite algebra witnessing the fact that the HV-procedure may fail without the assumption of finiteness (in this specific instance, the failure occurs in one of the vertical reductions).

Example 5.2.1. Let $\mathbf{A} = \langle A; F\rangle$ be the algebra defined by:

$$A := \{(i,j) \in (\omega+1)\times(\omega+1) : \; i \geq j\} \cup \{(\omega+1,\omega)\} \subseteq (\omega+2)\times(\omega+1),$$

$$F := \{f : A^n \to A : \; n \geq 1, f \text{ is an idempotent polymorphism of } \alpha \text{ and } \beta\},$$

where $\alpha = (0_{\omega+2} \otimes 1_{\omega+1}) \cap A^2$ and $\beta = (1_{\omega+2} \otimes 0_{\omega+1}) \cap A^2$. We can represent A as in the following figure:

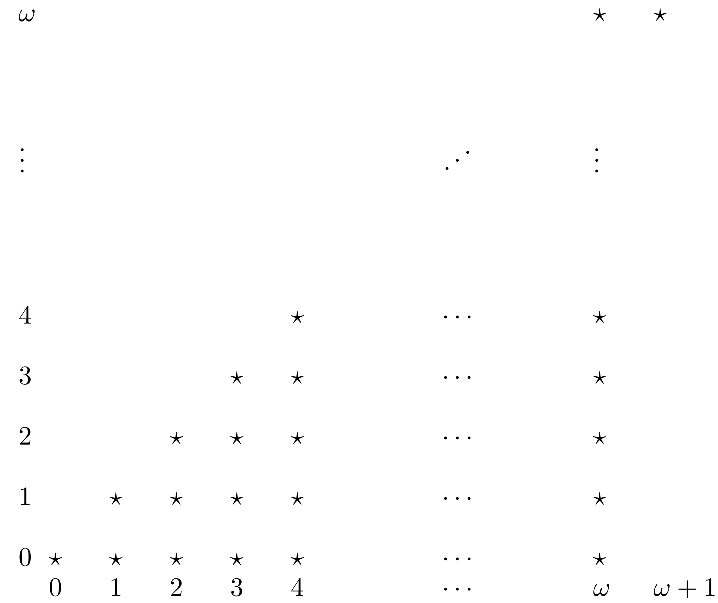

Figure 5.9: The countably infinite algebra **A**

It is straightforward to note that, for any $j \leq i < \omega$,

$$\begin{bmatrix} i \\ j \end{bmatrix} \;\beta\; \begin{bmatrix} \omega \\ j \end{bmatrix} \;\alpha\; \begin{bmatrix} \omega \\ \omega \end{bmatrix} \;\beta\; \begin{bmatrix} \omega+1 \\ \omega \end{bmatrix}.$$

If we also assume

$$\begin{bmatrix} i \\ j \end{bmatrix} \;\alpha\; \begin{bmatrix} i \\ j' \end{bmatrix} \;\beta\; \begin{bmatrix} \omega+1 \\ j' \end{bmatrix} \;\alpha\; \begin{bmatrix} \omega+1 \\ \omega \end{bmatrix},$$

for some $j' \in \omega+1$, then $(\omega+1, j') \in A$ forces $j' = \omega$, yielding in turn $(i,\omega) \in A$, which is impossible by definition of A itself.

Therefore, α and β are non-3-permuting congruences of the algebra $\mathbf{A}$. Using the same notation as the one in the proof of Theorem 5.2.7, if we apply the horizontal reductions described in the HV-procedure starting with $x = (0,0)$ and $y = (\omega+1,\omega)$, then we get that $Q = Q^{(0)} = A$ (i.e. the algebra $\mathbf{A}$ is already horizontally reduced), $S = \{(\omega, 0)\}$ and $T = \{(\omega,\omega)\}$.

On the other hand, by setting $r := (\omega, 0)$ and $p_i := (i, 0) \in x/\beta$, for $i < \omega$, we have that

$$b_1[p_i, r](p_i/\alpha) = \{\omega\} \times (i+1).$$

Hence, there is no $i < \omega$ such that $b_1[p_i, r](p_i/\alpha)$ is $\subseteq$-maximal, preventing us from being able to apply the first vertical reduction and find $\widehat{x}$.

The algebra **A** of Example 5.2.1 generates a variety that is clearly not congruence 3-permutable, but for which we do not know whether it admits or not M^+SHR_2. If **A** were not congruence n-permutable, for any $n \geq 2$ (in fact, we expect F contains Hagemann-Mitschke polymorphisms), then $\mathbf{HSP}(\mathbf{A})$ would admit SHR_2 by Theorem 4.2.3, and hence it would admit M^+SHR_2, by Theorem 5.2.6. This fact is implicitly linked to a previous appeal to a syntactic approach, given that Theorem 4.2.3 can be deduced with the helpful provision of Theorem 4.2.2, whose proof (see [42]) heavily deals with terms and syntactic constructions.

Nonetheless, should F contain Hagemann-Mitschke polymorphisms, we would not be able, for now, to produce in $\mathbf{HSP}(\mathbf{A})$ a generalized 2-dimensional special Hagemann relation with middle part anyways, not even via some syntactic procedure applicable, for instance, to some suitable free algebra.

More generally, we can reasonably ask the following question: can omitting M^+SHR_2 for idempotent varieties characterize congruence 3-permutability? In other words, can Theorem 5.2.8 be generalized also to non-locally finite idempotent varieties?

Unfortunately, despite multiple attempts, we do not know the answer to these questions yet. Theorem 5.2.8 can be obviously generalized to any idempotent variety containing at least one finite algebra failing to be congruence 3-permutable (so not necessarily a locally finite variety), but we have no argument, in this sense, valid for those non-congruence 3-permutable idempotent varieties containing only infinite failures of congruence 3-permutability (the variety generated by **A** of Example 5.2.1 might happen to be part of these).

A clue that it is not erroneous to think about $\Omega^{id}(M^+SHR_2)$ as a possible candidate for characterizing CP_3^{id} is the theorem that we are going to present next, which consistently generalizes Theorem 5.2.2. We are going to use a result discovered by J. Opršal while proving that idempotent congruence modularity is a prime Maltsev condition (see Theorem 4.7 of [31]). For this purpose, we first need a preliminary discussion, the details of which can be consulted in [31].

Let us then begin with the following definition.

Definition 5.2.4 ([31])**.** Let **A** and **B** be two similar algebras such that α and β denote the kernels of the projection maps from $\mathbf{A} \times \mathbf{B}$ onto, respectively, **A** and **B**. Let also $\gamma \in \mathrm{Con}(\mathbf{A} \times \mathbf{B})$ be a congruence with $\gamma < \alpha$, and for $a \in A$, denote the following equivalence of B by

$$\gamma^a := \left\{ (b_1, b_2) \in B^2 : \left(\begin{bmatrix} a \\ b_1 \end{bmatrix}, \begin{bmatrix} a \\ b_2 \end{bmatrix} \right) \in \gamma \right\}.$$

We say that γ is a *modularity blocker in* $\mathbf{A} \times \mathbf{B}$ if there exist a partition $\{H_\gamma, K_\gamma\}$ of A and an equivalence relation $\eta < 1_B$ of B, such that

$$\gamma^a = \begin{cases} 1_B & \text{if } a \in H_\gamma; \\ \eta & \text{if } a \in K_\gamma. \end{cases}$$

In such a case, we will sometimes say that γ is *induced by* $\{H_\gamma, K_\gamma\}$ *and* η.

The existence of modularity blockers characterizes idempotent non-congruence modular varieties, due to the following theorem.

Theorem 5.2.9 ([31])**.** *For an idempotent variety $\mathcal{V}$, the following are equivalent:*

1. *$\mathcal{V}$ is not congruence modular;*
2. *there exists a modularity blocker in $\mathbf{F}_\mathcal{V}(\{x, y\}) \times \mathbf{F}_\mathcal{V}(\{x, y\})$.*

Referring to Example 2.2.2, let us call

$$CM := \{\mathcal{V} : (\exists n \geq 2)[\mathcal{CM}_n \leq \mathcal{V}]\},$$

i.e. the Maltsev class of congruence modular varieties. Let us also denote by CM^{id} the class of (interpretability types of) idempotent varieties in CM.

We are then ready to state and prove the theorem that we have been preparing the background for.

Theorem 5.2.10. *Let $\mathcal{V}$ be an idempotent variety. If $\mathcal{V}$ omits M^+SHR_2, then $\mathcal{V}$ is congruence modular.*

In other words,

$$\Omega^{id}(M^+SHR_2) \subseteq CM^{id}.$$

Proof. Let us show the contrapositive statement: hence, let $\mathcal{V}$ be any idempotent variety which is not congruence modular and call $\mathbf{F}$ its algebra freely generated by $\{x, y\}$. By Theorem 5.2.9, there exists a modularity blocker γ in $\mathbf{F} \times \mathbf{F}$, which is to say $\gamma \in \mathrm{Con}(\mathbf{F} \times \mathbf{F})$, and for a partition $\{H_\gamma, K_\gamma\}$ of F and an equivalence relation $\eta < 1_F$, we have that, for all $a \in F$,

$$\text{if } a \in H_\gamma, \text{ then } \gamma^a = 1_F,$$

$$\text{if } a \in K_\gamma, \text{ then } \gamma^a = \eta.$$

Furthermore, call α and β respectively the kernels of the projections from $\mathbf{F} \times \mathbf{F}$ onto the first and the second coordinate, respectively, and also recall that $\gamma < \alpha$.

Since $\eta < 1_F$, there are at least two distinct equivalence η-classes; call them b_1/η and b_2/η, for some distinct $b_1, b_2 \in F$. Now, pick any $a \in K_\gamma$ and notice that for $i = 1, 2$

$$(a, b_i)/\beta \circ \gamma = \bigcup_{q \in F} (q, b_i)/\gamma = \left(\bigcup_{h \in H_\gamma} (h, b_i)/\gamma \right) \cup \left(\bigcup_{k \in K_\gamma} (k, b_i)/\gamma \right).$$

To prove this displayed expression, fix $i \in \{1, 2\}$ and immediately notice that the last equality on the right is obviously true since $\{H_\gamma, K_\gamma\}$ is a partition of F. For the other displayed equality, assume first $(e, b) \in (q, b_i)/\gamma$. Since $\gamma < \alpha$, then $e = q$, so $(e, b) = (q, b)$. Therefore,

$$\begin{bmatrix} a \\ b_i \end{bmatrix} \; \beta \; \begin{bmatrix} q \\ b_i \end{bmatrix} \; \gamma \; \begin{bmatrix} q \\ b \end{bmatrix},$$

showing that the inclusion " $\supseteq$ " holds.

Conversely, if $(e, b) \in (a, b_i)/\beta \circ \gamma$, then there exists $(q, c) \in F \times F$ satisfying

$$\begin{bmatrix} a \\ b_i \end{bmatrix} \; \beta \; \begin{bmatrix} q \\ c \end{bmatrix} \; \gamma \; \begin{bmatrix} e \\ b \end{bmatrix}.$$

This expression implies that $c = b_i$ and $q = e$, hence

$$\begin{bmatrix} q \\ b_i \end{bmatrix} = \begin{bmatrix} q \\ c \end{bmatrix} = \begin{bmatrix} e \\ c \end{bmatrix} \; \gamma \; \begin{bmatrix} e \\ b \end{bmatrix},$$

as expected.

At this point, notice that, for $h \in H_\gamma$,

$$(h, b_i)/\gamma = \{(h, b) : (b_i, b) \in \gamma^h = 1_F\} = \{h\} \times F;$$

whereas, for $k \in K_\gamma$,

$$(k, b_i)/\gamma = \{(k, b) : (b_i, b) \in \gamma^k = \eta\} = \{(k, b) : b \in b_i/\eta\} = \{k\} \times b_i/\eta,$$

for $i = 1, 2$. If we rewrite the above equality, then we get

$$S_i := (a, b_i)/\beta \circ \gamma = \left(\bigcup_{h \in H_\gamma} (h, b_i)/\gamma \right) \cup \left(\bigcup_{k \in K_\gamma} (k, b_i)/\gamma \right) = (H_\gamma \times F) \cup (K_\gamma \times b_i/\eta).$$

By idempotence, $S_i \leq \mathbf{F} \times \mathbf{F}$. Moreover, $b_1/\eta \cap b_2/\eta = \varnothing$, being two η-classes. If we call $X := K_\gamma =: Z$, $Y := H_\gamma$, $U := b_1/\eta$, $W := b_2/\beta$ and $V := F - (U \cup W)$, then the assumptions of Lemma 5.2.1 are satisfied (recall that X and Z are not required to be disjoint) and hence $\mathcal{V}$ admits M^-SHR_2 if $V = \varnothing$, or M^+SHR_2 otherwise. If we further invoke Theorem 5.2.6 in the case $V = \varnothing$, then we obtain that $\mathcal{V}$ admits M^+SHR_2, completing the proof. $\square$

To close this section, we are going to show that it is not possible to characterize congruence 3-permutability through the omission of 2-dimensional special Hagemann relations, not even with respect to locally finite idempotent varieties. In order to achieve such a result, we will prove that the Maltsev classes $\Omega^{id}(SHR_2)$ and $\Omega^{id}(M^+SHR_2)$ do not coincide. Indeed, we already know that

$$\Omega^{id}(M^-SHR_2) = \Omega^{id}(SHR_2) = \Omega^{id}(M^-SF_3) \supseteq \Omega^{id}(M^+SF_3) = \Omega^{id}(M^+SHR_2),$$

and our aim will be to define a finitely generated variety omitting M^-SF_3, which admits M^+SF_3 instead.

A pair of lemmas will be presented preliminarily. The first of them basically states that under certain assumptions a compatible relation between two homomorphic images of algebras has its shape preserved whenever pulled back to the original algebras themselves. More precisely,

Lemma 5.2.2. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be similar algebras and* $\varphi : \mathbf{A} \to \mathbf{C}$, $\psi : \mathbf{B} \to \mathbf{D}$ *be surjective homomorphisms. Moreover, let* $\{C_\gamma : \gamma \in \Gamma\}$ *and* $\{D_\delta : \delta \in \Delta\}$ *be partitions of, respectively,* C *and* D*, such that there exists* $\Sigma \subseteq \Gamma \times \Delta$*, with*

$$S_\Sigma := \bigcup_{(\gamma,\delta)\in\Sigma} C_\gamma \times D_\delta \leq \mathbf{C} \times \mathbf{D}.$$

Then, $\{\varphi^{-1}(C_\gamma) : \gamma \in \Gamma\}$ *and* $\{\psi^{-1}(D_\delta) : \delta \in \Delta\}$ *are partitions of, respectively,* A *and* B*, and furthermore*

$$(\varphi \times \psi)^{-1})(S_\Sigma) = \bigcup_{(\gamma,\delta)\in\Sigma} \varphi^{-1}(C_\gamma) \times \psi^{-1}(D_\delta) \leq \mathbf{A} \times \mathbf{B}.$$

Proof. Assume we have all the objects defined in the statement and call $A_\gamma := \varphi^{-1}(C_\gamma) \subseteq A$, $B_\delta := \psi^{-1}(D_\delta) \subseteq B$, for every $\gamma \in \Gamma$, $\delta \in \Delta$. It is elementary to prove that $\{A_\gamma : \gamma \in \Gamma\}$ and $\{B_\delta : \delta \in \Delta\}$ are partitions of A and B, respectively, hence we proceed to the next step. Call $R_\Sigma := (\varphi \times \psi)^{-1})(S_\Sigma)$, which is necessarily a subuniverse of $\mathbf{A} \times \mathbf{B}$, being the inverse image of a subuniverse of $\mathbf{C} \times \mathbf{D}$ through the homomorphism $\varphi \times \psi$. Moreover,

$$(a,b) \in R_\Sigma \;\Leftrightarrow\; (\varphi(a), \psi(b)) \in S_\Sigma \;\Leftrightarrow$$

$$\Leftrightarrow\; \exists(\gamma_0,\delta_0) \in \Sigma[(\varphi(a),\psi(b)) \in C_{\gamma_0} \times D_{\delta_0}] \;\Leftrightarrow$$

$$\Leftrightarrow\; \exists(\gamma_0,\delta_0) \in \Sigma[(a,b) \in \varphi^{-1}(C_{\gamma_0}) \times \psi^{-1}(D_{\delta_0})] = A_{\gamma_0} \times B_{\delta_0} \;\Leftrightarrow$$

$$\Leftrightarrow\; (a,b) \in \bigcup_{(\gamma,\delta)\in\Sigma} A_\gamma \times B_\delta,$$

showing that

$$R_\Sigma = \bigcup_{(\gamma,\delta)\in\Sigma} A_\gamma \times B_\delta.$$

□

On the other hand, the shape of a compatible relation between two algebras also gets preserved when restricting it to subalgebras, provided that some rather general hypotheses are satisfied.

Lemma 5.2.3. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be similar algebras such that* $\mathbf{C} \leq \mathbf{A}$ *and* $\mathbf{D} \leq \mathbf{B}$*. Also, let* $\{A_i : i \in I\}$ *and* $\{B_j : j \in J\}$ *be partitions of, respectively,* A *and* B*, such that* $A_i \cap C \neq \varnothing$ *and* $B_j \cap D \neq \varnothing$*, for all* $i \in I$*,* $j \in J$*. If there exists* $\Sigma \subseteq I \times J$ *such that*

$$S_\Sigma := \bigcup_{(i,j)\in\Sigma} A_i \times B_j \leq \mathbf{A} \times \mathbf{B},$$

then $\{A_i \cap C : i \in I\}$ *and* $\{B_j \cap D : j \in J\}$ *are also partitions of* C *and* D*, respectively, and*

$$S_\Sigma \cap (C \times D) = \bigcup_{(i,j)\in\Sigma} (A_i \cap C) \times (B_j \cap D) \leq \mathbf{C} \times \mathbf{D}.$$

Proof. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be algebras as described in the statement: for convenience, call $C_i := A_i \cap C$ and $D_j := B_j \cap D$, for all $i \in I$, $j \in J$.

Again, we omit the elementary proof of the fact that $\{C_i : i \in I\}$ is a partition of C and $\{D_j : j \in J\}$ is a partition of D. For the remaining part of the proof, call $R_\Sigma := S_\Sigma \cap (C \times D)$ and notice that it is a subuniverse of $\mathbf{C} \times \mathbf{D}$ for it is the intersection of two subuniverses.

Moreover,

$$R_\Sigma = S_\Sigma \cap (C \times D) =$$
$$= \left(\bigcup_{(i,j)\in\Sigma} A_i \times B_j \right) \cap (C \times D) = \bigcup_{(i,j)\in\Sigma} [(A_i \times B_j) \cap (C \times D)] =$$
$$= \bigcup_{(i,j)\in\Sigma} (A_i \cap C) \times (B_j \cap D) = \bigcup_{(i,j)\in\Sigma} C_i \times D_j,$$

concluding the proof. □

A special consequence of these fairly general lemmas is presented in another lemma which is stated below.

Lemma 5.2.4. *Let $\mathcal{V}$ be any variety. If $\mathcal{V}$ admits M^-SF_3, then a special failure of congruence 3-permutability without middle portion can be found as a subdirect product of $\mathbf{F}_\mathcal{V}(\{x, y, z\}) \times \mathbf{F}_\mathcal{V}(\{x, y\})$, whose universe extends the subuniverse generated by $\{(x, x), (y, x), (y, y), (z, y)\}$.*

Proof. Suppose $\mathcal{V}$ is a variety admitting M^-SF_3, which is to say there exist $\mathbf{P}, \mathbf{Q} \in \mathcal{V}$, partitions $\{X, Y, Z\}$ of P and $\{U, W\}$ of Q, and $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$, such that

$$S = (X \times U) \cup (Y \times Q) \cup (Z \times W).$$

In the respective non-empty sets in the partitions, fix any elements $a \in X$, $b \in Y$, $c \in Z$, $u \in U$ and finally $w \in W$, and consider the following subuniverses:

$$P' := \mathrm{Sg}^{\mathbf{P}}(\{a, b, c\}) \leq \mathbf{P},$$
$$Q' := \mathrm{Sg}^{\mathbf{Q}}(\{u, w\}) \leq \mathbf{Q}.$$

Immediately notice that $X' := P' \cap X$, $Y' := P' \cap Y$ and $Z' := P' \cap Z$ are non-empty, because they respectively contain a, b and c. Likewise, $U' := Q' \cap U$ contains u and $W' := Q' \cap W$ contains w, showing that these are non-empty as well.

By Lemma 5.2.3, we can deduce that $\{X', Y', Z'\}$ and $\{U', W'\}$ are still partitions of, respectively, P' and Q' and

$$S' := S \cap (P \times Q) = (X' \times U') \cup (Y' \times Q') \cup (Z' \times W').$$

Thus, $\mathbf{S}'$ is again a special failure of congruence 3-permutability without middle portion in $\mathcal{V}$.

At this point, note that $\mathbf{P}'$ and $\mathbf{Q}'$ are, respectively, 3-generated and 2-generated, which means they are homomorphic images of, respectively, $\mathbf{F}_3 := \mathbf{F}_\mathcal{V}(\{x, y, z\})$ and $\mathbf{F}_2 := \mathbf{F}_\mathcal{V}(\{x, y\})$. More precisely, if we define the maps

$$\{x, y, z\} \to \{a, b, c\} : \ x \rightsquigarrow a, \ y \rightsquigarrow b, \ z \rightsquigarrow c,$$
$$\{x, y\} \to \{u, w\} : \ x \rightsquigarrow u, \ y \rightsquigarrow w,$$

then these extend to surjective homomorphisms

$$\varphi : \mathbf{F}_3 \twoheadrightarrow \mathbf{P}',$$
$$\psi : \mathbf{F}_2 \twoheadrightarrow \mathbf{Q}'.$$

If we call $\widehat{X} := \varphi^{-1}(X')$, $\widehat{Y} := \varphi^{-1}(Y')$, $\widehat{Z} := \varphi^{-1}(Z')$, $\widehat{U} := \psi^{-1}(U')$ and $\widehat{W} := \psi^{-1}(W')$, then, by Lemma 5.2.2, we deduce that $\{\widehat{X}, \widehat{Y}, \widehat{Z}\}$ is a partition of F_3, $\{\widehat{U}, \widehat{W}\}$ is a partition of F_2 and

$$\widehat{S} := (\varphi \times \psi)^{-1}(S') = (\widehat{X} \times \widehat{U}) \cup (\widehat{Y} \times F_2) \cup (\widehat{Z} \times \widehat{W}).$$

Hence, $\widehat{\mathbf{S}} \leq_{sd} \mathbf{F}_3 \times \mathbf{F}_2$ is a special failure of congruence 3-permutability without middle portion in $\mathcal{V}$. Moreover, because $(x,x) \in \widehat{X} \times \widehat{U}$, $(y,x),(y,y) \in \widehat{Y} \times F_2$ and $(z,y) \in \widehat{Z} \times \widehat{W}$, we also get that

$$\mathrm{Sg}^{\mathbf{F}_3 \times \mathbf{F}_2}(\{(x,x),(y,x),(y,y),(z,y)\}) \leq \widehat{\mathbf{S}},$$

completing the proof. □

Let us now construct a finite algebra generating a variety which will be placed in $\Omega^{id}(SHR_2) - \Omega^{id}(M^+SHR_2)$ by an argument which will soon be developed. We also wish to point out that part of the claims made in the following example can be verified by the use of the UACalc software [14].

