Sensitive instances
of the Constraint Satisfaction Problem

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──── **Abstract** ────

We investigate the impact of modifying the constraining relations of a Constraint Satisfaction Problem (CSP) instance, with a fixed template, on the set of solutions of the instance. More precisely we investigate sensitive instances: an instance of the CSP is called sensitive, if removing any tuple from any constraining relation invalidates some solution of the instance. Equivalently, one could require that every tuple from any one of its constraints extends to a solution of the instance.

Clearly, any non-trivial template has instances which are not sensitive. Therefore we follow the direction proposed (in the context of strict width) by Feder and Vardi in [13] and require that only the instances produced by a local consistency checking algorithm are sensitive. In the language of the algebraic approach to the CSP we show that a finite idempotent algebra $\mathbf{A}$ has a $k + 2$ variable near unanimity term operation if and only if any instance that results from running the $(k, k + 1)$-consistency algorithm on an instance over $\mathbf{A}^2$ is sensitive.

A version of our result, without idempotency but with the sensitivity condition holding in a variety of algebras, settles a question posed by G. Bergman about systems of projections of algebras that arise from some subalgebra of a finite product of algebras.

Our results hold for infinite (albeit in the case of $\mathbf{A}$ idempotent) algebras as well and exhibit a surprising similarity to the strict width k condition proposed by Feder and Vardi. Both conditions can be characterized by the existence of a near unanimity operation, but the arities of the operations differ by 1.

1   Introduction

One important algorithmic approach to deciding if a given instance of the Constraint Satisfaction Problem (CSP) has a solution is to first consider whether it has a consistent set of local solutions. Clearly, the absence of local solutions will rule out having any (global) solutions, but in general having local solutions does not guarantee the presence of a solution. A major thrust of the recent research on the CSP has focused on coming up with suitable notions of local consistency and then characterizing those CSPs for which local consistency implies outright consistency or some stronger property. A good source for background material is the survey article [7].

Early results of Feder and Vardi [13] and also Jeavons, Cooper, and Cohen [16] establish that when a template (i.e., a relational structure) $\mathbb{A}$ has a special type of polymorphism, called a near unanimity operation, then not only will an instance of the CSP over $\mathbb{A}$ that has a suitably consistent set of local solutions have a solution, but that any partial solution of it can always be extended to a solution. The notion of local consistency that we investigate in this paper is related to that considered by these researchers but that, as we shall see, is weaker.

The following operations are central to our investigation.

▶ **Definition 1.** *An operation* $n(x_1, \ldots, x_{k+1})$ *on a set A of arity $k + 1$ is called a* near unanimity *operation on A if it satisfies the equalities*

$$n(b, a, a, \ldots, a) = n(a, b, a, \ldots, a) = \cdots = n(a, a, \ldots, a, b) = a$$

for all a, $b \in A$.

Near unanimity operations have played an important role in the development of universal algebra and first appeared in the 1970's in the work of Baker and Pixley [1] and Huhn [15]. More recently they have been used in the study of the CSP [13, 16] and related questions [2, 12]. The main results of this paper can be expressed in terms of the CSP and also in algebraic terms and we start by presenting them from both perspectives. In the concluding section, Section 6, a translation of parts of our results into a relational language is provided, along with some open problems.

1.1   CSP viewpoint

In their seminal paper, Feder and Vardi [13] introduced the notion of bounded width for the class of CSP instances over a finite template $\mathbb{A}$. Their definition of bounded width was presented in terms of the logic programming language DATALOG but there is an equivalent formulation using local consistency algorithms, also given in [13]. Given a CSP instance $\mathcal{I}$ and $k < l$, the (k, l)-consistency algorithm will produce a new instance having all k variable constraints that can be inferred by considering l variables at a time of $\mathcal{I}$. This algorithm rejects $\mathcal{I}$ if it produces an empty constraint. The class of CSP instances over a finite template $\mathbb{A}$ will have width (k, l) if the (k, l)-consistency algorithm rejects all instances from the class that do not have solutions, i.e., the (k, l)-consistency algorithm can be used to decide if a given instance from the class has a solution or not. The class has bounded width if it has width (k, l) for some $k < l$.

A lot of effort, in the framework of the algebraic approach to the CSP, has gone in to analyzing various properties of instances that are the outputs of these types of local consistency algorithms. On one end of the spectrum of the research is a rather wide class of

templates of bounded width [5] and on the other a very restrictive class of templates having bounded strict width [13].

To be more precise, we now formally introduce instances of the CSP.

▶ **Definition 2.** *An* instance $\mathcal{I}$ *of the* CSP *is a pair* $(V, \mathcal{C})$ *where* V *is a finite set of variables, and* $\mathcal{C}$ *is a set of constraints of the form* $((x_1, \ldots, x_n), R)$ *where all* x_i *are in* V *and* R *is an* n-*ary relation over (possibly infinite) sets* A_i *associated to each variable* x_i.