Example 5.2.2. Let $\mathbf{F} = \langle 5; p^{\mathbf{F}}, f^{\mathbf{F}} \rangle$ be the algebra of type $\langle 3, 2 \rangle$, where $p^{\mathbf{F}}$ and $f^{\mathbf{F}}$ are defined by the following tables:

$f^{\mathbf{F}}$	0	1	2	3	4
0	0	1	3	2	2
1	1	1	3	2	2
2	3	3	2	1	1
3	2	2	1	3	3
4	2	2	1	3	4

$p^{\mathbf{F}}$	0	1	2	3	4
$(0,0)$	0	1	1	1	1
$(0,1)$	0	0	0	0	0
$(0,2)$	0	0	0	0	0
$(0,3)$	0	0	0	0	0
$(0,4)$	0	0	0	0	0
$(1,0)$	1	1	1	1	1
$(1,1)$	1	1	1	1	1
$(1,2)$	1	1	1	1	1
$(1,3)$	1	1	1	1	1
$(1,4)$	1	1	1	1	1
$(2,0)$	2	2	2	2	2
$(2,1)$	2	2	2	2	2
$(2,2)$	2	2	2	2	2
$(2,3)$	2	2	2	2	2
$(2,4)$	2	2	2	2	2
$(3,0)$	3	3	3	3	3
$(3,1)$	3	3	3	3	3
$(3,2)$	3	3	3	3	3
$(3,3)$	3	3	3	3	3
$(3,4)$	3	3	3	3	3
$(4,0)$	4	4	4	4	4
$(4,1)$	4	4	4	4	4
$(4,2)$	4	4	4	4	4
$(4,3)$	4	4	4	4	4
$(4,4)$	3	3	3	3	4

Notice that both $p^{\mathbf{F}}$ and $f^{\mathbf{F}}$ are idempotent, and hence $\mathcal{F} := \mathbf{HSP}(\mathbf{F})$ is an idempotent variety as well.

It turns out that the algebra $\mathbf{F}$ itself is freely generated by $\{0, 4\}$. In other words,

$$\mathbf{F} \cong \langle \{x, p(x,x,y), f(x,y), p(y,y,x), y\}; p^{\mathbf{F}_{\mathcal{F}}(\{x,y\})}, f^{\mathbf{F}_{\mathcal{F}}(\{x,y\})} \rangle = \mathbf{F}_{\mathcal{F}}(\{x, y\}).$$

In fact, $p^{\mathbf{F}}$ and $f^{\mathbf{F}}$ are built in such a way that the following two equivalence relations on 5 are congruences of $\mathbf{F}$

$$\alpha = \{(0,0)\} \cup \{1,2,3\}^2 \cup \{(4,4)\},$$
$$\beta = \{0,1\}^2 \cup \{(2,2)\} \cup \{3,4\}^2$$

(equivalently, $p^{\mathbf{F}}$ and $f^{\mathbf{F}}$ are polymorphisms of α and β).

Moreover, notice that α and β do not 3-permute, because

$$0 \ \beta \ 1 \ \alpha \ 3 \ \beta \ 4,$$

but $(0, 4) \notin \alpha \circ \beta \circ \alpha$. More precisely, if we look at the subdirect representation of $\mathbf{F}$ in $\mathbf{F}/\alpha \times \mathbf{F}/\beta$, we have that such a representation is indeed a special failure of congruence 3-permutability with middle portion.

As a matter of fact, let $\mathbf{F}'$ be the subalgebra of $\mathbf{F}/\alpha \times \mathbf{F}/\beta$ which is isomorphic to $\mathbf{F}$. More precisely,

$$F/\alpha = \{\{0\}, \{1,2,3\}, \{4\}\}$$

and

$$F/\beta = \{\{0,1\}, \{2\}, \{3,4\}\},$$

and if we rename $\overline{0} = 0/\alpha$, $\overline{1} = 1/\alpha = \{1,2,3\}$, $\overline{2} = 4/\alpha = \{4\}$, $\widehat{0} = 0/\beta = \{0,1\}$, $\widehat{1} = 2/\beta = \{2\}$ and $\widehat{2} = 3/\beta = \{3,4\}$, we get that

$$F' = \{(\overline{0}, \widehat{0})\} \cup \{(\overline{1}, \widehat{0}), (\overline{1}, \widehat{1}), (\overline{1}, \widehat{2})\} \cup \{(\overline{2}, \widehat{2})\} =$$
$$= (\underbrace{\{\overline{0}\}}_{X} \times \underbrace{\{\widehat{0}\}}_{U}) \cup [\underbrace{\{\overline{1}\}}_{Y} \times (\underbrace{\{\widehat{0}\}}_{U} \cup \underbrace{\{\widehat{1}\}}_{V} \cup \underbrace{\{\widehat{2}\}}_{W})] \cup (\underbrace{\{\overline{2}\}}_{Z} \times \underbrace{\{\widehat{2}\}}_{W}),$$

which is a special failure of congruence 3-permutability with middle portion V. This shows that $\mathcal{F}$ admits M^+SF_3, i.e. $\mathcal{F} \notin \Omega^{id}(M^+SF_3) = \Omega^{id}(M^+SHR_2)$.

However, $\mathcal{F}$ is congruence 4-permutable, and its Hagemann-Mitschke terms are exactly

$$p_1(x,y,x) := p(x,y,z);$$
$$p_2(x,y,z) := f((x,z),y);$$
$$p_3(x,y,z) := p(z,y,x).$$

Eventually, we are going to show that, informally, there is no way to get rid of the middle portions whenever we have a special failure of congruence 3-permutability in $\mathcal{F}$, and the reason why this happens is explained in the next theorem, which represents a sufficient condition for the omission of $SHR_2 = M^-SHR_2$.

Beforehand, define $\mathcal{R}$ as the variety of type $\langle 3,3,3 \rangle$ and basic operation symbols $\langle p, q, r \rangle$, axiomatized by the following equations:

$$\begin{aligned} p(x,y,y) &\approx x; \\ p(x,x,y) &\approx q(x,y,y); \\ q(x,x,y) &\approx q(y,x,x); \\ q(x,y,x) &\approx q(y,x,y); \\ r(x, q(x,y,y), q(x,y,x)) &\approx q(x,x,y). \end{aligned}$$

This variety (or better, its interpretability type) represents a strong Maltsev condition in $\mathbf{L}$. Furthermore,

Theorem 5.2.11. *Let $\mathcal{V}$ be any idempotent variety. If $\mathcal{R} \leq \mathcal{V}$, then $\mathcal{V} \in \Omega^{id}(M^-SF_3) = \Omega^{id}(SHR_2)$.*

Proof. Let $\mathcal{V}$ be an idempotent variety and suppose $\mathcal{R} \leq \mathcal{V}$. Without loss of generality, we may call p, q and r three terms of $\mathcal{V}$ satisfying the equations axiomatizing $\mathcal{R}$. By idempotency, it will be sufficient to prove that $\mathcal{V}$ omits M^-SF_3, for also proving the omission of $M^-SHR_2 = SHR_2$ (Theorem 5.2.5).

Suppose further $\mathcal{V}$ admits M^-SF_3 and call $\mathbf{F}_3$ and $\mathbf{F}_2$ the free algebras in $\mathcal{V}$ generated by, respectively, $\{x, y, z\}$ and $\{x, y\}$: by Lemma 5.2.4, we may assume that a special failure of congruence 3-permutability without middle portion $\mathbf{S}$ can be found as a subdirect product of $\mathbf{F}_3 \times \mathbf{F}_2$. Moreover, $R \subseteq S$, where R denotes the subuniverse of $\mathbf{F}_3 \times \mathbf{F}_2$ generated by $\{(x, x), (y, x), (y, y), (z, y)\}$.

Let $\{X, Y, Z\}$ be the partition of F_3 and $\{U, W\}$ be the partition of F_2 such that

$$S = (X \times U) \cup (Y \times F_2) \cup (Z \times W).$$

Besides, by the proof of Lemma 5.2.4, we know that $(x, x) \in X \times U$, $y \in Y$, $(z, y) \in Z \times W$. By idempotence, $x/R, x/S, z/R$ and z/S are all subuniverses of $\mathbf{F}_2$ such that

$$x/R \subseteq x/S = U,$$

$$z/R \subseteq z/S = W,$$

implying that, in fact, $U, W \leq \mathbf{F}_2$. Moreover, we can easily verify that

$$x = p^{\mathbf{F}_2}(x, y, y)\ R\ p^{\mathbf{F}_2}(x, x, y) = q^{\mathbf{F}_2}(x, y, y),$$

$$z = p^{\mathbf{F}_2}(z, y, y)\ R\ p^{\mathbf{F}_2}(y, y, x) = q^{\mathbf{F}_2}(y, x, x),$$

which yield $(x, q^{\mathbf{F}_2}(x, y, y)) \in S$ and $(z, q^{\mathbf{F}_2}(y, x, x)) \in S$, or equivalently, $q^{\mathbf{F}_2}(x, y, y) = q(x, y, y) \in U$ and $q^{\mathbf{F}_2}(y, x, x) = q(y, x, x) \in W$.

After the previous observations, consider the term $t(x, y) := q(x, y, x) = q(y, x, y) \in F_2$: because F_2 is partitioned by $\{U, W\}$, then either $t(x, y) \in U$, or $t(x, y) \in W$.

If $t(x, y) \in U$, then, because also $x, q(x, y, y) \in U$ and $U \leq \mathbf{F}_2$, in particular U is closed under the action of the term operation $r^{\mathbf{F}_2}$, yielding

$$r^{\mathbf{F}_2}(x, q(x, y, y), t(x, y)) = r^{\mathbf{F}_2}(x, q(x, y, y), q(x, y, x)) =$$

$$= q^{\mathbf{F}_2}(x, x, y) = q(x, x, y) = q(y, x, x) \in U.$$

This contradicts the fact that $q(y, x, x) \in W$, as previously proven.

Therefore, $t(x, y) \in W$: by the same argument as above, since $y, q(y, x, x) \in W$, which is a subuniverse of $\mathbf{F}_2$, we get that, in particular

$$r^{\mathbf{F}_2}(y, q(y, x, x), t(x, y)) = r^{\mathbf{F}_2}(y, q(y, x, x), q(y, x, y)) =$$

$$= q^{\mathbf{F}_2}(y, y, x) = q(y, y, x) = q(x, y, y) \in W,$$

contradicting $q(x, y, y) \in U$.

In either case, we deduce a contradiction, which is to say it is erroneous to assume that there is a special failure of congruence 3-permutability without middle portion in $\mathcal{V}$, showing that $\mathcal{V}$ must omit M^-SF_3. $\square$

This theorem would imply nothing interesting if the variety $\mathcal{R}$ were equi-interpretable to the trivial variety. In fact, the following corollary guarantees that is not the case and allows us to finally reach the desired result of separating the classes $\Omega^{id}(SHR_2) = \Omega^{id}(M^-SF_3)$ and $\Omega^{id}(M^+SHR_2) = \Omega^{id}(M^+SF_3)$.

Corollary 5.2.3. *There exists an idempotent variety omitting M^+SHR_2 but admitting SHR_2. Therefore,*

$$\Omega^{id}(SHR_2) = \Omega^{id}(M^-SF_3) \supsetneq \Omega^{id}(M^+SF_3) = \Omega^{id}(M^+SHR_2).$$

Proof. We are going to prove that the variety $\mathcal{F}$ presented in Example 5.2.2 satisfies $\mathcal{R} \leq \mathcal{F}$. Also recall that $\mathcal{F}$ is idempotent. Moreover, because $\mathcal{F}$ is not a trivial variety, this implies that $\mathcal{R}$ is not trivial either.

Define the following terms of $\mathcal{F}$:

$$p(x, y, z) := p(x, y, z),$$

$$q(x, y, z) := f(f(x, z), y),$$

$$r(x, y, z) := f(x, z).$$

Because $\mathcal{F}$ is generated by $\mathbf{F}$, it suffices to show that the axioms of $\mathcal{R}$ hold in $\mathbf{F}$. By looking at the tables in Example 5.2.2, it is straightforward to verify that, for all $0 \leq i, j \leq 4$,

$$p^{\mathbf{F}}(i, j, j) = i,$$

$$f^{\mathbf{F}}(i, j) = f^{\mathbf{F}}(j, i),$$

which shows $p(x, y, y) \approx x$ and $f(x, y) \approx f(y, x)$ hold in $\mathcal{F}$. A bit more time is needed to verify that for $i, j \in 5$

$$p^{\mathbf{F}}(i, i, j) = q^{\mathbf{F}}(i, j, j),$$

which implies the validity of the corresponding equation in all of $\mathcal{F}$.

For the other identities, let us proceed as follows:

$$q(x, x, y) = f(f(x, y), x) \approx f(f(y, x), x) = q(y, x, x);$$

$$q(x, y, x) = f(f(x, x), y) \approx f(x, y) \approx f(y, x) \approx f(f(y, y), x) = q(y, x, y);$$

$$r(x, q(x, y, y), q(x, y, x)) = f(x, q(x, y, x)) =$$

$$= f(x, f(f(x, x), y)) \approx f(x, f(x, y)) \approx f(f(x, y), x) = q(x, x, y).$$

Hence, $\mathcal{R} \leq \mathcal{F}$, and by Theorem 5.2.11 we have that $\mathcal{F} \in \Omega^{id}(M^- SF_3) = \Omega^{id}(SHR_2)$. Moreover, we have already observed in Example 5.2.2 that $\mathcal{F}$ admits $M^+ SF_3$, and hence $M^+ SHR_2$ by Theorem 5.2.5, proving that $\mathcal{F} \notin \Omega^{id}(M^+ SHR_2)$ (which also yields $\mathcal{R} \notin \Omega(M^+ SHR_2)$). $\square$

To sum up, we have proven that locally finite idempotent congruence 3-permutability can be characterized by the omission of $M^+ SHR_2$ (Theorem 5.2.8) but not by the omission of SHR_2 (Corollary 5.2.3): in fact we have produced an example of a locally finite idempotent variety, namely $\mathcal{F}$ from Example 5.2.2, which is not congruence 3-permutable but omits SHR_2. Moreover, we have noticed that omitting $M^+ SHR_2$ for idempotent varieties implies congruence modularity (Theorem 5.2.10). Instead, the omission of SHR_2 does not imply congruence modularity, because there exists a variety in $\Omega^{id}(SHR_2)$ which is not congruence modular: as a matter of fact, consider again the variety $\mathcal{F}$ of Example 5.2.2 and recall that the generating algebra $\mathbf{F}$ is such that $\mathbf{F} = \mathbf{F}_{\mathcal{F}}(\{0, 4\})$. Now, consider the partition $\{H, K\}$ of F, defined by

$$H = \{0, 1, 2, 3\},$$

$$K = \{4\},$$

and the equivalence relation β on 5 as defined in Example 5.2.2 (which is, in fact, a congruence of $\mathbf{F}$). It is not hard to check that $\{H, K\}$ and β induce the modularity blocker $\gamma \in \mathrm{Con}(\mathbf{F} \times \mathbf{F})$, preventing $\mathcal{F}$ from being congruence modular (Theorem 5.2.9).

5.3 Some considerations on the case of 4-permutability

In the previous two sections we have analyzed the cases of congruence 2 and 3-permutability and we have seen that the primeness argument holding for idempotent congruence 2-permutability has a far more complex generalization to the case of locally finite idempotent congruence 3-permutability. As one could expect at this point, other complications arise for the case of congruence 4-permutability, even just considering certain kinds of failures in the idempotent setting. In this section, we are going to present some results which represent some primeness arguments for special cases of congruence 4-permutability. However, unlike locally finite idempotent non-congruence 3-permutable varieties, which turn out to always contain a uniform type of failure (namely a special failure of congruence 3-permutability with middle portion, due to Theorem 5.2.7 and Theorem 5.2.5), non-congruence 4-permutable varieties seem not to meet such a requirement: this fact has led us to conjecturing that congruence 4-permutability might be a non-prime strong Maltsev condition.

Let us then begin with considering a hierarchy (in a sense that will become clearer later) of algebras failing congruence 4-permutability, which deserve a specific name.

Definition 5.3.1. Let $\mathbf{P}$ and $\mathbf{Q}$ be two similar algebras, $k < \omega$ be a finite cardinal and let $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$. We say that $\mathbf{S}$ is a *special failure of congruence* 4*-permutability of genus* k if there exist (potentially empty) $M_1 \subsetneq P$, $M_2 \subsetneq Q$ and

- a partition $\{X, Y, Z\} \cup \{P_i : i \in k\}$ of $P - M_1$,
- a partition $\{U, V, W\} \cup \{Q_i : i \in k\}$ of $Q - M_2$,

such that

$$S = (X \times U) \cup [Y \times (Q - W)] \cup \bigcup_{i \in k} \left[P_i \times \left(V \cup \bigcup_{i \leq j < k} Q_j \right) \right] \cup (M_1 \times V) \cup [Z \times (V \cup W)].$$

For $i \in \{1, 2\}$, M_i is called the i^{th} *middle portion*: if M_i is non-empty (resp. empty), then we say that $\mathbf{S}$ is *with* (resp. *without*) i^{th} *middle portion.* If $\mathbf{S}$ is with (resp. without) both 1^{st} and 2^{nd} middle portion, then we say that it is *with* (resp. *without*) *middle portions.*

Note immediately that we have not excluded the case $k = 0 = \varnothing$, meaning that $\{P_i : i \in 0\} = \{Q_i : i \in 0\} = \varnothing$. In order to clarify the previous definition, below we provide a pictorial representation of a special failure of congruence 4-permutability of genus 3 with middle portions, from which one can easily figure out the shape in the general case.

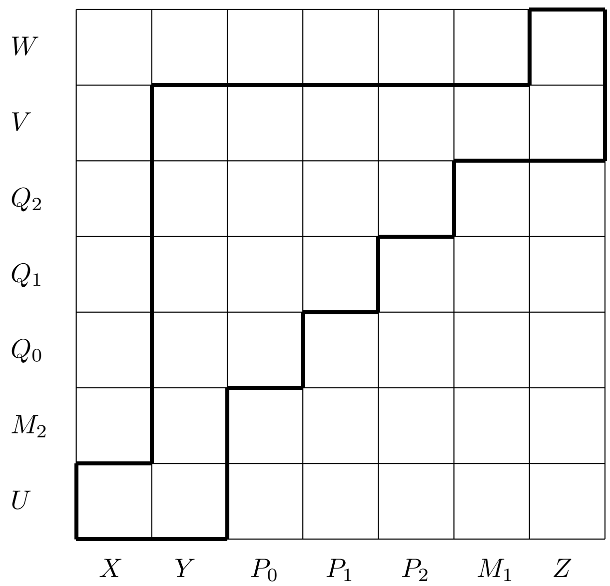

Figure 5.10: A special failure of congruence 4-permutability of genus 3 with middle portions

Notice that a standard argument using the kernels of the projection maps proves that a special failure of congruence 4-permutability of genus k is a non-congruence 4-permutable algebra for every $k \geq 0$. As a matter of fact, if $\mathbf{S}$ is as defined in Definition 5.3.1, then, by calling α and β the kernels of the projection maps from $P \times Q$ onto, respectively, P and Q, with domains restricted to S, we get that for any $(x, u) \in X \times U$, $(z, w) \in Z \times W$, $y \in Y$ and $v \in V$,

$$\begin{bmatrix} x \\ u \end{bmatrix} \ \beta \ \begin{bmatrix} y \\ u \end{bmatrix} \ \alpha \ \begin{bmatrix} y \\ v \end{bmatrix} \ \beta \ \begin{bmatrix} z \\ v \end{bmatrix} \ \alpha \ \begin{bmatrix} z \\ w \end{bmatrix},$$

yet $(x, u) \ \cancel{\alpha \circ \beta \circ \alpha \circ \beta} \ (z, w)$.

Likewise for special Hagemann relations and special failures of congruence 2 and 3-permutability, we can collect together all those (interpretability types of) varieties which do not present models with the shape of special failures of congruence 4-permutability.

Definition 5.3.2. For a variety $\mathcal{V}$ and a natural number $k \geq 0$, we say that $\mathcal{V}$ *admits* $M_1^{\pm} M_2^{\pm} SF_4^k$ if there exists $\mathbf{S} \in \mathcal{V}$ which is a special failure of congruence 4-permutability of genus k with (in case of $+$) or without (in case of $-$) 1^{st} or 2^{nd} middle portion; otherwise, we say that $\mathcal{V}$ *omits* $M_1^{\pm} M_2^{\pm} SF_4^k$. To abbreviate, we will use the conventions $M^+ SF_4^k := M_1^+ M_2^+ SF_4^k$ and $M^- SF_4^k := M_1^- M_2^- SF_4^k$.

Furthermore, we denote by $\Omega(M_1^{\pm} M_2^{\pm} SF_4^k)$ the class of varieties omitting $M_1^{\pm} M_2^{\pm} SF_4^k$, and by $\Omega^{id}(M_1^{\pm} M_2^{\pm} SF_4^k)$ the class of idempotent varieties in $\Omega(M_1^{\pm} M_2^{\pm} SF_4^k)$.

As a further clarification, for example a variety omitting $M_1^- M_2^+ SF_4^k$ has no special failures of congruence 4-permutability of genus k without 1^{st} middle portion and with 2^{nd} middle portion. We do not have a proof of the fact that these omission classes are Maltsev classes, even though we believe that a similar argument to the one used for proving Theorem 4.1.1 (although a lot messier) occurs for these classes as well.

At a varietal level, as it is the case for special failures of congruence 3-permutability, special failures of congruence 4-permutability with middle portions can always be produced out of special failures without middle portions in an idempotent setting. For this purpose, we are about to state a theorem which has the same flavor as Theorem 5.2.6 for congruence 3-permutability and generalizes in some sense the construction presented in its proof. In order to make this section more fluent and more focused on the results rather than on the reasonings that lead to them, we are going to omit the long proofs, which can be found in detail in Appendix A.

Theorem 5.3.1. *Let $k \geq 0$ be any integer, $\mathcal{V}$ be an idempotent variety and assume $\mathcal{V}$ admits $M_1^- M_2^{\pm} SF_4^k$ (resp. $M_1^{\pm} M_2^- SF_4^k$). Then $\mathcal{V}$ also admits $M_1^+ M_2^{\pm} SF_4^k$ (resp. $M_1^{\pm} M_2^+ SF_4^k$).*

In other words, for all $k \geq 0$,

$$\Omega^{id}(M_1^+ M_2^{\pm} SF_4^k) \subseteq \Omega^{id}(M_1^- M_2^{\pm} SF_4^k),$$

$$\Omega^{id}(M_1^{\pm} M_2^+ SF_4^k) \subseteq \Omega^{id}(M_1^{\pm} M_2^- SF_4^k).$$

Proof. See Section A.1 of Appendix A. □

As we mentioned earlier, Theorem 5.3.1 provides a procedure to build middle portions whenever these are missing in special failures of congruence 4-permutability, while the genus is maintained unchanged. Instead, the next result shows that in an idempotent variety it is possible to increase the genus of admitted special failures of congruence 4-permutability. More precisely,

Theorem 5.3.2. *For an idempotent variety $\mathcal{V}$ and an integer $k \geq 0$, if $\mathcal{V}$ admits $M_1^{\pm} M_2^{\pm} SF_4^k$, then $\mathcal{V}$ also admits $M_1^{\pm} M_2^+ SF_4^{k+1}$.*

In particular, for any $k \geq 0$,

$$\Omega^{id}(M^+ SF_4^{k+1}) \subseteq \Omega^{id}(M_1^{\pm} M_2^+ SF_4^{k+1}) \subseteq \Omega^{id}(M_1^{\pm} M_2^{\pm} SF_4^k) \subseteq \Omega^{id}(M^- SF_4^k).$$

Proof. See Section A.2 of Appendix A. □

Theorem 5.3.1 claims that, with respect to idempotent varieties, the omission classes of special failures of congruence 4-permutability form a chain as the genus varies in ω. If we just focus on omission classes of idempotent varieties of special failures with middle portions (this restriction is justified by Theorem 5.3.1), Theorem 5.3.2 claims that the following chain can be built up to eventually reach the class of idempotent congruence 4-permutable varieties:

$$\Omega^{id}(M^+SF_4^0) \supseteq \Omega^{id}(M^+SF_4^1) \supseteq \cdots \supseteq \Omega^{id}(M^+SF_4^k) \supseteq \Omega^{id}(M^+SF_4^{k+1}) \supseteq \cdots \supseteq CP_4^{id}.$$

The optimal situation would be that $CP_4^{id} = \bigcap_{k<\omega} \Omega^{id}(M^+SF_4^k)$, which would yield the primeness of congruence 4-permutability with respect to idempotent varieties, due to the following result.

Theorem 5.3.3. *Let $m, n < \omega$ be natural numbers, $k = \max\{n, m\}$ and $\mathcal{V}$ and $\mathcal{W}$ be idempotent varieties. If $\mathcal{V} \notin \Omega^{id}(M^+SF_4^m)$ and $\mathcal{W} \notin \Omega^{id}(M^+SF_4^n)$, then $\mathcal{V} \vee \mathcal{W} \notin \Omega^{id}(M^+SF_4^k)$.*

Proof. If $\mathcal{V}$ and $\mathcal{W}$ satisfy the conditions exposed in the statement, then finitely many applications of Theorem 5.3.2 yield that $\mathcal{V}, \mathcal{W} \notin \Omega^{id}(M^+SF_4^k)$. By a similar technique as the one invoked in Corollary 4.1.2 (which we omit), we get to deduce that also $\mathcal{V}\vee\mathcal{W} \notin \Omega^{id}(M^+SF_4^k)$, as desired. □

Unfortunately, CP_4^{id} seems to be strictly contained in $\bigcap_{k<\omega} \Omega^{id}(M^+SF_4^k)$, even just restricting to locally finite varieties. In fact, we have not been able to prove a theorem for locally finite idempotent congruence 4-permutability that is comparable to Theorem 5.2.7, although some observations have led us to imagine what a failure of congruence 4-permutability may be reduced to. We will provide then some examples and explain which obstacles are encountered when trying to prove a primeness argument even just for locally finite idempotent congruence 4-permutable varieties. Theorem 5.3.2 gives evidence of the fact that, in idempotent varieties, the presence of special failures of congruence 4-permutability with middle portions of genus $k \geq 0$ also yields the admission by the same variety of special failures of any higher genus h, for all $h \geq k$. As a matter of fact, the proof of Theorem 5.3.2 itself describes a procedure doing so. However, there is a wide collection of wild kinds of admissible failures, which seem not to be reducible to simpler ones, for which we have not found any procedure that may place these varieties in an omission class (or in a family of omission classes) of some uniform configuration(s).

Informally speaking, we can consider special failures of congruence 4-permutability of some genus $k > 1$, from which we have removed some Cartesian products of the form $P_i \times Q_j$, for $0 \leq i \leq j < k$. In order to deal with these objects more formally, we will provide the following definition.

Definition 5.3.3. Let k be a finite cardinal, $\mathbf{P}$ and $\mathbf{Q}$ be similar algebras and $\mathbf{S}$ be a subdirect product of $\mathbf{P} \times \mathbf{Q}$. Let further $\Sigma \subseteq (k \times k)^+$ [3] and let $\{X, Y, Z\} \cup \{P_i : i \in k\}$ be a partition of $P - M_1$ ($M_1 \subsetneq P$) and $\{U, V, W\} \cup \{Q_i : i \in k\}$ be a partition of $Q - M_2$ ($M_2 \subsetneq Q$). We say that $\mathbf{S}$ is a *Σ-holed special failure of congruence 4-permutability of genus k with/without i^{th} middle portion* (whether $M_i \neq \varnothing$ or $M_i = \varnothing$) for $i = 1, 2$, if

$$S = \left[(X \times U) \cup [Y \times (Q - W)] \cup \bigcup_{i\in k} \left[P_i \times \left(V \cup \bigcup_{i\leq j<k} Q_j \right) \right] \cup \right.$$

$$\left. \cup (M_1 \times V) \cup [Z \times (V \cup W)] \right] - \bigcup_{(i,j)\in\Sigma} P_i \times Q_j.$$

Notice that, for given $k < \omega$, a $\varnothing$-holed special failure of congruence 4-permutability of genus k is exactly a special failure of congruence 4-permutability of genus k, whereas any $(k\times k)^+$-holed special failure of congruence 4-permutability of genus k is indeed a special failure of congruence 4-permutability of genus 0 with both middle portions. Another specific example of such a special failure, precisely a $\{(0, 2)\}$-holed special failure of congruence 4-permutability of genus 3 with both middle portions, is pictured below

[3] $(k \times k)^+ := \{(i, j) \in k \times k : i \leq j\}$.

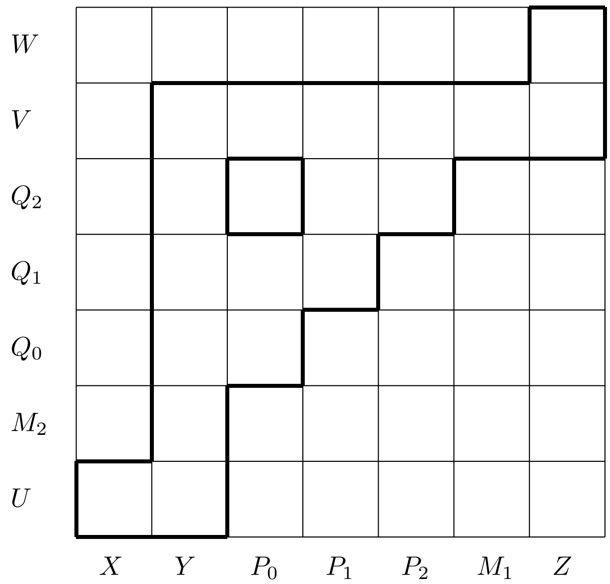

Figure 5.11: A $\{(0,2)\}$-holed special failure of congruence 4-permutability of genus 3 with middle portions

We have prepared the above background so as to pose the following question: given any pair of idempotent varieties $\mathcal{V}, \mathcal{W}$, finite cardinals n, m, $\Sigma \subseteq (n \times n)^+$ and $\Lambda \subseteq (m \times m)^+$, if $\mathcal{V}$ contains a Σ-holed special failure of congruence 4-permutability of genus n and $\mathcal{W}$ contains a Λ-holed special failure of congruence 4-permutability of genus m, does there exist a finite cardinal k, uniformly depending on n and m, and $\Gamma \subseteq (k \times k)^+$, uniformly depending on Σ and Λ, such that both $\mathcal{V}$ and $\mathcal{W}$ contain a Γ-holed special failure of congruence 4-permutability of genus k?

We do not know the answer to this question in the general case, yet we do in the particular case of $\Sigma = \Lambda = \varnothing$, for which the answer is affirmative and exactly corresponds to Theorem 5.3.2 and Theorem 5.3.3, where $k = \max\{m, n\}$ and $\Gamma = \varnothing$.

Indeed, the sets Σ and Λ, or in other words the positions of the holes, could be rather random and this fact is a clue strengthening our expectation that the answer to the previous question is generally negative. What this could entail is an instance of a pair of (locally finite) idempotent non-congruence 4-permutable varieties, whose coproduct (i.e. join in the lattice of interpretability types) does not contain any failure of congruence 4-permutability, namely is congruence 4-permutable. Such an example would finally prove that congruence 4-permutability is not a prime strong Maltsev condition, which we strongly believe and conjecture.

In this direction, Theorem 5.3.3 suggests that a potential example of a pair of idempotent varieties witnessing the failure of primeness for congruence 4-permutability need have at least one of the two non-congruence 4-permutable varieties belong to $\bigcap_{k<\omega} \Omega^{id}(M^+SF_4^k)$; some attempts have been made in this sense, unfortunately without succeeding. Nonetheless, we feel that another small light cone has been turned on and directed to an area which was still looking pretty dark and worth investigating.

Chapter 6

A family of strong Maltsev conditions implying congruence n-permutability

In this chapter we will be analyzing a family of strong Maltsev conditions which turn out to be stronger than congruence n-permutability at a varietal level and the motivation for this is given by the fact that most of the well known congruence n-permutable algebras and varieties that universal algebraists deal with, actually satisfy these strongest Maltsev conditions.

In the next section, we will provide the definition of these conditions and we will support with examples our claim that they are frequently met and used.

6.1 The axioms Δ_n and motivating examples

Let us begin with the following definition

Definition 6.1.1. Let $n > 1$ be an integer and $h_1, \ldots, h_{k_n}$ be 3-ary function symbols, for $k_n = \lfloor \frac{n}{2} \rfloor$. Define Δ_n as the set of the following equations:

- if n is odd, then Δ_n contains exactly

$$\begin{aligned} x &\approx h_1(x, y, y); \\ h_i(x, x, y) &\approx h_{i+1}(x, y, y) \text{ for } 1 \leq i < k_n; \\ h_{k_n}(x, x, y) &\approx h_{k_n}(y, y, x). \end{aligned}$$

- if n is even, then Δ_n contains exactly

$$\begin{aligned} x &\approx h_1(x, y, y); \\ h_i(x, x, y) &\approx h_{i+1}(x, y, y) \text{ for } 1 \leq i < k_n; \\ h_{k_n}(x, x, y) &\approx h_{k_n}(y, x, x). \end{aligned}$$

Let us call $\mathcal{D}_n$ the variety axiomatized by Δ_n, i.e. $\mathcal{D}_n = \mathrm{Mod}(\Delta_n)$, and henceforth refer to $h_1, \ldots, h_{k_n}$ as Δ_n-*terms.*

Furthermore, if a variety $\mathcal{V}$ satisfies $\mathcal{D}_n \leq \mathcal{V}$, for some $n > 1$, we will say that $\mathcal{V}$ *is* Δ_n.

Note that, for every $n > 1$, the variety $\mathcal{D}_n$ is idempotent and defines a strong Maltsev condition in the lattice of interpretability types. Also, notice that the variety $\mathcal{D}_2$ is equi-interpretable to the variety $\mathcal{HM}_2 = \mathcal{CP}_2$ (see Theorem 3.1.2), which is to say the strong Maltsev filter generated by $\mathcal{D}_2$ is exactly CP_2, the class of congruence 2-permutable varieties. As a matter of fact, the Maltsev term p_1 of $\mathcal{CP}_2$ satisfies $p_1(x,y,y) \approx x$ and $p_1(x,x,y) \approx y$, and hence interprets the Δ_2-term h_1 by setting

$$h_1(x,y,z) := p_1(x,y,z).$$

On the other hand, the Δ_2-term h_1 is actually a Maltsev term, because it satisfies

$$h_1(x,y,y) \approx x,$$

$$h_1(x,x,y) \approx h_1(y,x,x) \approx y.$$

The latter reasoning can be generalized to every $n \geq 2$, as we will see in the next theorem. However, we want to remark that the set $\{\mathcal{D}_n : n > 1\}$ is not a Maltsev condition, because we can show that it is not always the case that $\mathcal{D}_{n+1} \leq \mathcal{D}_n$ (we will justify this claim later). Nonetheless, a weaker fact is true, which is proven in the following theorem along with the connection to congruence n-permutability.

Theorem 6.1.1. *For $n \geq 2$, we have that*

$$\mathcal{D}_{n+2} \leq \mathcal{D}_n;$$

$$\mathcal{CP}_n \leq \mathcal{D}_n.$$

In other words, every Δ_n variety is both congruence n-permutable and Δ_{n+2}.

Proof. Fix $n > 1$ for the purpose of this proof and denote $k = \lfloor \frac{n}{2} \rfloor$. Let us first show that $\mathcal{D}_{n+2}$, whose basic operation symbols are $h'_1, \dots, h'_{k_n}, h'_{k_n+1}$, is interpretable in $\mathcal{D}_n$, with basic operation symbols $h_1, \dots, h_{k_n}$ (notice that $k_{n+2} = \lfloor \frac{n+2}{2} \rfloor = \lfloor \frac{n}{2} + 1 \rfloor = \lfloor \frac{n}{2} \rfloor + 1 = k_n + 1$).

Define

$$\begin{aligned} h'_1(x,y,z) &:= x \\ h'_i(x,y,z) &:= h_{i-1}(x,y,z) \text{ for } 2 \leq i \leq k_n + 1. \end{aligned}$$

We then have

$$\begin{aligned} h'_1(x,y,y) &\approx x; \\ h'_1(x,x,y) &\approx x \approx h_1(x,y,y) = h'_2(x,y,y); \\ h'_i(x,x,y) &= h_{i-1}(x,x,y) \approx h_i(x,y,y) = h'_{i+1}(x,y,y) \text{ for } 2 \leq i < k_n + 1; \\ h'_{k_n+1}(x,x,y) &= h_{k_n}(x,x,y) \approx h_{k_n}(y,u,x) = h'_{k_n+1}(y,u,x); \end{aligned}$$

where

$$u = \begin{cases} x & \text{for even } n \\ y & \text{for odd } n \end{cases}.$$

This proves that $\mathcal{D}_{n+2} \leq \mathcal{D}_n$.

For the other interpretation, define,

$$\begin{aligned} p_i(x,y,z) &:= h_i(x,y,z) \text{ for } 1 \leq i \leq k_n; \\ p_i(x,y,z) &:= h_{n-i}(z,y,x) \text{ for } k_n < i \leq n-1. \end{aligned}$$

Let us distinguish the two cases:

- If n is odd, then $k_n = \frac{n-1}{2}$, and hence

$$\begin{aligned} x &\approx h_1(x,y,y) = p_1(x,y,y); \\ p_i(x,x,y) &= h_i(x,x,y) \approx h_{i+1}(x,y,y) = p_{i+1}(x,y,y) \text{ for } 1 \leq i \leq k_n - 1; \\ p_{k_n}(x,x,y) &= h_{k_n}(x,x,y) \approx h_{k_n}(y,y,x) \overset{1}{=} p_{k_n+1}(x,y,y); \\ p_i(x,x,y) &= h_{n-i}(y,x,x) \approx h_{n-i-1}(y,y,x) \overset{2}{=} p_{i+1}(x,y,y) \text{ for } k_n < i \leq n-2; \\ p_{n-1}(x,x,y) &\overset{3}{=} h_1(y,x,x) \approx y. \end{aligned}$$

- If $n > 2$ is even (the case $n = 2$ was discussed previously), then $k_n = \frac{n}{2}$, and

$$\begin{aligned} x &\approx h_1(x,y,y) = p_1(x,y,y);\\ p_i(x,x,y) &= h_i(x,x,y) \approx h_{i+1}(x,y,y) = p_{i+1}(x,y,y) \text{ for } 1 \leq i \leq k_n - 1;\\ p_{k_n}(x,x,y) &= h_{k_n}(x,x,y) \approx h_{k_n}(y,x,x) \approx h_{k_n-1}(y,y,x) =^4 p_{k_n+1}(x,y,y);\\ p_i(x,x,y) &= h_{n-i}(y,x,x) \approx h_{n-i-1}(y,y,x) =^2 p_{i+1}(x,y,y) \text{ for } k_n < i \leq n-2;\\ p_{n-1}(x,x,y) &=^3 h_1(y,x,x) \approx y. \end{aligned}$$

In either case, $p_1, \dots, p_{n-1}$ are Hagemann-Mitschke terms witnessing congruence n-permutability, proving that $\mathcal{CP}_n \leq \mathcal{D}_n$. □

Theorem 6.1.1 yields that the families $\{\mathcal{D}_{2n} : n \geq 1\}$ and $\{\mathcal{D}_{2n+1} : n \geq 1\}$ are Maltsev conditions, although we cannot claim the same as far as the family $\{\mathcal{D}_n : n \geq 2\}$ is concerned, because of the following argument.

Consider the variety $\mathcal{C}_2$, whose unique function symbol is s, of type $\langle 2 \rangle$, axiomatized by

$$s(x,x) \approx x;$$

$$s(x,y) \approx s(y,x);$$

i.e. the variety generating the strong Maltsev filter of all varieties having a binary commutative idempotent term.

By definition of $\mathcal{D}_n$, for $n = 2m+1$ and $m \geq 1$ (i.e. with odd indexes), it is clear that

$$\mathcal{C}_2 \leq \mathcal{D}_{2m+1},$$

due to the equation $h_{k_n}(x,x,y) \approx h_{k_n}(y,y,x)$, which implies that the interpreted term $s(x,y) := h_{k_n}(x,x,y)$ satisfies $s(x,x) \approx x$ and $s(x,y) \approx s(y,x)$. Therefore, for any variety $\mathcal{V}$, whenever $\mathcal{V}$ is Δ_n, for odd $n \geq 3$, then $\mathcal{V}$ has to have a binary commutative idempotent term.

Thus, consider the following example.

Example 6.1.1. Let $\mathbf{Z}_n$ denote the cyclic group of order $n \geq 1$. It is well known that $\mathbf{Z}_n$ is congruence 2-permutable (every group is) thanks to the Maltsev term $p_1^{\mathbf{Z}_n}(x,y,z) := x - y + z$. Moreover, it is another known fact that every m-ary term operation $w^{\mathbf{Z}_n}$ of $\mathbf{Z}_n$ is of the form

$$w^{\mathbf{Z}_n}(x_1, \dots, x_m) = \sum_{i=1}^{m} a_i x_i,$$

where $a_1, \dots, a_m$ are integers and $a_i x_i$ stands for either $x_i + \ldots + x_i$ or $-x_i - \ldots - x_i$ with $|a_i|$ many addends, according to whether $a_i \geq 0$ or $a_i < 0$, respectively.

Fix any even $n \geq 2$ and assume that $\mathbf{Z}_n$ has a binary commutative idempotent term operation, call it $c^{\mathbf{Z}_n}$. By the above observation, $c^{\mathbf{Z}_n}$ has the form

$$c^{\mathbf{Z}_n}(x,y) = ax + by,$$

for some integers a, b. Because $c^{\mathbf{Z}_n}$ is idempotent then

$$a + b = 1 \mod n.$$

Moreover, commutativity yields that $a = b \mod n$: altogether, we obtain $2a = 1 \mod n$, which has no solution for n even.

[1] Notice that $n - k_n = n - \frac{n-1}{2} = \frac{n+1}{2} = \frac{n-1}{2} + 1 = k_n + 1$.
[2] Because $n - i - 1 = n - (i+1)$.
[3] Because $n - (n-1) = 1$.
[4] Notice that $k_n + 1 = \frac{n}{2} + 1 = n - \frac{n}{2} + 1 = n - (k_n - 1)$.

Therefore, if we call $\mathcal{Z}_n := \mathbf{HSP}(\mathbf{Z}_n)$, then we have that $\mathcal{C}_2 \not\leq \mathcal{Z}_{2m}$, for any $m \geq 1$, which in turn implies

$$\mathcal{D}_{2e+1} \not\leq \mathcal{Z}_{2m},$$

for any $e, m \geq 1$.

This fact has two direct consequences:

1. Because $\mathcal{Z}_n$ is congruence 2-permutable, and hence congruence n-permutable for every $n \geq 2$, then in particular it is congruence $(2e+1)$-permutable for every $e \geq 1$. Therefore, for each odd $n \geq 3$, there exist varieties (the $\mathcal{Z}_{2m}$'s themselves, $m \geq 1$) which are congruence n-permutable but not Δ_n, showing that for all $e \geq 1$,

$$\mathcal{D}_{2e+1} \not\leq \mathcal{CP}_{2e+1}.$$

   We do not know whether this also holds for even indexes.

2. Again, the fact that $\mathcal{Z}_{2m}$ is congruence 2-permutable, hence Δ_2, along with the observation of Theorem 6.1.1, allow us to deduce that $\mathcal{Z}_{2m}$ is Δ_{2e}, for every $m, e \geq 1$. However, since $\mathcal{D}_{2e+1} \not\leq \mathcal{Z}_{2m}$ for any $e, m \geq 1$, we get that

$$\mathcal{D}_{2e+1} \not\leq \mathcal{D}_{2e},$$

   for any $e \geq 1$ (the case of even indexes is discussed later on and yields opposite results).