A solution *of* $\mathcal{I}$ *is an evaluation* f *of variables such that, for every* $((x_1, \ldots, x_n), R) \in \mathcal{C}$ *we have* $(f(x_1), \ldots, f(x_n)) \in R$; *a partial solution* *is a partial function satisfying the same condition.*

The CSP *over a relational structure* $\mathbb{A}$, *written* CSP$(\mathbb{A})$, *is the class of* CSP *instances whose constraint relations are from* $\mathbb{A}$.

▶ **Example 3.** For $k > 1$, the template associated with the graph k-colouring problem is the relational structure $\mathbb{D}_{k\mathrm{colour}}$ that has universe $\{0, 1, \ldots, k-1\}$ and a single relation $\neq_k = \{(x, y) \mid x, y < k$ and $x \neq y\}$. The template associated with the HORN-3-SAT problem is the relational structure $\mathbb{D}_{\mathrm{horn}}$ that has universe $\{0, 1\}$ and two ternary relations R_0, R_1, where R_i contains all the triples but $(1, 1, i)$. It is known that CSP$(\mathbb{D}_{\mathrm{horn}})$ has width $(1, 2)$, that CSP$(\mathbb{D}_{2\mathrm{colour}})$ has width $(2, 3)$, and that for $k > 2$, CSP$(\mathbb{D}_{k\mathrm{colour}})$ does not have bounded width (see [7]).

Instances produced by the (k, l)-consistency algorithm have uniformity and consistency properties that we highlight.

▶ **Definition 4.** *The* CSP *instance* $\mathcal{I}$ *is* k-uniform *if all of its constraints are* k-ary *and every set of* k *variables is constrained by a single constraint.*

An instance is a (k, l)-instance *if it is* k-uniform *and for every choice of a set* W *of* l *variables no additional information about the constraints can be derived by restricting the instance to the variables in* W.

This last, very important, property can be rephrased in the following way: for every set $W \subseteq V$ of size l, every tuple in every constraint of $\mathcal{I}_{|W}$ participates in a solution to $\mathcal{I}_{|W}$ (where $\mathcal{I}_{|W}$ is obtained from $\mathcal{I}$ by removing all the variables outside of W and all the constraints that contain any such variables).

Consider the notion of strict width k introduced by Feder and Vardi [13, Section 6.1.2]. Let $\mathbb{A}$ be a template and let us assume, to avoid some technical subtleties, that every relation in $\mathbb{A}$ has arity at most k. The class CSP$(\mathbb{A})$ has *strict width* (k, l) if whenever the (k, l)-consistency algorithm does not reject an instance $\mathcal{I}$ from the class then "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints." In other words, if $\mathcal{I}$ is the result of applying the (k, l)-consistency algorithm to an instance of CSP$(\mathbb{A})$, then any partial solution of $\mathcal{I}$ can be extended to a solution. The template $\mathbb{A}$ is said to have *strict width* k if it has strict width (k, l) for some $l > k$.

A *polymorphism* of a template $\mathbb{A}$ is a function on A that preserves all of the relations of $\mathbb{A}$. Feder and Vardi prove the following.

▶ **Theorem 5** (see Theorem 25, [13]). *Let* $k > 1$ *and let* $\mathbb{A}$ *be a finite relational structure with relations of arity at most* k. *The class* CSP$(\mathbb{A})$ *has strict width* k *if and only if it has strict width* $(k, k+1)$ *if and only if* $\mathbb{A}$ *has a* $(k+1)$-*ary near unanimity operation as a polymorphism.*

Using this Theorem we can conclude that $\mathsf{CSP}(\mathbb{D}_{2\mathrm{colour}})$ from Example 3 has strict width 2 since the ternary majority operation preserves the relation $\neq_2$. In fact this operation preserves all binary relations over the set $\{0, 1\}$. On the other hand, $\mathsf{CSP}(\mathbb{D}_{\mathrm{horn}})$ does not have strict width k for any $k \geq 3$.

Following the algebraic approach to the CSP we replace templates $\mathbb{A}$ with algebras $\mathbf{A}$.

▶ **Definition 6.** *An* algebra $\mathbf{A}$ *is a pair* $(A, \mathcal{F})$ *where* A *is a non-empty set, called the* universe *of* $\mathbf{A}$ *and* $\mathcal{F} = (f_i \mid i \in I)$ *is a set of finitary operations on* A *called the* set of basic operations *of* $\mathbf{A}$. *The function that assigns the arity of the operation* f_i *to* i *is called the* signature *of* $\mathbf{A}$. *If* $t(x_1, \ldots, x_n)$ *is a term in the signature of* $\mathbf{A}$ *then the interpretation of* t *by* $\mathbf{A}$ *as an operation on* A *is called a* term operation *of* $\mathbf{A}$ *and is denoted by* $t^{\mathbf{A}}$.