Let us now consider a list of motivating examples (as anticipated in the title of this section), which will show that the most well known congruence n-permutable varieties are, in fact, Δ_n. Before that, let us also provide the following definition.

Definition 6.1.2. Let $n > 2$ be an integer and $\mathcal{V}$ a variety. We say that $\mathcal{V}$ is *sharply congruence n-permutable* if

$$\mathcal{CP}_n \leq \mathcal{V},\ \mathcal{CP}_{n-1} \not\leq \mathcal{V}.$$

On the other hand, we say that $\mathcal{V}$ is *purely congruence n-permutable* if

$$\mathcal{CP}_n \leq \mathcal{V},\ \mathcal{D}_n \not\leq \mathcal{V}.$$

Example 6.1.1 guarantees the existence of purely congruence n-permutable varieties for odd $n \geq 3$, although we are not able to demonstrate the existence of purely congruence n-permutable varieties, for any even $n > 3$. In the next examples we will also exhibit some well known instances of sharply congruence n-permutable varieties.

Example 6.1.2. We have already considered the case of Δ_2 varieties: we have proven that $\mathcal{CP}_2$ and $\mathcal{D}_2$ are equi-interpretable and it is hence worthwhile considering this as the first trivial example.

Example 6.1.3 ([29])**.** Implication algebras were first introduced by J.C. Abbott in [1], in order to formalize the logical connective of implication of the classical propositional logic. A. Mitschke used these structures to prove that there exist sharply congruence 3-permutable varieties, after E.T. Schmidt in [34] had provided the characterization of congruence n-permutability via a Maltsev condition (as presented in Theorem 3.1.1). We will use a different but equivalent notation from the one in [29] for implication algebras, which is closer to the intuition of implication in the logical sense.

The variety of implication algebras $\mathcal{I}$ is the class of models of type $\langle 2, 0\rangle$, whose unique basic operation symbols are denoted by $\to$ (binary) and 1 (constant), axiomatized by

$$(x \to y) \to x \approx x;$$

$$(x \to y) \to y \approx (y \to x) \to x;$$

$$x \to (y \to z) \approx y \to (x \to z);$$

$$x \to x \approx 1;$$

$$x \to 1 \approx 1;$$
$$1 \to x \approx x;$$

[the latter three equations are in fact a consequence of the first three, yet we have preferred to include them in the axioms].

The following 3-element algebra $\mathbf{I} = \langle \{a, b, 1\}; \to^{\mathbf{I}}, 1^{\mathbf{I}} \rangle$, where $1^{\mathbf{I}} = 1$ and $\to^{\mathbf{I}}$ is defined in the table below, turns out to be an implication algebra:

$\to^{\mathbf{I}}$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

It is not hard to see that

$$\alpha := \{a, 1\}^2 \cup \{(b, b)\},$$
$$\beta := \{(a, a)\} \cup \{b, 1\}^2,$$

are congruences of $\mathbf{I}$ for which $(a, b) \in \alpha \circ \beta$ but $(a, b) \notin \beta \circ \alpha$. This is to say $\mathbf{I}$ is not congruence 2-permutable, and so $\mathcal{I}$ is not as well. However, if we define

$$h_1(x, y, z) := (z \to y) \to x,$$

then we can verify that

$$h_1(x, y, y) = (y \to y) \to x \approx 1 \to x \approx x,$$
$$h_1(x, x, y) = (y \to x) \to x \approx (x \to y) \to y = h_1(y, y, x),$$

showing that $\mathcal{I}$ is Δ_3.

Thus, $\mathcal{I}$ is an example of a Δ_3 sharply congruence 3-permutable variety.

Example 6.1.4 ([33])**.** This example deals with the so called *lower BCK-semilattices* (briefly $BCK(\wedge)$'s), which were used by J.G. Raftery in [33] to provide a counterexample to a question that had been raised by H.P. Gumm and A. Ursini in [17] regarding the existence of non-congruence 3-permutable ideal determined varieties (for more details see the already mentioned article [17]). As a matter of fact, Raftery shows that the variety of $BCK(\wedge)$'s, call it $\mathcal{BCK}(\wedge)$, is sharply congruence 4-permutable.

Let us then define the variety of lower BCK-semilattices as the class of algebras $\mathbf{A} = \langle A; \wedge^{\mathbf{A}}, \cdot^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ of type $\langle 2, 2, 0 \rangle$, such that $\langle A; \wedge^{\mathbf{A}} \rangle$ is a semilattice and the following equations also hold (the symbol $\cdot$ is omitted and denoted by juxtaposition):

$$0x \approx 0;$$
$$x0 \approx x;$$
$$xx \approx 0;$$
$$((xy)(xz))(zy) \approx 0;$$
$$(x \wedge y)x \approx x;$$
$$x \wedge (y(yx)) \approx y(yx).$$

A non-straightforward argument, which can be found in the proof of the main theorem of [33] and which we omit, shows that the variety $\mathcal{BCK}(\wedge)$ is not congruence 3-permutable. However, if we define the terms

$$h_1(x, y, z) := x(yz),$$
$$h_2(x, y, z) := (x(xy)) \wedge (z(zy)),$$

we can verify the following:

$$h_1(x, y, y) = x(yy) \approx x0 \approx x;$$
$$h_1(x, x, y) = x(xy) \approx y \wedge (x(xy)) \approx (x(xy)) \wedge y \approx (x(xy)) \wedge y0 \approx (x(xy)) \wedge y(yy) = h_2(x, y, y);$$
$$h_2(x, x, y) = (x(xx)) \wedge (y(yx)) \approx (y(yx)) \wedge (x(xx)) = h_2(y, x, x).$$

This means that $\mathcal{D}_4 \leq \mathcal{BCK}(\wedge)$ and hence provides an example of a Δ_4 sharply congruence 4-permutable variety.

Example 6.1.5 ([35],[13]). In this example we will present those varieties which can be considered the first instance of sharply congruence n-permutable varieties for every $n \geq 3$. When Schmidt first found out in [34] the Maltsev condition describing congruence n-permutability, he did not provide any argument about the separation of the varieties $\mathcal{S}_n$'s that we introduced in Theorem 3.1.1. In the later paper [35], though, he built a family of varieties, the models of which he called *n-Boolean algebras*, generalizing the already well known Boolean algebras (1-Boolean algebras in this setting), which turned out to be sharply congruence $(n+1)$-permutable for all $n \geq 2$. Let us present their definition.

For $n \geq 1$, define the class of *n-Boolean algebras*, denoted by $\mathcal{B}_n$, as the class of all algebras with basic operation symbols $\langle \wedge, \vee, f_1 \ldots, f_n, o_0, \ldots, o_n \rangle$ and type $\langle 2, 2, \underbrace{1, \ldots, 1}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n+1 \text{ times}} \rangle$, such that, for all $\mathbf{B} \in \mathcal{B}_n$, $\langle B; \wedge, \vee \rangle$ is a distributive lattice, and it satisfies the following equations:

$$x \vee o_0 \approx x;$$

$$x \wedge o_n \approx x;$$

$$[(x \vee o_{i-1}) \wedge o_i] \vee f_i(x) \approx o_i \text{ for } 1 \leq i \leq n;$$

$$[(x \vee o_{i-1}) \wedge o_i] \wedge f_i(x) \approx o_{i-1} \text{ for } 1 \leq i \leq n.$$

Thus $\mathcal{B}_n$ is a variety for all $n \geq 1$.

Let us show that, for $n > 1$, the variety $\mathcal{B}_n$ is not congruence n-permutable. First, let us consider the reducts of the algebras defined in Example 3.8 of [22] and explicitly described in the proof of Theorem 8.4 of [13]. More precisely, for $n > 1$, let $\mathbf{K}_n$ be the algebra $\langle n; h_1^{\mathbf{K}_n}, \ldots, h_{n-1}^{\mathbf{K}_n} \rangle$ such that $h_i^{\mathbf{K}_n}$ is defined, for $1 \leq i < n$, as

$$h_i^{\mathbf{K}_n}(x, y, z) := (x \wedge z) \vee (x \wedge f_{n-i}(y)) \vee (z \wedge f_i(y)),$$

where $\langle n; \vee, \wedge \rangle$ is the n-chain $\mathbf{C}_n$ (as defined in the proof of Theorem 4.2.3) and $f_i(0) = \ldots = f_i(i-1) = i$, $f_i(i) = \ldots = f_i(n-1) = i-1$.

It is quite straightforward to check that, for $n > 2$, the algebra $\mathbf{B}_n := \langle n; \vee, \wedge, f_1, \ldots, f_{n-1}, o_0, \ldots, o_{n-1} \rangle$ is a model in $\mathcal{B}_{n-1}$, for $\vee^{\mathbf{B}_n} = \vee^{\mathbf{C}_n}$, $\wedge^{\mathbf{B}_n} = \wedge^{\mathbf{C}_n}$, $f_i^{\mathbf{B}_n} = f_i$, for $1 \leq i \leq n-1$, and $o_i^{\mathbf{B}_n} = i$, for $0 \leq i \leq n-1$. Moreover, the non-$(n-1)$-permuting congruences of $\mathbf{C}_n$, mentioned after the proof of Theorem 4.2.3, are also congruences of $\mathbf{B}_n$, showing that $\mathcal{B}_{n-1}$ is not congruence $(n-1)$-permutable, for any $n > 2$.

Going back to the reduct $\mathbf{K}_n$ of the algebra $\mathbf{B}_n$, in [13] the authors claim that $h_1, \ldots, h_{n-1}$ are Hagemann-Mitschke terms: we are going to prove that $h_1, \ldots, h_{\lfloor \frac{n}{2} \rfloor}$ are, in fact, Δ_n-terms for $\mathcal{K}_n := \mathbf{HSP}(\mathbf{K}_n)$.

Fix $n > 2$. Let us begin with proving that $h_i^{\mathbf{K}_n}(z, y, x) = h_{n-i}^{\mathbf{K}_n}(x, y, z)$.

$$\begin{aligned} h_i^{\mathbf{K}_n}(z, y, x) =& (z \wedge x) \vee (z \wedge f_{n-i}(y)) \vee (x \wedge f_i(y)) = \\ =& (x \wedge z) \vee (x \wedge f_{n-i}(y)) \vee (z \wedge f_i(y)) = h_{n-i}^{\mathbf{K}_n}(x, y, z), \end{aligned}$$

considering that $n - (n-i) = i$.

Furthermore,

if n is even, then

$$\begin{aligned} h_{\frac{n}{2}}^{\mathbf{K}_n}(x, x, y) =& (x \wedge y) \vee (x \wedge f_{\frac{n}{2}}(x)) \vee (y \wedge f_{\frac{n}{2}}(x)) = \\ =& (y \wedge x) \vee (y \wedge f_{\frac{n}{2}}(x)) \vee (x \wedge f_{\frac{n}{2}}(x)) = h_{\frac{n}{2}}^{\mathbf{K}_n}(y, x, x), \end{aligned}$$

considering that $n - \frac{n}{2} = \frac{n}{2}$;

if n is odd, then the proof proceeds case by case. Call $k = k_n = \frac{n-1}{2}$.

- $x = y$: straightforward;

- $x < y < k$:

$$\begin{aligned} h_k^{\mathbf{K}_n}(x,x,y) =&(x \wedge y) \vee (x \wedge f_{k+1}(x)) \vee (y \wedge f_k(x)) = \\ =&x \vee (x \wedge k+1) \vee (y \wedge k) = x \vee x \vee y = \\ =&x \vee y \vee x = x \vee (y \wedge k+1) \vee (x \wedge k) = \\ =&(y \wedge x) \vee (y \wedge f_{k+1}(y)) \vee (x \wedge f_k(y)) = h_k^{\mathbf{K}_n}(y,y,x); \end{aligned}$$

- $x < k \leq y$:

$$\begin{aligned} h_k^{\mathbf{K}_n}(x,x,y) =&(x \wedge y) \vee (x \wedge f_{k+1}(x)) \vee (y \wedge f_k(x)) = \\ =&x \vee (x \wedge k+1) \vee (y \wedge k) = x \vee x \vee k = k = \\ =&x \vee k \vee x = \begin{cases} x \vee (k \wedge k+1) \vee (x \wedge k-1) & \text{if } y = k; \\ x \vee (y \wedge k) \vee (x \wedge k-1) & \text{if } y > k. \end{cases} = \\ =&(y \wedge x) \vee (y \wedge f_{k+1}(y)) \vee (x \wedge f_k(y)) = h_k^{\mathbf{K}_n}(y,y,x); \end{aligned}$$

- $k \leq x < y$:

$$\begin{aligned} h_k^{\mathbf{K}_n}(x,x,y) =&(x \wedge y) \vee (x \wedge f_{k+1}(x)) \vee (y \wedge f_k(x)) = \\ =&\begin{cases} x \vee (k \wedge k+1) \vee (y \wedge k-1) & \text{if } x = k; \\ x \vee (x \wedge k) \vee (y \wedge k-1) & \text{if } x > k. \end{cases} = \\ =&x \vee k \vee k-1 = x \vee (y \wedge k) \vee (x \wedge k-1) = \\ =&(y \wedge x) \vee (y \wedge f_{k+1}(y)) \vee (x \wedge f_k(y)) = h_k^{\mathbf{K}_n}(y,y,x); \end{aligned}$$

Since $\mathbf{K}_n$ generates $\mathcal{K}_n$, the identities hold in the whole variety, showing that $\mathcal{K}_n$ is congruence n-permutable and that $\mathbf{HSP}(\mathbf{B}_n) \subseteq \mathcal{B}_{n-1}$ is, as well. Therefore, $\mathcal{K}_n$ is an example of a Δ_n sharply congruence n-permutable variety, for all $n \geq 3$.

Indeed, in [35], the author essentially proves that the terms $h_1, \ldots, h_{n-1}$ in the language of $\mathcal{B}_{n-1}$ are Hagemann-Mitsckhe terms. If it is the case that $\mathcal{B}_{n-1} = \mathbf{HSP}(\mathbf{B}_n)$ (which we do not know), then by the previous argument $\mathcal{B}_{n-1}$ is Δ_n with Δ_n-terms $h_1, \ldots, h_{k_n}$. Otherwise, we can still argue that $\mathcal{B}_{n-1}$ is a Δ_n variety, only because $\mathcal{B}_{n-1}$ is congruence n-permutable and has a binary commutative idempotent term (i.e. $\vee$ or $\wedge$), as we will prove in a later theorem in the general case.

Example 6.1.6. The algebra $\mathbf{F}$ defined in Example 5.2.2 is another instance of an algebra generating a Δ_4 variety. As a matter of fact, the variety $\mathcal{F}$ has the following two terms

$$h_1(x,y,z) := p(x,y,z),$$

$$h_2(x,y,z) := f(f(x,z),y),$$

satisfying $h_1(x,y,y) \approx x$, $h_1(x,x,y) \approx h_2(x,y,y)$, and finally

$$h_2(x,x,y) = f(f(x,y),x) \approx f(f(y,x),x) = h_2(y,x,x).$$

Hence $\mathcal{D}_4 \leq \mathcal{F}$. Moreover, $\mathcal{F}$ is not congruence 3-permutable.

Also, the variety $\mathcal{R}$ used in Theorem 5.2.11 is Δ_4 due to the terms

$$h_1(x,y,z) := p(x,y,z),$$

$$h_2(x,y,z) := q(x,y,z),$$

and the identities certifying this are part of the axiomatizing equations of $\mathcal{R}$.

Therefore, both $\mathcal{F}$ and $\mathcal{R}$ are Δ_4 sharply congruence 4-permutable varieties.

Example 6.1.7 ([32])**.** Polin's famous variety has been the first example of a non-congruence modular variety whose members have congruence lattices satisfying some non-trivial lattice identity. In [32], the author provides an equational theory for the variety he is building, yet we will consider a simplified version of his argument, based on the construction of a finitely generated variety, whose generating algebra is exactly the algebra that Polin used to show the failure of congruence modularity.

Let $\mathbf{P}_i := \langle \{0,1\}; \wedge, u, v, 0, 1\rangle$ $(i = 1, 2)$ be two algebras of type $\langle 2, 1, 1, 0, 0\rangle$, where $\langle \{0,1\}; \wedge\rangle$ is a semilattice with $0 < 1$, and the other operations are defined as

$$0^{\mathbf{P}_1} = 0^{\mathbf{P}_2} = 0,$$

$$1^{\mathbf{P}_1} = 1^{\mathbf{P}_2} = 1,$$

$$u^{\mathbf{P}_1}(x) = v^{\mathbf{P}_2}(x) = 1 - x,$$

$$v^{\mathbf{P}_1}(x) = 1,$$

$$u^{\mathbf{P}_2}(x) = x,$$

for $x \in \{0, 1\}$ (see also Exercise 9.20.6 of [19]). Call $\mathcal{P} := \mathbf{HSP}(\mathbf{P}_1 \times \mathbf{P}_2)$. It turns out that $\mathbf{Con}(\mathbf{P}_1 \times \mathbf{P}_2) \cong \mathbf{N}_5$, which yields $\mathcal{P}$ is not congruence modular and hence not congruence 3-permutable either, by Theorem 5.2.2.

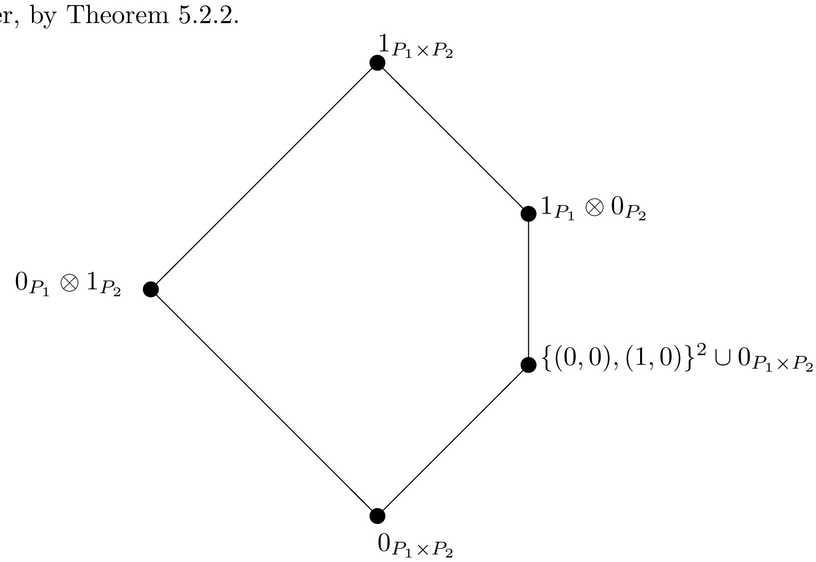

Figure 6.1: The congruence lattice of Polin's algebra $\mathbf{P}_1 \times \mathbf{P}_2$

Now call a, b the following binary terms of $\mathcal{P}$

$$a(x, y) := x \wedge u(y),$$

$$b(x, y) := x \wedge v(y),$$

and further define

$$h_1(x, y, z) := b(x, b(y, z)),$$

$$h_2(x, y, z) := u(a(y, x)) \wedge u(a(y, z)) \wedge u(a(u(x), z)).$$

It is a simple exercise to prove that h_1, h_2 are Δ_4-terms for $\mathcal{P}$, by verifying that the corresponding identities hold in both $\mathbf{P}_1$ and $\mathbf{P}_2$.

Therefore, the variety $\mathcal{P}$ is Δ_4 and sharply congruence 4-permutable.

Example 6.1.8 ([12])**.** Example 2.1 of [12] exhibits a variety which is congruence 3-permutable, indeed Δ_3, and non-congruence 2-permutable. We will present the same example with a slightly different notation.

Let $\mathbf{A} = \langle 2; p^{\mathbf{A}}, q^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle 2; p^{\mathbf{B}}, q^{\mathbf{B}} \rangle$ be two algebras of the same type, whose universes are $2 = \{0, 1\}$ and the operations are defined as

$$p^{\mathbf{A}}(x, y, z) = q^{\mathbf{B}}(x, y, z) := x + y + z \mod 2;$$

$$q^{\mathbf{A}}(x, y, z) = p^{\mathbf{B}}(x, y, z) := \max\{x, y, z\};$$

for $x, y, z \in \{0, 1\}$. Let $\mathcal{V}$ be the variety generated by $\mathbf{A} \times \mathbf{B}$. It is not hard to verify that the algebra $\mathbf{S} \leq \mathbf{A} \times \mathbf{B}$ whose universe is $\{0, 1\}^2 - \{(0, 0)\}$ is a special failure of congruence 2-permutability, preventing $\mathcal{V}$ from being congruence 2-permutable. Nevertheless, $\mathcal{V}$ is congruence 3-permutable by M. Valeriote's result referred to as Theorem 6.1 in [12].

Moreover, it is not hard to directly verify on $\mathbf{A} \times \mathbf{B}$ that the term $s(x, y) := p(x, x, q(x, x, y))$ is a binary commutative idempotent term of $\mathcal{V}$; this fact yields that $\mathcal{V}$ is a Δ_3-variety, as will be shown more generally in Theorem 6.2.2.

In the next section, we will also provide a few examples of purely congruence n-permutable varieties for some values of $n \geq 3$.

6.2 Decomposability and further properties

So far, we have proven a few properties of Δ_n varieties, among which is the fact that being Δ_n is stronger that being congruence n-permutable, for $n \geq 2$. Moreover, we have observed that Δ_n varieties are also Δ_{n+2}, for every $n \geq 2$, and it is not generally the case that Δ_n varieties are Δ_{n+1}: more precisely, it is never the case that being Δ_n implies being Δ_{n+1}, for even $n \geq 2$. Instead, if we consider the odd case, that property holds. In addition, we are also able to show that congruence n-permutability implies being Δ_m, for some $m \geq n$.

Theorem 6.2.1. *For $n \geq 2$,*

$$\mathcal{D}_{2n} \leq \mathcal{D}_{2n-1},$$

$$\mathcal{D}_{2(n-1)} \leq \mathcal{CP}_n.$$

In other words, for every $n \geq 2$, every congruence n-permutable variety is also $\Delta_{2(n-1)}$ and being Δ_n, for odd n, implies being Δ_{n+1} as well.

Proof. Fix any $n \geq 2$. Let us first prove that $\mathcal{D}_{2n} \leq \mathcal{D}_{2n-1}$. Notice that $k_{2n-1} = n - 1$ and $k_{2n} = n$.

Thus, suppose $h_1, \ldots, h_{n-1}$ are Δ_{2n-1}-terms for $\mathcal{D}_{2n-1}$. Then, define the following terms

$$h'_i(x, y, z) := h_i(x, y, z) \text{ for } 1 \leq i \leq n - 1;$$

$$h'_n(x, y, z) := h_{n-1}(z, z, x).$$

It is clear that, for $1 \leq i \leq n - 2$ when $n > 2$, or for $i = 1$ when $n = 2$, we get

$$h'_1(x, y, y) = h_1(x, y, y) \approx x;$$

$$h'_i(x, x, y) = h_i(x, x, y) \approx h_{i+1}(x, y, y) = h'_{i+1}(x, y, y).$$

Moreover,

$$h'_{n-1}(x, x, y) = h_{n-1}(x, x, y) \approx h_{n-1}(y, y, x) = h'_n(x, y, y);$$

$$h'_n(x, x, y) = h_{n-1}(y, y, x) \approx h_{n-1}(x, x, y) = h'_n(y, x, x);$$

proving that $h'_1, \ldots, h'_n$ are Δ_{2n}-terms, as desired.

For the second statement, first notice that the case $n = 2$ is already known, so suppose $n > 2$.

Given the Hagemann-Mitschke terms $p_1, \ldots, p_{n-1}$ of $\mathcal{CP}_n$, define the following interpretation (note $k_{2(n-1)} = n - 1$)

$$h_i(x, y, z) := p_i(x, y, z) \text{ for } 1 \leq i \leq n - 2;$$

$$h_{n-1}(x, y, z) := p_{n-1}(x, y, p_{n-1}(z, y, y)).$$

We can verify that the following identities hold:

$$h_1(x,y,y) = p_1(x,y,y) \approx x;$$

$$h_i(x,x,y) = p_i(x,x,y) \approx p_{i+1}(x,y,y) \text{ for } 1 \leq i < n-2;$$

$$h_{n-2}(x,x,y) = p_{n-2}(x,x,y) \approx p_{n-1}(x,y,y) \approx p_{n-1}(x,y,p_{n-1}(y,y,y)) = h_{n-1}(x,y,y);$$

$$h_{n-1}(x,x,y) = p_{n-1}(x,x,p_{n-1}(y,x,x)) \approx p_{n-1}(y,x,x) \approx p_{n-1}(y,x,p_{n-1}(x,x,x)) = h_{n-1}(y,x,x).$$

This is to say $h_1, \ldots, h_{n-1}$ are $\Delta_{2(n-1)}$-terms, showing that $\mathcal{D}_{2(n-1)} \leq \mathcal{CP}_n$. ☐

This theorem, together with Theorem 6.1.1, have a direct consequence which is stated in the following corollary

Corollary 6.2.1. *For any variety $\mathcal{V}$, $\mathcal{V}$ is congruence n-permutable, for some $n \geq 2$, if and only if $\mathcal{V}$ is Δ_m, for some $m \geq 2$.*

In other words,

$$CP_\omega = \{\mathcal{V} : (\exists n \geq 2)[\mathcal{D}_n \leq \mathcal{V}]\}.$$

Proof. If $\mathcal{V}$ is congruence n-permutable for some $n \geq 2$, then $\mathcal{V}$ is $\Delta_{2(n-1)}$ by Theorem 6.2.1. On the other hand, a Δ_n variety ($n \geq 2$) is also congruence n-permutable by Theorem 6.1.1, completing the proof. ☐

Likewise for congruence n-permutability, for which there exist characterizations making use of reflexive binary compatible relations on suitable free algebras (e.g. Corollaries 3.1.1, 3.1.2, 3.1.3), there is a similar description for having Δ_n-terms for a variety. Such a characterization is presented in the next lemma.

Lemma 6.2.1. *Let $n \geq 2$, $\mathcal{V}$ be a variety and let $\mathbf{F}$ denote the free algebra in $\mathcal{V}$ generated by $\{x, y\}$. Then, the following statements are equivalent:*

1. *$\mathcal{D}_n \leq \mathcal{V}$;*
2. *there exist $u, v \in F$ such that*

$$(x,u) \in R \circ^{k_n - 1} R \text{ and } (u,v,v) \in S,$$

where $k_n = \lfloor \frac{n}{2} \rfloor$, $R = \mathrm{Sg}^{\mathbf{F}^2}(\{(x,x),(y,x),(y,y)\})$, $S = \mathrm{Sg}^{\mathbf{F}^3}(\{(x,x,y),(y,x,a),(y,y,x)\})$ and

$$a = \begin{cases} x & \text{for even } n; \\ y & \text{for odd } n. \end{cases}$$

Proof. The proof of this lemma is fairly standard, but we present it anyway.

Throughout the proof, keep $n \geq 2$ and a variety $\mathcal{V}$ fixed and let $\mathbf{F}$, $R \leq \mathbf{F} \times \mathbf{F}$, $S \leq \mathbf{F} \times \mathbf{F} \times \mathbf{F}$, k_n and a be defined as in the statement.

Suppose first $\mathcal{V}$ is Δ_n and let $h_1, \ldots, h_{k_n}$ be the terms witnessing it. Only for the sake of this proof, call $h_0(x,y,z) := x$. Let us also define

$$u := h_{k_n}(x,y,y),$$

$$v := h_{k_n}(x,x,y).$$

u, v are obviously elements of F. Moreover

$$x = h_1^{\mathbf{F}}(x,y,y) \; R \; h_1^{\mathbf{F}}(x,x,y) \; R \; \cdots \; h_{k_n-1}^{\mathbf{F}}(x,y,y) \; R \; h_{k_n-1}^{\mathbf{F}}(x,x,y) = h_{k_n}^{\mathbf{F}}(x,y,y) = u,$$

with $k_n - 1$ occurrences of R (where 0 occurrences of R means $x = u$), showing that $(x,u) \in R \circ^{k_n - 1} R$. Also, because $v = h_{k_n}^{\mathbf{F}}(x,x,y) = h_{k_n}^{\mathbf{F}}(y,a,x)$, then we have

$$\begin{bmatrix} u \\ v \\ v \end{bmatrix} = \begin{bmatrix} h_{k_n}^{\mathbf{F}}(x,y,y) \\ h_{k_n}^{\mathbf{F}}(x,x,y) \\ h_{k_n}^{\mathbf{F}}(y,a,x) \end{bmatrix} = h_{k_n}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ a \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \end{bmatrix} \right) \in S,$$

proving that (1) implies (2).

Conversely, assume (2) holds: since $(x, u) \in R \circ^{k_n - 1} R$, then there exist $f_0, \ldots, f_{k_n-1} \in F$, with $f_0 = x$ and $f_{k_n-1} = u$, such that

$$(f_i, f_{i+1}) \in R \text{ for } 0 \leq i < k_n - 1,$$

which is to say there exist ternary term operations of $\mathbf{F}$, $h_1^{\mathbf{F}}, \ldots, h_{k_n-1}^{\mathbf{F}}$ such that, for all $0 \leq i \leq k_n - 2$,

$$\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix} = h_{i+1}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right) = \begin{bmatrix} h_{i+1}^{\mathbf{F}}(x, y, y) \\ h_{i+1}^{\mathbf{F}}(x, x, y) \end{bmatrix}.$$

Also, because $(u, v, v) \in S$, then there exists another term operation of $\mathbf{F}$, $h_{k_n}^{\mathbf{F}}$, such that

$$\begin{bmatrix} u \\ v \\ v \end{bmatrix} = h_{k_n}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ a \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} h_{k_n}^{\mathbf{F}}(x, y, y) \\ h_{k_n}^{\mathbf{F}}(x, x, y) \\ h_{k_n}^{\mathbf{F}}(y, a, x) \end{bmatrix}.$$

If we put all these facts together, we can deduce that the following equalities hold in $\mathbf{F}$:

$$x = h_1^{\mathbf{F}}(x, y, y),$$

$$h_i^{\mathbf{F}}(x, x, y) = h_{i+1}^{\mathbf{F}}(x, y, y) \text{ for } 1 \leq i \leq k_n - 1,$$

$$h_{k_n}^{\mathbf{F}}(x, x, y) = v = h_{k_n}^{\mathbf{F}}(x, a, y).$$

These yield that the whole variety $\mathcal{V}$ must satisfy the corresponding identities involving $h_1, \ldots, h_{k_n}$, proving that $\mathcal{V}$ is Δ_n. □

Lemma 6.2.1 is a useful tool because it ensures that the existence of Δ_n-terms for a variety can be checked directly within the 2-generated free algebra of the variety itself. On the other hand, it is not always computationally feasible to build the 2-generated free algebra or the relations R and S, not even in locally finite varieties: this difficulty led us to wondering whether there might exist a polynomial time algorithm for checking the existence of Δ_n-terms, at least with respect to finitely generated idempotent varieties, as it occurs for congruence n-permutability (as M. Valeriote and R. Willard show in [42]). We are not able to answer this question in general, but we can whenever $n \geq 3$ is odd. This fact is an indirect consequence of the theorem below, which also provides a decomposability (hence non-primeness) argument for being Δ_n with respect to odd values of $n \geq 3$.

Recall that $\mathcal{C}_2$ is the variety of algebras having a binary commutative idempotent term.

Theorem 6.2.2. *In the lattice of interpretability types, for every $n \geq 2$,*

$$\mathcal{CP}_n \vee \mathcal{C}_2 \geq \mathcal{D}_n.$$

If, further, n is odd, then equality holds, i.e. for all $m \geq 1$

$$\mathcal{CP}_{2m+1} \vee \mathcal{C}_2 = \mathcal{D}_{2m+1}.$$

Proof. We already know that $\mathcal{CP}_2 = \mathcal{D}_2$, which obviously implies that $\mathcal{CP}_2 \vee \mathcal{C}_2 = \mathcal{D}_2 \vee \mathcal{C}_2 \geq \mathcal{D}_2$; hence we may assume $n > 2$.

Recall that the basic operations of $\mathcal{CP}_n$ are $p_1, \ldots, p_{n-1}$ (which are Hagemann-Mitschke terms) and the basic operation of $\mathcal{C}_2$ is s. Hence define, for $1 \leq i \leq k_n$, the following terms in the language of $\mathcal{CP}_n \vee \mathcal{C}_2$:

$$h_i(x, y, z) := s(p_i(x, y, z), p_{n-i}(z, y, x)).$$

Let us prove that the previous equality defines an interpretation from $\mathcal{D}_n$ to $\mathcal{CP}_n \vee \mathcal{C}_2$.

Let us start with noticing that the following identities always hold, regardless of n being even or odd:

$$h_1(x, y, y) = s(p_1(x, y, y), p_{n-1}(y, y, x)) \approx s(x, x) \approx x;$$

$$h_i(x,x,y) = s(p_i(x,x,y), p_{n-i}(y,x,x)) \approx$$
$$\approx s(p_{i+1}(x,y,y), p_{n-i-1}(y,y,x)) = s(p_{i+1}(x,y,y), p_{n-(i+1)}(y,y,x)) = h_{i+1}(x,y,y),$$

for $1 \leq i < k_n$.

Moreover, if n is even, then $k_n = \frac{n}{2}$, which yields

$$h_{k_n}(x,x,y) = s(p_{k_n}(x,x,y), p_{n-k_n}(y,x,x)) = s(p_{k_n}(x,x,y), p_{k_n}(y,x,x)) \approx$$
$$\approx s(p_{k_n}(y,x,x), p_{k_n}(x,x,y)) = s(p_{k_n}(y,x,x), p_{n-k_n}(x,x,y)) = h_{k_n}(y,x,x).$$

Instead, for n odd, then $k_n = \frac{n-1}{2}$ (hence $n - k_n = \frac{n+1}{2} = k_n + 1$) implying

$$h_{k_n}(x,x,y) = s(p_{k_n}(x,x,y), p_{n-k_n}(y,x,x)) = s(p_{k_n}(x,x,y), p_{k_n+1}(y,x,x)) \approx$$
$$\approx s(p_{k_n+1}(y,x,x), p_{k_n}(x,x,y)) \approx$$
$$\approx s(p_{k_n}(y,y,x), p_{k_n+1}(x,y,y)) = s(p_{k_n}(y,y,x), p_{n-k_n}(x,y,y)) = h_{k_n}(y,y,x).$$

In either case we get the interpretation we meant, showing that $\mathcal{D}_n \leq \mathcal{CP}_n \vee \mathcal{C}_2$. In the odd case, in addition, we have that $\mathcal{D}_n$ is both congruence n-permutable (Theorem 6.1.1) and has a binary idempotent commutative term (namely $h_{k_n}(x,x,y)$), as observed in the previous section: this means that also $\mathcal{D}_{2m+1} \geq \mathcal{CP}_{2m+1} \vee \mathcal{C}_2$, for every $m \geq 1$, yielding the equality. $\square$

As previously anticipated, one consequence of this theorem is that to check the existence of Δ_n-terms for odd $n \geq 3$ with respect to idempotent finitely generated varieties can be done in polynomial time. As a matter of fact, given a finite idempotent algebra $\mathbf{A}$ and odd $n \geq 3$, one can check in polynomial time whether $\mathbf{HSP}(\mathbf{A})$ is congruence n-permutable using the Valeriote-Willard algorithm in [42], and then check the existence of a binary commutative idempotent term which can be also done in polynomial time, given the result of [3] (a binary commutative term is in fact a cyclic term of arity 2): if the answer is affirmative in both cases, then $\mathbf{HSP}(\mathbf{A})$ is Δ_n.

Another interesting consequence is expressed in the next corollary. Before stating that, let us call, for $n \geq 2$

$$D_n := \{\mathcal{V} : \mathcal{D}_n \leq \mathcal{V}\},$$
$$D_{2\omega} := \bigcup_{n\geq 1} D_{2n},$$
$$D_{2\omega+1} := \bigcup_{n\geq 1} D_{2n+1},$$

and further call C_2 the strong Maltsev filter in $\mathbf{L}$ generated by $\mathcal{C}_2$.

Corollary 6.2.2. *For odd $n \geq 3$, being Δ_n for a variety is a decomposable strong Maltsev condition. More precisely,*

$$D_{2m+1} = CP_{2m+1} \cap C_2,$$

for all $m \geq 1$, and hence

$$D_{2\omega+1} = CP_\omega \cap C_2.$$

Proof. For fixed odd $n \geq 3$, Theorem 6.2.2 ensures that any variety $\mathcal{V}$ satisfying $\mathcal{CP}_n \vee \mathcal{C}_2 \leq \mathcal{V}$ is in fact in D_{2n+1}. The other inclusion is straightforward, yielding that $CP_{2n+1} \cap C_2 = D_{2n+1}$. Moreover, since for instance $\mathcal{Z}_2 \notin D_n$, then $CP_n \supsetneq D_n$. On the other hand, the variety $\mathcal{S}$ of semilattices is an example of a variety having a binary idempotent commutative term which is not Δ_n (in fact, for any $k \geq 2$, $\mathcal{S}$ is not congruence k-permutable, hence it is not Δ_k either): this also shows that $C_2 \supsetneq D_n$, implying the property of decomposability. Finally, the last displayed equality is a direct consequence of the previous one. $\square$

Unlike the odd case, we cannot claim a similar property for $\mathcal{D}_n$ whenever n is even. The main reason is that we are not able to separate CP_{2m} and D_{2m}, for any $m \geq 1$: in fact, if we could claim that $CP_{2m} \supsetneq D_{2m}$, then Theorem 6.2.2 would imply that D_{2m} is a non-prime strong Maltsev class. Unfortunately, the question of the separation of the two above mentioned classes has not been solved and remains open.

To close this section, we want to provide some examples of locally finite sharply and purely congruence n-permutable varieties, for some odd values of $n > 2$.

Before that, let us state an instance of Lemma 4.4 of [2] for the case of commutative terms.

Lemma 6.2.2 ([2]). *Let* $\mathbf{A}$ *be a finite idempotent algebra. Then, the following are equivalent:*

1. $\mathbf{A}$ *has a binary commutative term operation;*
2. *For any* $R \leq \mathbf{A} \times \mathbf{A}$*, if* R *is symmetric then there exists* $a \in A$ *such that* $(a, a) \in R$.

We will mention Lemma 6.2.2 while building the next example.

Example 6.2.1. For $n \geq 3$, let $\mathbf{A}_n$ be the algebra with universe $A_n := n + 1$, whose basic operations are all the idempotent polymorphisms of the relation $R_n \subseteq A_n \times A_n$, defined by

$$R_n := \{(0,1)\} \cup \bigcup_{i=1}^{n-1} \{(i, i-1), (i, i+1)\} \cup \{(n, n-1)\}.$$

Also, call $\mathcal{A}_n$ the variety generated by $\mathbf{A}_n$.

It is straightforward to check that, by definition, R_n does not contain any pair of the form (a, a), for any $a \in A_n$: by Lemma 6.2.2, this implies that $\mathbf{A}_n$ does not have any binary commutative (idempotent) term operation, meaning $\mathcal{C}_2 \not\leq \mathcal{A}_n$, for any $n \geq 3$.

Moreover, the variety $\mathcal{A}_n$ is not congruence $(n-1)$-permutable for the following reason: consider the algebra $\mathbf{R}_n \in \mathcal{A}_n$ and let α and β be the kernels of the projection maps (first and second respectively) restricted to R_n. Define further, for $n \geq 3$,

$$(u_n, v_n) = \begin{cases} (n-1, n) \text{ for odd } n; \\ (n, n-1) \text{ for even } n. \end{cases}.$$

Notice that for all $n \geq 3$,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} \alpha \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdots \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

showing that $(0,1)\ \beta \circ^{n-1} \alpha\ (u_n, v_n)$. Nonetheless, if we assumed that $(0,1)\ \alpha \circ^{n-1} \beta\ (u_n, v_n)$, then either one of the following cases would occur:

- there exists even $i \in A_n$ such that $(i, i+1)\ \alpha \circ \beta\ (i+2, i+3)$, which in particular yields $(i, i+3) \in R_n$, against the definition of R_n itself;
- there exists even $i \in A_n$, such that $(i, i-1)\ \beta \circ \alpha\ (i+2, i+1)$, implying $(i+2, i-1) \in R_n$, again contradicting its definition.

The contradictions obtained in either case show that $(0,1)\ \cancel{\alpha \circ^{n-1} \beta}\ (u_n, v_n)$, meaning that α and β do not $(n-1)$-permute and hence proving that $\mathcal{A}_n$ is not congruence $(n-1)$-permutable.

If we consider the case $n = 3$, we have that $\mathcal{A}_3$ is also congruence 3-permutable. Indeed, if $p^{\mathbf{A}_3}$ is the operation defined as follows

$p^{\mathbf{A}_3}$	0	1	2	3
$(0,0)$	0	1	2	1
$(0,1)$	0	0	0	0
$(0,2)$	0	1	0	1
$(0,3)$	0	0	0	0
$(1,0)$	1	1	1	1
$(1,1)$	0	1	0	1
$(1,2)$	1	1	1	1
$(1,3)$	0	1	0	1
$(2,0)$	2	1	2	1
$(2,1)$	0	2	0	2
$(2,2)$	2	1	2	1
$(2,3)$	0	2	0	2
$(3,0)$	3	1	1	1
$(3,1)$	0	3	0	1
$(3,2)$	1	1	3	1
$(3,3)$	0	1	0	3

then the terms $p_1(x,y,z) := p(x,y,z)$ and $p_2(x,y,z) := p(p(z,x,y),y,p(x,x,y))$ are Hagemann-Mitschke terms witnessing congruence 3-permutability for $\mathcal{A}_3$, showing that such a variety is sharply congruence 3-permutable. Moreover, since $\mathcal{A}_3$ does not have any binary commutative idempotent term, then, by Corollary 6.2.2, it is also purely congruence 3-permutable.

Even for the case $n = 5$, we can find Hagemann-Mitschke terms witnessing congruence 5-permutability of $\mathcal{A}_5$, showing that $\mathcal{A}_5$ is sharply and purely congruence 5-permutable as well.

This reasoning is probably generalizable, in a way that lets us guess $\mathcal{A}_{2m+1}$ is an example of a sharply and purely congruence $(2m+1)$-permutable variety, for every $m \geq 1$.

On the other hand, for even values of n, $\mathcal{A}_n$ is likely to be already Δ_n, as is the case for $n = 4$. However, it could be possible to consider suitable reducts of $\mathcal{A}_n$ so as to obtain purely congruence n-permutable varieties: such reducts would still have no binary commutative idempotent terms (a necessary condition for a congruence n-permutable variety not to be Δ_n, as exposed in Theorem 6.2.2), due to how the relation R_n has been defined.

Chapter 7

Conclusion

We conclude this thesis by discussing some possible future directions regarding some open questions that have further emerged in the analysis of the topics treated so far.

Let us begin with the arguments discussed in Chapter 4 and, particularly, let us focus on the omission classes $\Omega(SHR_n)$, for $n \geq 1$. We have proven in Theorem 4.1.1 that these classes are in fact Maltsev classes, which is equivalent to saying that at the level of varieties, the omission of special Hagemann relations of dimension n, for fixed $n \geq 1$, can be characterized by a Maltsev condition, namely by the presence of terms in the language of those varieties satisfying some equations. Unfortunately, the proof of Theorem 4.1.1 is not constructive and, in fact, it eventually invokes the compactness theorem to deduce the finite presentability condition. Due to this, we have not been able to have access to the actual terms and equations characterizing $\Omega(SHR_n)$, nor do we know whether such a condition is a *strong* Maltsev condition or not. We then pose the following question.

Problem 1. For fixed $n \geq 1$, is $\Omega(SHR_n)$ a strong Maltsev class? In any case, what does the (strong) Maltsev condition characterizing $\Omega(SHR_n)$ look like?

We have no clue about that, except for the fact that congruence $(n+1)$-permutability implies those Maltsev conditions. Moreover, we can expect that, as it is frequently the case in these frameworks, the terms and equations for a variety $\mathcal{V} \in \Omega(SHR_n)$ witnessing the satisfaction of such a Maltsev conditions ought to be discovered by studying suitable free algebras of $\mathcal{V}$. As a matter of fact, for any fixed $n \geq 1$, if a variety $\mathcal{V}$ admits SHR_n, then Lemma 5.2.3 and Lemma 5.2.2 ensure that a special Hagemann relation of dimension n in $\mathcal{V}$ can be found as a subalgebra of $\mathbf{F}_{\mathcal{V}}(\{x_0, \ldots, x_{n+1}\}) \times \mathbf{F}_{\mathcal{V}}(\{x_0, \ldots, x_{n+1}\})$. Therefore, by a contrapositive reasoning, if a variety $\mathcal{V}$ omits SHR_n, then the second power of $\mathbf{F}_{\mathcal{V}}(\{x_0, \ldots, x_{n+1}\})$ will not have any n-dimensional special Hagemann relation as a subalgebra: this property could somehow become crucial for finding the desired terms and equations holding in the whole variety.