The CSP *over* $\mathbf{A}$, *written* $\mathsf{CSP}(\mathbf{A})$, *is the class of* CSP *instances whose constraint relations are amongst those relations over* A *that are preserved by the operations of* $\mathbf{A}$ *(i.e., they are subuniverses of powers of* $\mathbf{A}$*).*

A number of important questions about the CSP can be reduced to considering templates that have all of the singleton unary relations [7]; the algebraic counterpart to these types of templates are the *idempotent algebras*.

▶ **Definition 7.** *An operation* $f : A^n \to A$ *on a set* A *is* idempotent *if* $f(a, a, \ldots, a) = a$ *for all* $a \in A$. *An algebra* $\mathbf{A}$ *is* idempotent *if all of its basic operations are.*

It follows that if $\mathbf{A}$ is idempotent then every term operation of $\mathbf{A}$ is an idempotent operation. As demonstrated in Example 22, several of the results in this paper do not hold in the absence of idempotency.

The characterization of strict width in Theorem 5 has the following consequence in terms of algebras.

▶ **Corollary 8.** *Let* $k > 1$ *and let* $\mathbb{A}$ *be a finite relational structure with relations of arity at most* k. *Let* $\mathbf{A}$ *be the algebra with the same universe as* $\mathbb{A}$ *whose basic operations are exactly the polymorphisms of* $\mathbb{A}$. *The following are equivalent:*
1. $\mathbf{A}$ *has a near unanimity term operation of arity* $k + 1$;
2. *in every* $(k, k + 1)$*-instance over* $\mathbf{A}$, *every partial solution extends to a solution.*

The implication "1 implies 2" in Corollary 8 remains valid for general algebras, not necessarily coming from finite relational structures with restricted arities of relations. However, the converse implication fails even if $\mathbf{A}$ is assumed to be finite and idempotent.

▶ **Example 9.** Consider the rather trivial algebra $\mathbf{A}$ that has universe $\{0, 1\}$ and no basic operations. If $\mathcal{I}$ is a $(2, 3)$-instance over $\mathbf{A}$ then since, as noted just after Theorem 5, every binary relation over $\{0, 1\}$ is invariant under the ternary majority operation on $\{0, 1\}$ it follows that every partial solution of $\mathcal{I}$ can be extended to a solution. Of course, $\mathbf{A}$ does not have a near unanimity term operation of any arity.

What this example demonstrates is that in general, for a fixed k, the k-ary constraint relations arising from an algebra do not capture that much of the structure of the algebra. Example 22 provides further evidence for this.

Our first theorem shows that for finite idempotent algebras $\mathbf{A}$, by considering a slightly bigger set of $(k, k + 1)$-instances, over $\mathsf{CSP}(\mathbf{A}^2)$, rather than over $\mathsf{CSP}(\mathbf{A})$, we can detect the presence of a $(k + 1)$-ary near unanimity term operation. Moreover, it is enough to consider only instances with $k + 2$ variables. We note that every $(k, k + 1)$-instance over $\mathbf{A}$ can be easily encoded as a $(k, k + 1)$-instance over $\mathbf{A}^2$.

▶ **Theorem 10.** *Let* **A** *be a finite, idempotent algebra and $k > 1$. The following are equivalent:*

1. **A** *(or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k + 1$;*
2. *in every $(k, k + 1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k + 1)$-instance over $\mathbf{A}^2$ **on** $k + 2$ **variables**, every partial solution extends to a solution.*

In Theorem 20 we extend our result to infinite idempotent algebras by working with local near unanimity term operations.

Going back the original definition of strict width: "it should be possible to obtain a solution by greedily assigning values to the variables one at a time while satisfying the inferred k-constraints" we note that the requirement that the assignment should be greedy is rather restrictive. The main theorem of this paper investigates an arguably more natural concept where the assignment need not be greedy.

▶ **Definition 11.** *An instance of the* CSP *is called* sensitive, *if removing any tuple from any constraining relation invalidates some solution of the instance.*

In other words, an instance is sensitive if every tuple in every constraint of the instance extends to a solution. For $(k, k + 1)$-instances, being sensitive is equivalent to the instance being a (k, n)-instance, where n is the number of variables present in the instance. We provide the following characterization.