Another convenient aspect of accessing the actual Maltsev condition for $\Omega(SHR_n)$ could be related to the following: we have proven in Theorem 4.1.2 that omitting $\Omega(SHR_n)$ is a prime Maltsev class, for every $n \geq 1$, and if we specialize to the case $n = 1$, we already know that $\Omega^{id}(SHR_1) = CP_2^{id}$ (Theorem 5.1.2). For the general case, we easily deduce that $CP_2 \subseteq \Omega(SHR_1)$, yet the inverse inclusion is still an open question[1], which we highlight below.

Problem 2. Does the following equality hold:

$$CP_2 = \Omega(SHR_1)?$$

A positive answer to this question would necessarily yield the primeness of congruence 2-permutability and hence a possibly alternative proof of Steven Tschantz's unpublished result of [41]. On the other hand, Problem 2 is not likely to be generalizable, meaning that it might be the case that, for each $n \geq 2$, $CP_{n+1} \subsetneq \Omega(SHR_n)$, and we have certain evidence of this for the case of congruence 3-permutability. As a matter of fact, we have proven that $\Omega^{id}(SHR_2) \supsetneq$

[1]In Appendix B, added after the external reviewer's comments, this question is answered negatively.

$\Omega^{id}(M^+SHR_2) \supseteq CP_3^{id}$ by exhibiting an instance of a variety in $\Omega(SHR_2) - \Omega(M^+SHR_2)$, namely the variety $\mathcal{F}$ from Example 5.2.2. Also, for the case of congruence 4-permutability we have observed in Section 5.3 that CP_4 looks rather far from being characterized by $\Omega(SHR_3)$, even only at the level of idempotent varieties. Therefore, for higher values of $n \geq 3$, it seems improbable that the omission of SHR_n can capture and characterize congruence $(n+1)$-permutability.

Nevertheless, due to Theorem 5.2.8 and Theorem 5.2.10, a plausible question to ask about congruence 3-permutability is the following

Problem 3. Does either one of the following equalities hold

$$CP_3 = \Omega(M^+SHR_2)?^2$$

$$CP_3^{id} = \Omega^{id}(M^+SHR_2)?$$

We suspect that neither is true, thinking as a possible counterexample of the variety generated by the algebra $\mathbf{A}$ from Example 5.2.1. Even for the Maltsev class $\Omega(M^+SHR_2)$, it could be convenient to find a termwise characterization, which could also turn out to be helpful to verify or deny the validity of Problem 3.

As far as congruence 4-permutability is concerned, we previously mentioned at the end of Section 5.3 that we conjecture the non-primeness of this strong Maltsev condition. We invite the reader to examine that part in order to be provided with more details and explanations on the reasons why we have been led to believing so. In this current section, we are going to state the open problem for completeness and suggest a potential pair of varieties which could be used for proving the non-primeness argument.

Problem 4. Is CP_4^{id} (and hence CP_4) a non-prime strong Maltsev class?

We again remark that we conjecture an affirmative answer. More concretely, let $\mathbf{A} \leq_{sd} \mathbf{B} \times \mathbf{C}$ be a special failure of congruence 4-permutability with middle portions of genus 5, whose basic operations are the idempotent polymorphisms of the kernels of the projections onto B and C restricted to A; likewise, let $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$ be a Σ-holed special failure of congruence 4-permutability with middle portions of genus 4, where $\Sigma = \{(0,1),(0,3),(1,2),(2,3)\}$ and again the basic operations of $\mathbf{S}$ are the idempotent polymorphisms of the kernels of the projection maps onto P and Q restricted to S. If we call $\mathcal{V} = \mathbf{HSP}(\mathbf{A})$ and $\mathcal{W} = \mathbf{HSP}(\mathbf{S})$, then we presume $\mathcal{V} \vee \mathcal{W} \in CP_4^{id}$, although neither $\mathcal{V}$ nor $\mathcal{W}$ is congruence 4-permutable. Should this pair of varieties not work for this purpose, one could try to build another pair of special failures of congruence 4-permutability (perhaps with *all* the polymorphisms of the non-4-permutable kernels of the projections as basic operations), at least one of which be Σ-holed for some suitable Σ, such that there is no possible chance to find any failure of congruence 4-permutability in the coproduct of the varieties they generate.

In any case, it is also worth mentioning that, if $\mathcal{V}$ and $\mathcal{W}$ happen to be non-congruence 4-permutable varieties satisfying $\mathcal{V} \vee \mathcal{W} \in CP_4$, then by Corollary 3.4 of [31], recalled in Chapter 5 as Theorem 5.0.3, either $\mathcal{V}$ or $\mathcal{W}$ is not linear. This result is indeed a particular consequence of Theorem 5.3.3, due to the fact that non-congruence 4-permutable linear varieties admit $M^-SF_4^0$, as Proposition 3.3, along with Lemma 2.1 of [31], imply.

To sum up, the situation we have been profiling so far is the following: if we focus on idempotent varieties, we know that CP_2^{id} is a prime filter in $\mathbf{L}^{id}$, and so we expect for CP_3^{id} as well (Theorem 5.2.8 guarantees the primeness only with respect to locally finite idempotent varieties). On the other hand, we are also aware of the fact that CP_ω^{id} is a prime filter in $\mathbf{L}^{id}$ (Theorem 4.2.2 or Theorem 4.2.3). Despite this, we expect that $n = 4$ is the first value for which CP_n^{id} is a non-prime class. This described scenario might look a little singular, yet an actual instance of something analogous occurs in the case of idempotent congruence modularity. Indeed, if we call CM_n (resp. CM_n^{id}) the class of varieties (resp. idempotent varieties) $\mathcal{V}$ satisfying $\mathcal{CM}_n \leq \mathcal{V}$ for $n \geq 2$ (see Example 2.2.2), then it is known that $CM_2 = CP_2$, and hence $CM_2^{id} = CP_2^{id}$, which are prime filters within, respectively, $\mathbf{L}$ and $\mathbf{L}^{id}$. Regarding CM_3

[2] In Appendix B, added after the external reviewer's comments, this question is answered negatively.

and CM_3^{id}, the question whether these classes are prime or not is still open, whereas for $n \geq 4$, L. Sequeira showed in his Ph.D. thesis [36] that CM_n^{id} (and hence CM_n) is not prime. However, J. Opršal has proven in [31] that $CM^{id} = \bigcup_{2 \leq n < \omega} CM_n^{id}$ is a prime filter in $\mathbf{L}^{id}$, making the case of idempotent congruence modularity a peculiar example, likely to be mimicked by the Maltsev condition of idempotent congruence n-permutability, for some $n \geq 2$.

The last topic we want to discuss further so as to close with another open question is relative to the family of strong Maltsev conditions presented in Chapter 6. We again point out that the variety $\mathcal{D}_n$ ($n \geq 2$) as defined in Definition 6.1.1, generates a strong Maltsev filter in the lattice of interpretability types that contains the types of many varieties which universal algebraists frequently deal with, as we have shown by a list of examples. We have also proven that the class D_{2n+1} is properly contained in CP_{2n+1}, for any $n \geq 1$ and, in fact, a decomposability property has come out in Theorem 6.2.2. On the other hand, the class D_{2n} has surprisingly turned out to behave differently from the odd counterpart, meaning that we have not even been able to separate it from CP_{2n}, for any $n \geq 2$ (instead we know that $D_{2n} \subseteq CP_{2n}$). Thus, we pose the following question

Problem 5. For each $n \geq 2$, do the classes D_{2n} and CP_{2n} coincide?

We expect that the answer to this question is negative for every $n \geq 2$, yet we are going to discuss some consequences of both scenarios. Let us then fix some $n \geq 2$ and consider the possible cases.

If the two classes do coincide, then the variety $\mathcal{CP}_{2n}$ is a Δ_{2n} variety, which essentially implies that some suitable compositions of the Hagemann-Mitschke terms $p_1, \ldots, p_{2n-1}$ can produce some interpretation of the Δ_{2n}-terms $h_1, \ldots, h_n$: this fact would yield that congruence $2n$-permutability of a variety $\mathcal{V} \in CP_{2n}$ can be captured by half of the terms that are instead needed according to the Hagemann-Mitschke's characterization, also avoiding a growth in arity, which is still kept fixed at 3.

Else, if $D_{2n} \subsetneq CP_{2n}$, then necessarily the variety $\mathcal{CP}_{2n}$ cannot be Δ_{2n}; if one were to prove this fact, there could be a way to simulate the technique of K. Fichtner's in [11], where he syntactically proved that $\mathcal{CM}_k \not\leq \mathcal{CM}_{k+1}$, for every $k \geq 2$. Although this path is theoretically possible, it looks rather hard in terms of complexity. The optimal method to separate the two classes would be to build a finitely generated congruence $2n$-permutable variety $\mathcal{V}$, which is not Δ_{2n}. By Theorem 6.2.2, such a $\mathcal{V}$ must have no binary idempotent commutative term, making possible reducts of the varieties $\mathcal{A}_{2n}$, discussed in Example 6.2.1, good candidates for lying in $CP_{2n} - D_{2n}$.

Appendix A

The proofs of Theorems 5.3.1 and 5.3.2

This appendix is meant to contain the proofs of Theorem 5.3.1 and Theorem 5.3.2, which have only been stated in Section 5.3. Let us see those proofs in detail.

A.1 Proof of Theorem 5.3.1

Fix $k \geq 0$ any integer and let $\mathcal{V}$ be an idempotent variety containing $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$ being a special failure of congruence 4-permutability of genus k without 2^{nd} middle portion. Let $\{X, Y, Z\} \cup \{P_i : i \in k\}$ and $\{U, V, W\} \cup \{Q_i : i \in k\}$ be the partitions of, respectively, $P - M_1$ and Q, where M_1 is a potentially empty subset of P, such that the equality displayed in Definition 5.3.1 holds. At this point, likewise for Theorem 5.2.6, the proof consists of building some algebras $\mathbf{P}', \mathbf{Q}', \mathbf{S}' \in \mathcal{V}$ such that $\mathbf{S}' \leq \mathbf{P}' \times \mathbf{Q}'$ is a special failure of congruence 4-permutability of genus k with 2^{nd} middle portion. Such a proof proceeds nearly as the one of Theorem 5.2.6, hence we will omit some details.

Fix any elements in both U and Z and call them $0 \in U$, $d \in Z$; define the following subset of $S^2 \times [(P \times Q) \times S]$ as

$$S' = \left\{ \begin{bmatrix} ((p,0),(r,e)) \\ ((q,j),(d,i)) \end{bmatrix} : (p,0),(r,e),(p,j),(r,j),(r,i),(d,i) \in S, \right.$$
$$\left. \exists f \in Q[(p,f),(q,f),(r,f) \in S], \exists a \in P[(a,e),(a,j),(a,i) \in S] \right\}.$$

Notice that, in this definition, $(p,0),(r,e),(d,i) \in S$, whereas $(q,j) \in P \times Q$. Moreover, $S' \leq \mathbf{S}^2 \times [(\mathbf{P} \times \mathbf{Q}) \times \mathbf{S}]$, due to the idempotence of $\mathcal{V}$.

Define also the following two sets:

$$P' := [(X \cup Y) \times \{0\}] \times [S - (X \times U)] \subseteq S \times S,$$

$$Q' := [P \times (Q - W)] \times [\{d\} \times (V \cup W)] \subseteq (P \times Q) \times S,$$

and notice that they both are subuniverses of the respective algebras. As a matter of fact, it is not hard to show that

$$X \cup Y = 0/S^{-1};$$
$$S - (X \times U) = (d,v)/\beta \circ \alpha;$$
$$Q - W = y/S;$$
$$V \cup W = d/S;$$

where $v \in V$, $y \in Y$, $\alpha = (0_P \otimes 1_Q) \cap S$ and $\beta = (1_P \otimes 0_Q) \cap S$. These equalities, along with the idempotence of $\mathcal{V}$, yield that $\mathbf{P}', \mathbf{Q}' \in \mathcal{V}$.

Let us then proceed with considering the following sets

$$X' = [X \times \{0\}] \times [Y \times (Q - W)],$$
$$Y' = [Y \times \{0\}] \times [Y \times (Q - W)],$$
$$Z' = [Y \times \{0\}] \times [Z \times (V \cup W)],$$
$$P'_i = [Y \times \{0\}] \times \left[P_i \times \left(V \cup \bigcup_{j=i}^{k-1} Q_j\right)\right] \ (i \in k),$$
$$M'_1 = [Y \times \{0\}] \times [M_1 \times V],$$

and

$$U' = [(X \cup Y) \times U] \times [\{d\} \times V],$$
$$V' = [(P - X) \times V] \times [\{d\} \times V],$$
$$W' = [(P - X) \times V] \times [\{d\} \times W],$$
$$Q'_i = [(P - X) \times Q_i] \times [\{d\} \times V] \ (i \in k),$$
$$M'_2 = \left[\left[\left(M_1 \cup Z \cup \bigcup_{i \in k} P_i\right) \times U\right] \cup \left[X \times \left(V \cup \bigcup_{i \in k} Q_i\right)\right]\right] \times [\{d\} \times V].$$

Notice that $\{X', Y', Z'\} \cup \{P'_i : i \in k\}$ is a partition of $P' - M'_1$ (M'_1 is empty if and only if M_1 is) and $\{U', V', Z'\} \cup \{Q'_i : i \in k\}$ is a partition of $Q' - M'_2$, where M'_2 is certainly non-empty. Thus, we aim to prove that

$$S' = (X' \times U') \cup [Y' \times (Q' - W')] \cup \bigcup_{i \in k} \left[P'_i \times \left(V' \cup \bigcup_{j=i}^{k-1} Q'_j\right)\right] \cup (M'_1 \times V') \cup [Z' \times (V' \cup W')].$$

Let us first prove that each operand on the right hand side of the above equality is contained in S'.

- $X' \times U' \subseteq S'$: let $\big((p,0),(r,e)\big) \in X'$ and $\big((q,j),(d,i)\big) \in U'$, which imply $p \in X$, $r \in Y$, $e \notin W$, $q \in X \cup Y$, $j \in U$ and $i \in V$. Therefore $(p,0),(p,j) \in X \times U \subseteq S$, $(r,e) \in Y \times (Q - W) \subseteq S$, $(r,j) \in Y \times U \subseteq S$, $(r,i) \in Y \times V \subseteq S$ and $(d,i) \in Z \times V \subseteq S$; in addition, since $p, r, q \in X \cup Y$, by choosing, for instance, $0 \in U$, we get that $(p,0),(r,0),(q,0) \in S$. On the other hand, since $e, j, i \in Q - W$, then for any $y \in Y$, $(y,e),(y,j),(y,i) \in S$, showing that
$$\begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,j),(d,i)\big) \end{bmatrix} \in S'.$$
- $Y' \times (Q' - W') \subseteq S'$: let $\big((p,0),(r,e)\big) \in Y'$ and $\big((q,j),(d,i)\big) \in Q' - W'$, meaning that $p, r \in Y$, $e, j \in Q - W$, $q \in P$ and $i \in V$. In such case, $(p,j),(r,j) \in Y \times (Q - W) \subseteq S$ and further $(r,i) \in Y \times V \subseteq S'$. Moreover, if $q \in X$, then $(p,0),(r,0),(q,0) \in S$, whereas if $q \notin X$, then for any $v \in V$, $(q,v) \in S$, and hence $(p,v),(r,v),(q,v) \in S$. On the other hand, for any $y \in Y$, $(y,e),(y,j),(y,i) \in S$ being $Y \times (Q - W), Y \times V \subseteq S$. This also shows that
$$\begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,j),(d,i)\big) \end{bmatrix} \in S'.$$
- Fix $i \in k$ (provided $k \neq 0$) and let
$$\vec{s} = \begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,a),(d,b)\big) \end{bmatrix} \in P'_i \times (V' \cup Q'_i \cup \ldots \cup Q'_{k-1});$$
i.e. $p \in Y$, $r \in P_i$, $e, a \in V \cup Q_i \cup \ldots \cup Q_{k-1}$, $q \notin X$ and $b \in V$. In such case, we have that $(p,0),(p,a) \in Y \times (Q - W) \subseteq S$, $(r,e),(r,a) \in P_i \times (V \cup Q_i \cup \ldots \cup Q_{k-1}) \subseteq S$, $(r,b) \in P_i \times V \subseteq S$ and $(d,b) \in Z \times V \subseteq S$. Moreover, for any $v \in V$, $(p,v),(r,v),(q,v) \in (P - X) \times V \subseteq S$, finally proving that $\vec{s} \in S'$.

- $M_1' \times V' \subseteq S'$: assume $M_1 \neq \varnothing$ and let $\big((p,0),(r,e)\big) \in M_1'$ and $\big((q,j),(d,i)\big) \in V'$. This means $p \in Y$, $r \in M_1$, $e,j,i \in V$ and $q \notin X$, yielding that $(p,0) \in Y \times U \subseteq S$, $(p,j) \in Y \times V \subseteq S$, $(r,e),(r,j),(r,i) \in M_1 \times V \subseteq S$ and $(d,i) \in Z \times V \subseteq S$. Moreover, notice that for any $v \in V$, $(p,v) \in Y \times V \subseteq S$, $(r,v) \in M_1 \times V \subseteq S$ and $(q,v) \in (P-X) \times V \subseteq S$, again showing that
$$\begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,j),(d,i)\big) \end{bmatrix} \in S'.$$

- $Z' \times (V' \cup W') \subseteq S'$: again, let
$$\vec{s} = \begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,j),(d,i)\big) \end{bmatrix} \in Z' \times (V' \cup W'),$$
meaning that $p \in Y$, $r \in Z$, $e,i \in V \cup W$, $q \notin X$ and $j \in V$. Hence, $(p,0) \in S$, $(p,j) \in Y \times V \subseteq S$ and $(r,j),(r,e),(r,i),(d,i) \in Z \times (V \cup W) \subseteq S$. Moreover, for any $v \in V$, we also have that $(p,v),(r,v),(q,v) \in (P-X) \times V \subseteq S$, implying $\vec{s} \in S'$.

For the inverse inclusion, let us pick any element in S', for example
$$\vec{s} = \begin{bmatrix} \big((p,0),(r,e)\big) \\ \big((q,a),(d,b)\big) \end{bmatrix} \in S',$$
and let us notice that, by definition, $0 \in U$ and $(p,0) \in S$ yield $p \in X \cup Y$; likewise, since $d \in Z$ and $(d,b) \in S$, then $b \in V \cup W$.

Suppose $p \in X$: firstly, the fact that $(p,a) \in S$ and $(q,a) \in S$ forces $a \in U$ and hence $q \in X \cup Y$; secondly, because there exists $h \in Q$ such that $(p,h),(r,h),(q,h) \in S$, then $h \in U$, implying also $r \in X \cup Y$ and hence forcing $e \notin W$. Moreover, $a \in V \cup W$ and $(q,a) \in S$ force $q \notin X$, showing that $q \in Y$. Finally, since $(r,b),(d,b) \in S$ and $d \in Z$, then $b \in (V \cup W) - W = V$, proving that $\vec{s} \in X' \times U'$.

By the same technique as the one invoked in the above reasoning, we can prove that, if $b \in W$, then $\vec{s} \in Z' \times W'$.

Let us then consider the remaining case where $p \in Y$ and $b \in V$: because $(p,a) \in S$, then $a \in Q - W$, for which we need distinguish a few cases.

- If $a \in U$, since $(r,a) \in S$, then $r \in X \cup Y$; however, because $(r,b) \in S$ and $b \in V$, then $r \in Y$, which in turn forces $e \in Q - W$. Hence, if $q \in X \cup Y$, then $\vec{s} \in Y' \times U'$; else, whenever $q \in M_1 \cup Z \cup \bigcup_{i \in k} P_i$, then $\vec{s} \in Y' \times M_2'$.
- If $a \in V$, since $(r,a) \in S$, then $r \in P - X$, which yields a list of subcases:
  - if $r \in Y$, then $e \in Q - W$, implying that $\vec{s} \in Y' \times (V' \cup M_2')$;
  - if $r \in P_m$, for some $m \in k$, then $e \in V \cup \bigcup_{i=m}^{k-1} Q_i$, showing that $\vec{s} \in P_m' \times V'$;
  - if $r \in M_1$ (provided $M_1 \neq \varnothing$), then $e \in V$ and hence $\vec{s} \in M_1' \times V'$;
  - if $r \in Z$, then $e \in V \cup W$ and hence $\vec{s} \in Z' \times V'$.
- If $a \in Q_m$, for some $m \in k$, then it has to be the case that $r \in Y$ or $r \in P_n$, for some $0 \leq n \leq m$. If $r \in Y$, since $(p,f),(q,f),(r,f) \in S$ for some $f \in Q$, and $p \in Y$, then $\vec{s} \in Y' \times M_2'$ (for $q \in X$), or $\vec{s} \in Y' \times Q_m'$ (for $q \in P - X$). Instead, whenever $r \in P_n$, then $e \in V \cup \bigcup_{j=n}^{k-1} Q_j$ and an analogous reasoning leads to deducing that $\vec{s} \in P_n' \times Q_m'$.

Once the equality has been proven, we have indeed shown that $\mathbf{S}' \in \mathcal{V}$ is a special failure of congruence 4-permutability of genus k with 2^{nd} middle portion. In particular, this proves that $\Omega^{id}(M_1^{\pm} M_2^{+} SF_4^k) \subseteq \Omega^{id}(M_1^{\pm} M_2^{-} SF_4^k)$.

On the other hand, if we start off with a special failure of congruence 4-permutability of genus k without 1^{st} middle portion, call it $\mathbf{S} \in \mathcal{V}$, then we can consider the algebra $\mathbf{S}^{-1}$ with universe S^{-1}, which becomes a special failure of congruence 4-permutability of genus k without 2^{nd} middle portion. By using the procedure described above, we can then build $(\mathbf{S}^{-1})' \in \mathcal{V}$ being a special failure of congruence 4-permutability of genus k with 2^{nd} middle portion. At this point, by turning back to $((\mathbf{S}^{-1})')^{-1}$, we get that the latter is a special failure of congruence 4-permutability of genus k with 1^{st} middle portion. This also shows that $\Omega^{id}(M_1^{+} M_2^{\pm} SF_4^k) \subseteq \Omega^{id}(M_1^{-} M_2^{\pm} SF_4^k)$, completing the proof.

A.2 Proof of Theorem 5.3.2

Let $\mathcal{V}$ be an idempotent variety and let us first prove the statement for $k = 0$. Hence, assume $\mathcal{V}$ admits $M_1^{\pm}M_2^{\pm}SF_4^0$, namely there exists a special failure of congruence 4-permutability of genus 0 in $\mathcal{V}$, call it $\mathbf{S}$, where $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$, $\{X, Y, Z\}$ is a partition of $P - M_1$ ($M_1 \subsetneq P$) and $\{U, V, W\}$ is a partition of $Q - M_2$ ($M_2 \subsetneq Q$), such that

$$S = (X \times U) \cup [Y \times (Q - W)] \cup (M_1 \times V) \cup [Z \times (V \cup W)].$$

Furthermore, call, as usual, $\alpha = (0_P \otimes 1_Q) \cap S$ and $\beta = (1_P \otimes 0_Q) \cap S$ in $\operatorname{Con} \mathbf{S}$.

Immediately notice that, by idempotence, $U \cup M_2 \cup V$ is a subuniverse of $\mathbf{Q}$, being

$$U \cup M_2 \cup V = Q - W = y/S,$$

for any $y \in Y$. Likewise, $Y \cup M_1 \cup Z = P - X = v/S^{-1} \leq \mathbf{P}$, for any $v \in V$. Therefore, the following two sets are subuniverses of $\mathbf{S}$:

$$S_1 = S \cap [(Y \cup M_1 \cup Z) \times Q] = [Y \times (Q - W)] \cup [M_1 \times V] \cup [Z \times (V \cup W)] \leq \mathbf{S},$$

$$S_2 = S \cap [P \times (U \cup M_2 \cup V)] = (X \times U) \cup [Y \times (Q - W)] \cup [(M_1 \cup Z) \times V] \leq \mathbf{S},$$

and obviously $\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{V}$.

At this point, define the following binary relation on S:

$$R := (\alpha \circ \beta) \cap (S_1 \times S_2),$$

and notice that $R \leq \mathbf{S} \times \mathbf{S}$, and hence $\mathbf{R} \in \mathcal{V}$. In fact, $\mathbf{R} \leq_{sd} \mathbf{S}_1 \times \mathbf{S}_2$. Moreover, it is not hard to show that the following equality holds

$$R = \big[[Y \times (Q - W)] \times S_2\big] \cup \big[[(M_1 \times V) \cup [Z \times (V \cup W)]] \times [(P - X) \times V]\big].$$

By convenience, let us call $\alpha_i = \alpha_{|S_i}$ and $\beta_i = \beta_{|S_i}$, for $i = 1, 2$. By definition of α and β, notice that, in $\operatorname{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$,

$$(\alpha_1 \otimes \alpha_2) \cap (\beta_1 \otimes \beta_2) = (\alpha_1 \cap \beta_1) \otimes (\alpha_2 \cap \beta_2) = 0_{S_1} \otimes 0_{S_2} = 0_{S_1 \times S_2};$$

in particular $(\alpha_1 \otimes \alpha_2)_{|R} \cap (\beta_1 \otimes \beta_2)_{|R} = 0_R$ in $\operatorname{Con} \mathbf{R}$. The latter fact yields, by Theorem 1.2.1, that

$$\mathbf{R} \cong \mathbf{R}' \leq_{sd} \mathbf{R}/(\alpha_1 \otimes \alpha_2)_{|R} \times \mathbf{R}/(\beta_1 \otimes \beta_2)_{|R},$$

where $R' = \{\big((\vec{s}_1, \vec{s}_2)/(\alpha_1 \otimes \alpha_2)_{|R}, (\vec{s}_1, \vec{s}_2)/(\beta_1 \otimes \beta_2)_{|R}\big) : (\vec{s}_1, \vec{s}_2) \in R\}$. We will refer to any element in R' as $(\vec{s}_1, \vec{s}_2)'$, for $(\vec{s}_1, \vec{s}_2) \in R$. What we aim to show is that $\mathbf{R}'$ is a special failure of congruence 4-permutability of genus 1; in particular,

$$R' = (X' \times U') \cup [Y' \times (R/(\beta_1 \otimes \beta_2)_{|R} - W')] \cup [P_0' \times (Q_0' \cup V')] \cup (M_1' \times V') \cup [Z' \times (V' \cup W')],$$

where $\{X', Y', Z', P_0'\}$ is a partition of $R/(\alpha_1 \otimes \alpha_2)_{|R} - M_1'$ and $\{U', V', W', Q_0'\}$ is a partition of $R/(\beta_1 \otimes \beta_2)_{|R} - M_2'$, having defined[1]

$$X' := \big[[Y \times (Q - W)] \times (X \times U)\big] \wr (\alpha_1 \otimes \alpha_2)_{|R},$$

$$Y' := \big[[Y \times (Q - W)] \times [Y \times (Q - W)]\big] \wr (\alpha_1 \otimes \alpha_2)_{|R},$$

$$P_0' := \big[[Y \times (Q - W)] \times [(M_1 \cup Z) \times V]\big] \wr (\alpha_1 \otimes \alpha_2)_{|R},$$

$$M_1' := \big[(M_1 \times V) \times [(P - X) \times V]\big] \wr (\alpha_1 \otimes \alpha_2)_{|R},$$

$$Z' := \big[[Z \times (V \cup W)] \times [(P - X) \times V]\big] \wr (\alpha_1 \otimes \alpha_2)_{|R},$$

[1] To recall the adopted notation, see Section 1.1.

and

$$\begin{aligned}
U' &:= \big[[Y\times(Q-W)]\times[(X\cup Y)\times U]\big]\wr(\beta_1\otimes\beta_2)_{|R},\\
M_2' &:= \big[[Y\times(Q-W)]\times(Y\times M_2)\big]\wr(\beta_1\otimes\beta_2)_{|R},\\
Q_0' &:= \big[[Y\times(U\cup M_2)]\times[(P-X)\times V]\big]\wr(\beta_1\otimes\beta_2)_{|R},\\
V' &:= \big[[(P-X)\times V]\times[(P-X)\times V]\big]\wr(\beta_1\otimes\beta_2)_{|R},\\
W' &:= \big[(Z\times W)\times[(P-X)\times V]\big]\wr(\beta_1\otimes\beta_2)_{|R}.
\end{aligned}$$

Notice that M_i' is empty if and only if M_i is, for $i\in\{1,2\}$, whereas all the other sets are definitely non-empty.

To prove that the union covers the entire set, suppose we pick any $(\vec{s}_1,\vec{s}_2)/(\alpha_1\otimes\alpha_2)_{|R}\in R/(\alpha_1\otimes\alpha_2)_{|R}$, meaning $(\vec{s}_1,\vec{s}_2)\in R$; this yields

$$(\vec{s}_1,\vec{s}_2)\in\big[[Y\times(Q-W)]\times(X\times U)\big]\cup\big[[Y\times(Q-W)]\times[Y\times(Q-W)]\big]\cup$$

$$\cup\big[[Y\times(Q-W)]\times[(M_1\cup Z)\times V]\big]\cup\big[(M_1\times V)\times[(P-X)\times V]\big]\cup\big[[Z\times(V\cup W)]\times[(P-X)\times V]\big],$$

and hence $(\vec{s}_1,\vec{s}_2)/(\alpha_1\otimes\alpha_2)_{|R}\in X'\cup Y'\cup P_0'\cup M_1'\cup Z'$. An analogous reasoning, which we omit, holds for the other partition.

Regarding the disjointness, we are going to prove just one of the cases, the technique being the same for all of them. Suppose for instance $X'\cap Y'=\varnothing$ and then let

$$\big((p,q),(r,s)\big)/(\alpha_1\otimes\alpha_2)_{|R}\in X'\cap Y'.$$

By definition of X' and Y', there exist $\big((y,t),(x,u)\big)\in[Y\times(Q-W)]\times(X\times U)$ and $\big((y',q'),(y'',q'')\big)\in[Y\times(Q-W)]\times[Y\times(Q-W)]$ satisfying

$$\begin{bmatrix}(y,t)\\(x,u)\end{bmatrix}\ (\alpha_1\otimes\alpha_2)_{|R}\ \begin{bmatrix}(p,q)\\(r,s)\end{bmatrix}\ (\alpha_1\otimes\alpha_2)_{|R}\begin{bmatrix}(y',q')\\(y'',q'')\end{bmatrix}.$$

In particular, we deduce that $x=r=y''$, implying $x=y''\in X\cap Y$, which contradicts the disjointness of X and Y. The verifications for the other remaining mutual cases and for U',V',W',Q_0' and M_2' are basically the same and we leave them to the reader.

The last step towards proving that $\mathbf{R}'$ is a special failure of congruence 4-permutability of genus 1 consists of verifying the above displayed equality. We are going to distinguish several cases and explicitly prove them one by one, this being the crucial part of the whole proof.

- $X'\times U'\subseteq R'$: suppose $\big((p,q),(r,s)\big)/(\alpha_1\otimes\alpha_2)_{|R}\in X'$ and $\big((a,b),(c,d)\big)/(\beta_1\otimes\beta_2)_{|R}\in U'$, meaning $(p,q),(a,b)\in Y\times(Q-W)$, $(r,s)\in X\times U$ and $(c,d)\in(X\cup Y)\times U$. Notice that $(p,b)\in Y\times(Q-W)$, $(r,d)\in X\times U$ and hence $\big((p,b),(r,d)\big)\in R$. Moreover,

  $$\begin{bmatrix}(p,q)\\(r,s)\end{bmatrix}\ (\alpha_1\otimes\alpha_2)_{|R}\ \begin{bmatrix}(p,b)\\(r,d)\end{bmatrix}\ (\beta_1\otimes\beta_2)_{|R}\ \begin{bmatrix}(a,b)\\(c,d)\end{bmatrix},$$

  showing that

  $$\Big(\big((p,q),(r,s)\big)/(\alpha_1\otimes\alpha_2)_{|R},\big((a,b),(c,d)\big)/(\beta_1\otimes\beta_2)_{|R}\Big)=\big((p,b),(r,d)\big)'\in R'.$$

- $X'\times(R/(\beta_1\otimes\beta_2)_{|R}-U')\subseteq(R/(\alpha_1\otimes\alpha_2)_{|R}\times R/(\beta_1\otimes\beta_2)_{|R})-R'$: suppose instead there exists $\Big(\big((p,q),(r,s)\big)/(\alpha_1\otimes\alpha_2)_{|R},\big((a,b),(c,d)\big)/(\beta_1\otimes\beta_2)_{|R}\Big)\in X'\times(R/(\beta_1\otimes\beta_2)_{|R}-U')$ [in particular $(a,b)\notin Y\times(Q-W)$ or $(c,d)\notin(X\cup Y)\times U$] such that $\Big(\big((p,q),(r,s)\big)/(\alpha_1\otimes\alpha_2)_{|R},\big((a,b),(c,d)\big)/(\beta_1\otimes\beta_2)_{|R}\Big)\in R'$. Thus, there exists $\big((e,f),(g,h)\big)\in R$ with

  $$\begin{bmatrix}(p,q)\\(r,s)\end{bmatrix}\ (\alpha_1\otimes\alpha_2)_{|R}\ \begin{bmatrix}(e,f)\\(g,h)\end{bmatrix}\ (\beta_1\otimes\beta_2)_{|R}\ \begin{bmatrix}(a,b)\\(c,d)\end{bmatrix},$$

  implying that $(e,f)=(p,b)\in Y\times(Q-W)$ and $(g,h)=(r,d)\in X\times U$. Therefore, there are two possible scenarios:

– if $c \notin X \cup Y$, then we contradict that $(c,d) \in S_2$;
– if $a \notin Y$, since $(a,b) \in S_1$, then it must be the case that $a \in M_1 \cup Z$, forcing $b \in V$ and contradicting the fact that $(a,b)\ \alpha \circ \beta\ (c,d)$.

In any case, a contradiction arises, proving that the initially displayed inclusion need hold.

- $Y' \times (R/(\beta_1 \otimes \beta_2)_{|R} - W') \subseteq R'$: pick any $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) \in Y' \times (R/(\beta_1 \otimes \beta_2)_{|R} - W')$, i.e. $\big((p,q),(r,s)\big) \in [Y \times (Q-W)]^2$ and $\big((a,b),(c,d)\big) \notin (Z \times W) \times [(P-X) \times V]$. Because $(c,d) \in S_2$, it is always the case that $d \in Q-W$, implying in particular $(r,d) \in Y \times (Q-W) \subseteq S_2$. On the other hand, if we assume $(a,b) \in Z \times W$, then the fact that $(a,b)\ \alpha \circ \beta\ (c,d)$ (recall $d \notin W$) forces $(c,d) \in (P-X) \times V$, contradicting $\big((a,b),(c,d)\big) \notin (Z \times W) \times [(P-X) \times V]$. Therefore, $(a,b) \notin Z \times W$: then the fact that $(a,b) \in S_1$ forces $b \in Q-W$, hence showing that also $(p,b) \in Y \in (Q-W) \subseteq S_1$.

  To sum up, we have proven that
  $$\begin{bmatrix}(p,q)\\(r,s)\end{bmatrix}\ (\alpha_1 \otimes \alpha_2)_{|R}\ \begin{bmatrix}(p,b)\\(r,d)\end{bmatrix}\ (\beta_1 \otimes \beta_2)_{|R}\ \begin{bmatrix}(a,b)\\(c,d)\end{bmatrix},$$
  yielding
  $$\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) = \big((p,b),(r,d)\big)' \in R'.$$

- $Y' \times W' \subseteq (R/(\alpha_1 \otimes \alpha_2)_{|R} \times R/(\beta_1 \otimes \beta_2)_{|R}) - R'$: for the sake of contradiction, assume there exists $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) \in (Y' \times W') \cap R'$. As in the above case, we may deduce that $\big(\big((p,b),(r,d)\big)' \in R'$ and hence $\big(\big((p,b),(r,d)\big) \in R$, which in particular implies $(p,b) \in S_1$, although $(p,b) \in Y \times W \subseteq (P \times Q) - S$. This contradiction shows that $Y' \times W' \subseteq (R/(\alpha_1 \otimes \alpha_2)_{|R} \times R/(\beta_1 \otimes \beta_2)_{|R}) - R'$.

- $P'_0 \times (Q'_0 \cup V') \subseteq R'$: let $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) \in P'_0 \times (Q'_0 \cup V')$, which is to say $(p,q) \in Y \times (Q-W)$, $(r,s) \in (M_1 \cup Z) \times V$, $(a,b) \in [Y \times (U \cup M_2)] \cup [(P-X) \times V]$ and $(c,d) \in (P-X) \times V$. Therefore, $(p,b) \in Y \times (U \cup M_2 \cup V) \subseteq S_1$ and $(r,d) \in (M_1 \cup Z) \times V \subseteq S_2$, and hence $(p,b)\ \alpha \circ \beta\ (r,d)$. Furthermore, $\big((p,b),(r,d)\big) \in \big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R} \cap \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}$, proving that $\big((p,b),(r,d)\big)' \in R'$.

- $P'_0 \times (U' \cup M'_2 \cup W') \subseteq (R/(\alpha_1 \otimes \alpha_2)_{|R} \times R/(\beta_1 \otimes \beta_2)_{|R}) - R'$: assume there exists $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) \in [P'_0 \times (U' \cup M'_2 \cup W')] \cap R'$. Again, we deduce that $\big(\big((p,b),(r,d)\big) \in R$, where necessarily $p \in Y$ and $r \in M_1 \cup Z$: hence, if $b \in Q-W$, then $d \in U \cup M_2$, which in turn yields $(r,d) \in (M_1 \cup Z) \times (U \cup M_2)$, preventing (r,d) from belonging to S_2. Otherwise, when $b \in W$, then $(p,b) \in Y \times W$, contradicting $(p,b) \in S_1$. In either case, we have obtained a contradiction, as desired.

- $M'_1 \times V' \subseteq R'$ (we are assuming $M_1 \neq \varnothing$): consider any $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) \in M'_1 \times V'$, which is to say $(p,q) \in M_1 \times V$ and $(r,s),(a,b),(c,d) \in (P-X) \times V$. It is straightforward to realize that $(p,q) \in M_1 \times V$ and $(r,d) \in (P-X) \times V$, and hence
  $$\begin{bmatrix}(p,q)\\(r,s)\end{bmatrix}\ (\alpha_1 \otimes \alpha_2)_{|R}\ \begin{bmatrix}(p,b)\\(r,d)\end{bmatrix}\ (\beta_1 \otimes \beta_2)_{|R}\ \begin{bmatrix}(a,b)\\(c,d)\end{bmatrix},$$
  showing that
  $$\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big) = \big((p,b),(r,d)\big)' \in R'.$$

- $M'_1 \times (R/(\alpha_1 \otimes \alpha_2)_{|R} - V') \subseteq (R/(\alpha_1 \otimes \alpha_2)_{|R} \times R/(\beta_1 \otimes \beta_2)_{|R}) - R'$: if we assume that there exists one element in $[M'_1 \times (R/(\alpha_1 \otimes \alpha_2)_{|R} - V')] \cap R'$, say $\big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\big)$, then a reasoning that we have invoked several times ensures that $\big((p,b),(r,d)\big) \in R$, in particular $(p,b) \in S_1$ and $(r,d) \in S_2$. Since $p \in M_1$, then $b \in V$; in addition the fact that $(p,b) \in M_1 \times V$ and $(p,b)\ \alpha \circ \beta\ (r,d)$ forces $d \in V$: all these constraints together determine that $a \in X$ or $c \in X$, contradicting either $(a,b) \in S_1$ or $(c,d) \in S_2$.

- $Z' \times (V' \cup W') \subseteq R'$: let $\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R} \in Z'$ and $\big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R} \in V' \cup W'$, meaning $(p,q) \in Z \times (V \cup W)$, $(r,s),(c,d) \in (P-X) \times V$ and $(a,b) \in [(P-X) \times V] \cup (Z \times W)$. In any case, we get that $(p,b) \in Z \times (V \cup W)$ and $(r,d) \in (P-X) \times V$, showing also that $\big((p,b),(r,d)\big) \in R$ and hence

$$\Big(\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R}, \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R}\Big) = \big((p,b),(r,d)\big)' \in R'.$$

- $Z' \times [R/(\beta_1 \otimes \beta_2)_{|R} - (V' \cup W')] \subseteq (R/(\alpha_1 \otimes \alpha_2)_{|R} \times R/(\beta_1 \otimes \beta_2)_{|R}) - R'$: suppose there exists $\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R} \in Z'$ and $\big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R} \notin V' \cup W'$ such that

$$\big((p,q),(r,s)\big)/(\alpha_1 \otimes \alpha_2)_{|R} \; R' \; \big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R},$$

which implies $(p,b)\ R\ (r,d)$. The latter forces $b \in V \cup W$, which in turn forces $(r,d) \in (P-X) \times V$. Moreover, since $(a,b) \in S_1$, then $a \notin X$: this, along with the fact that $\big((a,b),(c,d)\big)/(\beta_1 \otimes \beta_2)_{|R} \notin V' \cup W'$, yields that $c \in X$, contradicting $(c,d) \in S_2$.

This completes the proof that $\mathbf{R}'$ is a special failure of congruence 4-permutability of genus 1. Should $\mathbf{R}'$ not have i^{th} middle portion $(i = 1,2)$, Theorem 5.3.1 guarantees that $\mathcal{V}$ also contains a special failure with i^{th} middle portion, as required.

The second part of the proof consists of considering any case $k \geq 1$. Thus, suppose $\mathcal{V}$ admits $M_1^{\pm} M_2^{\pm} SF_4^k$, witnessed by the special failure $\mathbf{S} \leq_{sd} \mathbf{P} \times \mathbf{Q}$, where the partitions involved are $\{X,Y,Z\} \cup \{P_i : i \in k\}$ of $P - M_1$ ($M_1 \subsetneq P$) and $\{U,V,W\} \cup \{Q_i : i \in k\}$ of $Q - M_2$ ($M_2 \subsetneq Q$). Many techniques we are going to use are similar to the ones exposed in the first part of the proof, so we will feel free to omit certain repetitive details.

Let α and β be the congruences of $\mathbf{S}$ defined as in the previous part. Moreover, notice that the following is a subuniverse of $\mathbf{S}$ by idempotence:

$$Y \cup P_0 = q/S^{-1} \leq \mathbf{P},$$

for any $q \in Q_0$, also yielding

$$T := [(Y \cup P_0) \times Q] \cap S = [Y \times (Q - W)] \cup \left[P_0 \times \left(V \cup \bigcup_{i \in k} Q_i\right)\right] \leq \mathbf{S}.$$

At this point, define the following relation R on $S \times S$:

$$R := (\alpha \circ \beta) \cap (S \times T),$$

which turns out to be a subuniverse of $\mathbf{S} \times \mathbf{S}$, and hence $\mathbf{R} \in \mathcal{V}$. Again, let us notice that $\alpha' := (\alpha \otimes \alpha_{|T})_{|R}$ and $\beta' := (\beta \otimes \beta_{|T})_{|R}$ are congruences of $\mathbf{R}$ intersecting at 0_R, which implies, by Theorem 1.2.1,

$$\mathbf{R} \cong \mathbf{R}' \leq_{sd} \mathbf{R}/\alpha' \times \mathbf{R}/\beta',$$

where $R' = \{(\vec{p},\vec{r})' := \big((\vec{p},\vec{r})/\alpha', (\vec{p},\vec{r})/\beta'\big) : (\vec{p},\vec{r}) \in R\}$ and the displayed isomorphism is given by $(\vec{p},\vec{r}) \rightsquigarrow (\vec{p},\vec{r})'$.

Indeed, we can rather straightforwardly observe that the following equality holds for R:

$$\begin{gathered} R = [(X \times U) \times (Y \times U)] \cup \\ \cup \big[[Y \times (Q - W)] \times T\big] \cup \bigcup_{i \in k} \left[\left[P_i \times \left(V \cup \bigcup_{j=i}^{k-1} Q_j\right)\right] \times \left[(Y \cup P_0) \times \left(V \cup \bigcup_{j=i}^{k-1} Q_j\right)\right]\right] \cup \\ \cup \big[[(M_1 \times V) \cup [Z \times (V \cup W)]] \times [(Y \cup P_0) \times V]\big]. \end{gathered}$$

Next, we are going to prove that $\mathbf{R}'$ is a special failure of congruence 4-permutability of genus $k+1$ which always does have the 2^{nd} middle portion, independently of $\mathbf{S}$ having it or not. To do so, we aim to show that

$$R' = (X' \times U') \cup [Y' \times (R/\beta' - W')] \cup \bigcup_{i \in k+1} \left[P_i' \times \left(V' \cup \bigcup_{j=i}^{k} Q_j'\right)\right] \cup [M_1' \times V] \cup [Z' \times (V' \cup W')],$$

where $\{X', Y', Z'\} \cup \{P'_i : i \in k+1\}$ is a partition of $R/\alpha' - M'_1$ and $\{U', V', W'\} \cup \{Q'_i : i \in k+1\}$ is a partition of $R/\beta' - M'_2$, with

$$X' := [(X \times U) \times (Y \times U)] \wr \alpha',$$

$$Y' := [Y \times (Q - W)]^2 \wr \alpha',$$

$$P'_0 := \left[[Y \times (Q - W)] \times \left[P_0 \times \left(V \cup \bigcup_{i \in k} Q_i\right)\right]\right] \wr \alpha',$$

$$P'_i := \left[\left[P_{i-1} \times \left(V \cup \bigcup_{j=i-1}^{k-1} Q_j\right)\right] \times \left[(Y \cup P_0) \times \left(V \cup \bigcup_{j=i-1}^{k-1} Q_j\right)\right]\right] \wr \alpha', \text{ for } 1 \leq i \leq k,$$

$$M'_1 := \big[(M_1 \times V) \times [(Y \cup P_0) \times V]\big] \wr \alpha',$$

$$Z' := \big[[Z \times (V \cup W)] \times [(Y \cup P_0) \times V]\big] \wr \alpha'$$

and

$$U' := \big[[(X \cup Y) \times U] \times (Y \times U)\big] \wr \beta',$$

$$M'_2 := \Big[\big[[Y \times [Q - (U \cup W)]] \times (Y \times U)\big] \cup \big[[Y \times (Q - W)] \times (Y \times M_2)\big]\Big] \wr \beta',$$

$$Q'_0 := \left[[Y \times (U \cup M_2)] \times \left[(Y \cup P_0) \times \left(V \cup \bigcup_{i \in k} Q_i\right)\right]\right] \wr \beta',$$

$$Q'_i := \left[\left[\left[\left(Y \cup \bigcup_{j=0}^{i-1} P_j\right) \times \left(V \cup \bigcup_{j=i-1}^{k-1} Q_j\right)\right] \times [(Y \cup P_0) \times Q_{i-1}]\right] \cup\right.$$

$$\left. \cup \left[\left[\left(Y \cup \bigcup_{j=0}^{i-1} P_j\right) \times Q_{i-1}\right] \times \left[(Y \cup P_0) \times \left(V \cup \bigcup_{j=i}^{k-1} Q_j\right)\right]\right]\right] \wr \beta', \text{ for } 1 \leq i \leq k,$$

$$V' := \big[[(P - X) \times V] \times [(Y \cup P_0) \times V]\big] \wr \beta',$$

$$W' := \big[(Z \times W) \times [(Y \cup P_0) \times V]\big] \wr \beta'.$$

Notice that, on the one hand, $M'_1 = \varnothing$ if and only if $M_1 = \varnothing$; on the other hand, M'_2 cannot be empty, given that the block $\big[[Y \times [Q - (U \cup W)]] \times (Y \times U)\big]$ is not empty (whereas the other block $\big[[Y \times (Q - W)] \times (Y \times M_2)\big]$ is empty if and only if M_2 is).

In order to show that the above mentioned collections are partitions, notice that none of the sets defined in the previous list are empty, except, possibly, for M'_1, as already discussed. Moreover, if E' is any one of those sets, since that is defined as $E' = H \wr \gamma$, for some set $H \subseteq R$ and $\gamma \in \{\alpha', \beta'\}$, it is straightforward to prove that they satisfy the covering property, this being satisfied in advance by the H's, which cover the whole R.

Finally, an argument similar to the one expressed in the first part of the proof shows that those sets are disjoint in their corresponding extensions. Such an argument proceeds as follows: assume $E' := H \wr \gamma$, $F' := G \wr \gamma$ are any two subsets of R/γ from the above list ($\gamma \in \{\alpha', \beta'\}$), such that $E' \cap F' \neq \varnothing$, and let $\vec{r}/\gamma$ be any element in common, meaning $\vec{r} \in H \cap G$. By construction, H and G are of the form

$$H = (A_1 \times A_2) \times (A_3 \times A_4) \text{ or } H = [(A_1 \times A_2) \times (A_3 \times A_4)] \cup [(C_1 \times C_2) \times (C_3 \times C_4)],$$

$$G = (B_1 \times B_2) \times (B_3 \times B_4) \text{ or } G = [(B_1 \times B_2) \times (B_3 \times B_4)] \cup [(D_1 \times D_2) \times (D_3 \times D_4)],$$

and $\vec{r} = \big((r_1, r_2), (r_3, r_4)\big)$, for some sets A_i's, B_i's, C_i's, D_i's and elements r_i's. Therefore, we have that, for all $i = 1, 2, 3, 4$, $r_i \in A_i \cap B_i$, or $r_i \in A_i \cap D_i$, or $r_i \in C_i \cap B_i$ or $r_i \in C_i \cap D_i$, which, in any case, yield a contradiction since it is always possible to find $j \in \{1, 2, 3, 4\}$ such that $A_j \cap B_j = \varnothing$, or $A_j \cap D_j = \varnothing$, or $C_j \cap B_j = \varnothing$ or $C_j \cap D_j = \varnothing$ (this fact can be verified by direct inspection). As an example of this, suppose we need to verify that X' and P'_i are disjoint

$(0 < i < k)$; then it is sufficient to notice that $X \cap P_{i-1} = \varnothing$ and the reasoning above does the rest of the job.

To verify the equality involving R', we are going to proceed case by case and, in each of them, we are going to prove an inclusion of the form $E' \times F \subseteq R'$ or $E' \times F \subseteq (R/\alpha' \times R/\beta') - R'$; thus, in order to omit repetitions, an element from the left hand side of those inclusions will be denoted by $\bar{e} = \big(\vec{t}/\alpha', \vec{f}/\beta'\big)$, with $\vec{t} = \big((p,q),(r,s)\big)$ and $\vec{f} = \big((a,b),(c,d)\big)$, whereas the particular element $\big((p,b),(r,d)\big)$ will be referred to as $\vec{m}$.

- $X' \times U' \subseteq R'$: if $\bar{e} \in X' \times U'$, then $(p,q) \in X \times U$, $(r,s) \in Y \times U$, $(a,b) \in (X \cup Y) \times U$ and $(c,d) \in Y \times U$, which implies $\vec{m} \in (X \times U) \times (Y \times U) \subseteq R$. Moreover, being $\vec{t}\ \alpha'\ \vec{m}\ \beta'\ \vec{f}$, we deduce that
$$\bar{e} = \vec{m}' \in R'.$$

- $X' \times (R/\beta' - U') \subseteq (R/\alpha' \times R/\beta') - R'$: suppose $\bar{e}$ is an element of both $X' \times (R/\beta' - U')$ and R'. This in particular means $(p,q) \in X \times U$, $(r,s) \in Y \times U$ and that either one of the following cases has to occur: $(a,b) \notin (X \cup Y) \times U$ or $(c,d) \notin Y \times U$. On the other hand, $\bar{e} \in R'$, meaning there must be some element $\vec{g} \in R$ such that $\bar{e} = \vec{g}'$. It turns out that $\vec{g} = \vec{m} = \big((p,b),(r,d)\big) \in R$: since $(p,b) \in S$ and $p \in X$, then $b \in U$, which also forces $a \in X \cup Y$ (recall $(a,b) \in S$). Therefore, the other possible case need occur, namely $(c,d) \notin Y \times U$. Because, $(p,b)\ \alpha \circ \beta\ (r,d)$, we get that $(r,d) \in (X \cup Y) \times U$, in particular $d \in U$, which necessarily implies $c \notin Y$. The fact that $(c,d) \in T$ forces $c \in P_0$, contradicting the fact that $T \cap (P_0 \times U) = \varnothing$.

- $Y' \times (R/\beta' - W') \subseteq R'$: if $\vec{t}/\alpha' \in Y'$ and $\vec{f}/\beta' \notin W'$, then $(p,q),(r,s) \in Y \times (Q - W)$, while for $\vec{f} \in R$, either $(a,b) \notin Z \times W$, or $(c,d) \notin (Y \cup P_0) \times V$. Because $(c,d) \in T$, then $c \in Y \cup P_0$ and it must also be the case that $d \in Q - W$: this yields that $(r,d) \in Y \times (Q - W) \subseteq T$. If $b \in W$, then the fact that $(a,b) \in S$ implies $a \in Z$, forcing in turn $d \in V$ and contradicting the fact that $\vec{f}/\beta' \notin W'$. Therefore, we deduce that $b \notin W$, which implies $(r,b) \in Y \times (Q - W) \subseteq S$, $\vec{m} \in R$, and hence $\bar{e} = \vec{m}' \in R'$.

- $Y' \times W' \subseteq (R/\alpha' \times R/\beta') - R'$: if $\vec{t}/\alpha' \in Y'$ and $\vec{f}/\beta' \in W'$, then $(p,q),(r,s) \in Y \times (Q - W)$, $(a,b) \in Z \times W$ and $(c,d) \in (Y \cup P_0) \times V$, yielding that $(p,b) \in Y \times W \subseteq (P \times Q) - S$ and $(r,d) \in Y \times V$, preventing $\vec{m}$ from belonging to R and hence making it impossible for $\vec{m}'$ to be in R'.

- $P_0' \times (U' \cup M_2') \subseteq (R/\alpha' \times R/\beta') - R'$: first notice that the following holds
$$U' \cup M_2' = \big[[Y \times (Q - W)] \times [Y \times (U \cup M_2)]\big] \wr \beta'.$$
Hence, suppose $\bar{e} \in [P_0' \times (U' \cup M_2')] \cap R'$, i.e. $(p,q) \in Y \times (Q - W)$, $(r,s) \in P_0 \times (V \cup \bigcup_{i \in k} Q_i)$ and $(a,b) \in Y \times (Q - W)$, $(c,d) \in Y \times (U \cup M_2)$, with $(a,b)\ \alpha \circ \beta\ (c,d)$. We deduce that $\bar{e} = \vec{m}' \in R$, in particular $\vec{m} \in R$. The latter is impossible since $(r,d) \in P_0 \times (U \cup M_2) \subseteq (P \times Q) - S$. Such a contradiction shows that $\bar{e}$ cannot lie in R', as desired.

- $P_0' \times Q_0' \subseteq R'$: the fact that $p \in Y$, $b \in U \cup M_2$, $r \in P_0$ and $d \in V \cup \bigcup_{i \in k} Q_i$ implies the validity of the displayed inclusion.

- $P_0' \times Q_i' \subseteq R'$, for $1 \leq i \leq k$: if $\bar{e} \in P_0' \times Q_i'$, then $(p,q) \in Y \times (Q - W)$, $(r,s) \in P_0 \times (V \cup \bigcup_{j \in k} Q_j)$, $a \in Y \cup \bigcup_{j \in 0}^{i-1} P_j$, $b \in V \cup \bigcup_{j=i-1}^{k-1} Q_j$, $c \in Y \cup P_0$ and $d \in V \cup \bigcup_{j=i-1}^{k-1} Q_j$. Therefore,
$$(p,b) \in Y \times \left(V \cup \bigcup_{j=i-1}^{k-1} Q_j \right) \subseteq S,$$
$$(r,d) \in P_0 \times \left(V \cup \bigcup_{j=i-1}^{k-1} Q_j \right) \subseteq T,$$
and further $(p,b)\ \alpha \circ \beta\ (r,d)$, showing that $\vec{m} \in R$ and $\bar{e} = \vec{m}' \in R'$.

- $P_0' \times V' \subseteq R'$: the inclusion is proven by the fact that $(p,b) \in Y \times V$ and $(r,d) \in P_0 \times V$, implying that $(p,b)\ R\ (r,d)$ and hence $\bar{e} = \vec{m}' \in R'$.

- $P_0' \times W' \subseteq (R/\alpha' \times R/\beta') - R'$: whenever $\vec{t}/\alpha' \in P_0'$ and $\vec{f}/\beta' \in W'$, if also $\bar{e} = (\vec{t}/\alpha', \vec{f}/\beta') \in R'$, then in particular $\vec{m}' \in R'$, namely $\vec{m} \in R$: this fact in turn yields $(p,b) \in S$, against the fact that $(p,b) \in Y \times W \subseteq (P \times Q) - S$.

- $P_i' \times (U' \cup M_2') \subseteq (R/\alpha' \times R/\beta') - R'$, for $1 \leq i \leq k$: if $\bar{e} \in [P_i' \times (U' \cup M_2')] \cap R'$, then in particular $p \in P_{i-1}$, $b \in Q - W$, $r \in Y \cup P_0$ and $d \in U \cup M_2$. The fact that $\vec{m} \in R$ ($\bar{e} = \vec{m}' \in R'$) and $(p,b) \in S$ force $b \in V \cup \bigcup_{j=i-1}^{k-1} Q_j$. Moreover, since $d \in U \cup M_2$, then $r \in Y$. In such a scenario, $(p,b)\ \cancel{\alpha \circ \beta}\ (r,d)$, in contradiction with the previous deduction that $\vec{m} \in R$.

- $P_i' \times Q_0' \subseteq (R/\alpha' \times R/\beta') - R'$, for $1 \leq i \leq k$: if we assume $\bar{e} = \vec{m}' \in (P_i' \times Q_0') \cap R'$, then the contradiction comes out when we realize that $p \in P_{i-1}$ and $b \in U \cup M_2$, preventing (p,b) from being an element of S.

- $P_i' \times Q_j' \subseteq (R/\alpha' \times R/\beta') - R'$, for $2 \leq i \leq k$, $1 \leq j < i$: if we assume $\vec{m}' = \bar{e} \in (P_i' \times Q_j') \cap R'$, then $p \in P_{i-1}$, $r \in Y \cup P_0$ and two cases arise for b and d
  - if $b \in Q_{j-1}$, then $(p,b) \notin S$, since $j-1 < i-1$;
  - if $d \in Q_{j-1}$, then $b \in V \cup \bigcup_{n=i}^{k-1} Q_n$ (we have excluded the case $b \in Q_n$ for $j-1 \leq n < i$, otherwise deducing a contradiction as the one already deduced in the previous case), meaning that $(p,b)\ \cancel{\alpha \circ \beta}\ (r,d)$.

  In either case a contracting argument leads to proving that $\bar{e}$ cannot lie in R', as desired.

- $P_i' \times Q_j' \subseteq R'$, for $1 \leq i \leq k$ and $i \leq j \leq k$: if $\bar{e} \in P_i' \times Q_j'$, then, in particular, $p \in P_{i-1}$, $r \in Y \cup P_0$ and either one of the two following cases occurs
  - $b \in V \cup \bigcup_{n=j-1}^{k-1} Q_n$ and $d \in Q_{j-1}$: hence, we get that $(p,b) \in P_{i-1} \times (V \cup \bigcup_{n=j-1}^{k-1} Q_n) \subseteq S$ (because $i-1 \leq j-1$), $(r,d) \in (Y \cup P_0) \times Q_{j-1} \subseteq T$ and finally $(p,b)\ R\ (r,d)$;
  - $b \in Q_{j-1}$ and $d \in V \cup \bigcup_{n=j}^{k-1} Q_n$: even in this case, $(p,b) \in P_{i-1} \times Q_{j-1} \subseteq S$ (because $i-1 \leq j-1$), $(r,d) \in (Y \cup P_0) \times (V \cup \bigcup_{n=j}^{k-1} Q_n) \subseteq T$ and finally $(p,b)\ R\ (r,d)$.

  Either case shows that $\bar{e} = \vec{m}' \in R'$.

- $P_i' \times W' \subseteq (R/\alpha' \times R/\beta') - R'$, for $1 \leq i \leq k$: By assuming that $\bar{e} = \vec{m}' \in R'$, we easily obtain the two contradicting conjunction $(p,b) \in S$ and $(p,b) \in P_{i-1} \times W$.

- $(M_1' \cup Z') \times V' \subseteq R'$: by picking $\bar{e} = \vec{m}' \in (M_1' \cup Z') \times V'$, we straightforwardly deduce that $(p,b) \in (M_1 \cup Z) \times V$ and $(r,d) \in (Y \cup P_0) \times V$, yielding $\bar{e} = \vec{m}' \in R'$.

- $(M_1' \cup Z') \times [R/\beta' - (V' \cup W')] \subseteq (R/\alpha' \times R/\beta') - R'$: if the set on the left hand side contains $\bar{e} = \vec{m}' \in R'$, then $p \in M_1 \cup Z$, $r \in Y \cup P_0$. The fact that $(p,b) \in S$ forces $b \in V \cup W$, which in turn forces $d \in V$ (because $\big((p,b),(r,d)\big) \in R$): the only possible case, then, is $\vec{f}/\beta' \in V' \cup W'$, contradicting the initial assumption.

- $M_1' \times W' \subseteq (R/\alpha' \times R/\beta') - R'$ (provided $M_1 \neq \varnothing$): if $\bar{e} = \vec{m}' \in (M_1' \times W') \cap R'$, then in particular $(p,b) \in S \cap (M_1 \times W) = \varnothing$, which is a contradiction. Therefore the displayed inclusion does have to hold.

- $Z' \times W' \subseteq R'$: for any $\bar{e} \in Z' \times W'$, we get that $(p,b) \in Z \times W \subseteq S$, $(r,d) \in (Y \cup P_0) \times V \subseteq T$ and also $(p,b)\ \alpha \circ \beta\ (r,d)$. As a result, $\vec{m} \in R$, and because $\vec{m}' = \bar{e}$ we have that $\bar{e} \in R'$.

The above exposed verifications conclude the proof of the fact that $\mathbf{R}'$ is a special failure of congruence 4-permutability of genus $k+1$ with 2^{nd} middle portion and with or without 1^{st} middle portion, depending on whether M_1' is empty or not. The rest of the statement involving the omission classes follows easily from what we have proven so far and what Theorem 5.3.1 states.

Appendix B

A separation argument for CP_n and $\Omega(SHR_{n-1})$

[1] During the oral examination held on May 29^{th}, 2018, we learnt from Dr. Á. Szendrei's comments that she has answered our question about the separation of the classes CP_n and $\Omega(SHR_{n-1})$ ($n \geq 2$).

In the concluding chapter of this thesis, we pose in Problem 2 the question whether the equality $CP_2 = \Omega(SHR_1)$ holds. We already know that, in general, $CP_n \subseteq \Omega(SHR_{n-1})$, for every $n \geq 2$; Dr. Szendrei's argument indeed shows that the inclusion is strict, in particular answering negatively the question in Problem 2 (it is also worth mentioning that, in Section 5.2, we provide a proof, in Corollary 5.2.3, of the fact that $CP_3^{id} \subsetneq \Omega^{id}(SHR_2)$).

Before presenting her argument, we need to include a definition that can be found in Exercise 3.12(4) of [19].

Definition B.0.1. For $k \geq 1$ and an algebra $\mathbf{A} = \langle A; \{f^{\mathbf{A}} : f \in F\}\rangle$, the k^{th} *matrix power of* $\mathbf{A}$ is the algebra $\mathbf{A}^{[k]} := \langle A^k; \{f^{\mathbf{A}^k} : f \in F\} \cup \{d^{\mathbf{A}^{[k]}}, p^{\mathbf{A}^{[k]}}\}\rangle$, such that $d^{\mathbf{A}^{[k]}}$ is k-ary, $p^{\mathbf{A}^{[k]}}$ is unary and

$$d^{\mathbf{A}^{[k]}}\left(\begin{bmatrix} x_1^1 \\ \vdots \\ x_k^1 \end{bmatrix}, \dots, \begin{bmatrix} x_1^k \\ \vdots \\ x_k^k \end{bmatrix}\right) = \begin{bmatrix} x_1^1 \\ \vdots \\ x_k^k \end{bmatrix},$$

$$p^{\mathbf{A}^{[k]}}\big((x_1, x_2, \dots, x_k)\big) = (x_2, \dots, x_k, x_1).$$

For a class K of similar algebras, also define $K^{[k]} = \{\mathbf{A}^{[k]} : \mathbf{A} \in K\}$.

It turns out that, whenever $\mathcal{V}$ is a variety, then $\mathcal{V}^{[k]}$ is a variety as well, for every $k \geq 1$. Moreover, a crucial property that we need in the next argument is that, whenever $R \leq \mathbf{A}^{[2]} \times \mathbf{A}^{[2]}$, then $R \cong E \otimes E$, for some $E \leq \mathbf{A}^2$.

That being said, in order to show that $CP_n \subsetneq \Omega(SHR_{n-1})$, for all $n \geq 2$, it is sufficient to prove that $\mathcal{S}ets^{[2]} \in \Omega(SHR_1)$ (and hence in $\Omega(SHR_k)$, for every $k \geq 1$, due to Theorem 4.1.3), although $\mathcal{S}ets^{[2]} \notin CP_\omega$ (also notice that $\mathcal{S}ets^{[2]}$ is not idempotent).

Dr. Szendrei's argument proceeds by first proving that $\mathcal{S}ets^{[2]}$ is not congruence n-permutable for any $n \geq 2$, since it cannot realize any Hagemann-Mitschke terms. As a matter of fact, the only idempotent terms of $\mathcal{S}ets^{[2]}$ are the projections and $d(x, y)$, for $x \neq y$, preventing the axiomatizing equations of $\mathcal{CP}_n$ from being satisfied, for every $n \geq 2$.

Furthermore, if $\mathcal{S}ets^{[2]}$ admits SHR_1, then there exists a 1-dimensional special Hagemann relation $R \leq \mathbf{A}^{[2]} \times \mathbf{A}^{[2]}$, for some $\mathbf{A} \in \mathcal{S}ets$. In turn, because of the previous observation, there exists $E \leq \mathbf{A}^2$, such that $R \cong E \otimes E$. However, the proof of Theorem 4.1.1 ensures that $E = A^2$, and hence $R = A^2 \times A^2$, contradicting its definition and showing that $\mathcal{S}ets^{[2]} \in \Omega(SHR_k)$, for every $k \geq 1$.

[1] This appendix has been added after the oral defence and external reviewer's comments.

An analogous argument on the same variety $\mathcal{S}ets^{[2]}$ can be provided to show that $CP_3 \subsetneq \Omega(M^+SHR_2)$, answering negatively also the first question of Problem 3 (while the second question still remains open).

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