▶ **Theorem 12.** *Let* **A** *be a finite, idempotent algebra and $k > 1$. The following are equivalent:*

1. **A** *(or equivalently $\mathbf{A}^2$) has a near unanimity term operation of arity $k + 2$;*
2. *every $(k, k + 1)$-instance over $\mathbf{A}^2$ is sensitive;*
3. *every $(k, k + 1)$-instance over $\mathbf{A}^2$ **on** $k + 2$ **variables** is sensitive.*

Exactly as in Theorem 10 we can consider infinite algebras at the cost of using local near unanimity term operations (see Theorem 21).

In conclusion we investigate a natural property of instances motivated by the definition of strict width and provide a characterization of this new condition in algebraic terms. A surprising conclusion is that the new concept is, in fact, very close to the strict width concept, i.e., for a fixed k one characterization is equivalent to a near unanimity operation of arity $k + 1$ and the second of arity $k + 2$.

1.2 Algebraic viewpoint

Our work has as an antecedent the papers of Baker and Pixley [1] and of Bergman [8] on algebras having near unanimity term operations. In these papers the authors considered subalgebras of products of algebras and systems of projections associated with them. Baker and Pixley showed that in the presence of a near unanimity term operation, such a subalgebra is closely tied with its projections onto small sets of coordinates.

▶ **Definition 13.** *A* variety of algebras *is a class of algebras of the same signature that is closed under taking homomorphic images, subalgebras, and direct products. For* **A** *an algebra, $\mathcal{V}(\mathbf{A})$ denotes the smallest variety that contains* **A** *and is called the* variety generated by **A**. *A variety $\mathcal{V}$ has a near unanimity term of arity $k + 1$ if there is some $(k + 1)$-ary term in the signature of $\mathcal{V}$ whose interpretation in each member of $\mathcal{V}$ is a near unanimity operation.*

Here is one version of the Baker-Pixley Theorem:

▶ **Theorem 14** (see Theorem 2.1 from [1]). *Let* **A** *be an algebra and $k > 1$. The following are equivalent:*

1. **A** *has a* $(k+1)$*-ary near unanimity term operation;*
2. *for every* $r > k$ *and every* $\mathbf{A}_i \in \mathcal{V}(\mathbf{A})$, $1 \leq i \leq r$, *every subalgebra* $\mathbf{R}$ *of* $\prod_{i=1}^{r} \mathbf{A}_i$ *is **uniquely** determined by the projections of R on all products $A_{i_1} \times \cdots \times A_{i_k}$ for* $1 \leq i_1 < i_2 < \cdots < i_k \leq r$;
3. *the same as condition 2, with r set to $k+1$.*

In other words, an algebra has a $(k+1)$-ary near unanimity term operation if and only if every subalgebra of a product of algebras from $\mathcal{V}(\mathbf{A})$ is uniquely determined by its system of k-fold projections into its factor algebras. A natural question, extending the result above, was investigated by Bergman [8]: when does a given "system of k-fold projections" arise from a product algebra?

Note that such a system can be viewed as a k-uniform CSP instance: indeed, following the notation of Theorem 14, we can introduce a variable x_i for each $i \leq r$ and a constraint $((x_{i_1}, \ldots, x_{i_k}); \mathrm{proj}_{i_1,\ldots,i_k} R)$ for each $1 \leq i_1 < i_2 < \cdots < i_k \leq r$. In this way the original relation R consists of solutions of the created instance (but in general will not contain all of them). In this particular instance, different variables can be evaluated in different algebras. Note that the instance is sensitive, if and only if it "arises from a product algebra" in the sense investigated by Bergman.

We will say that $\mathcal{I}$ is a CSP instance *over the variety $\mathcal{V}$ (denoted $\mathcal{I} \in \mathsf{CSP}(\mathcal{V})$)* if all the constraining relations of $\mathcal{I}$ are algebras in $\mathcal{V}$. In the language of the CSP, Bergman proved the following:

▶ **Theorem 15** ([8])**.** *If $\mathcal{V}$ is a variety that has a $(k+1)$-ary near unanimity term then every $(k, k+1)$-instance over $\mathcal{V}$ is sensitive.*

In commentary that Bergman provided on his proof of this theorem he noted that a stronger conclusion could be drawn from it and he proved the following theorem. We note that this theorem anticipates the results from [13] and [16] dealing with templates having near unanimity operations as polymorphisms.

▶ **Theorem 16** ([8])**.** *Let $k > 1$ and $\mathcal{V}$ be a variety. The following are equivalent:*
1. $\mathcal{V}$ *has a $(k+1)$-ary near unanimity term;*
2. *any partial solution of a $(k, k+1)$-instance over $\mathcal{V}$ extends to a solution.*

We present a proof of this theorem in Appendix A.

Theorem 15 provides a partial answer to the question that Bergman posed in [8], namely that in the presence of a $(k+1)$-ary near unanimity term, a necessary and sufficient condition for a k-fold system of algebras to arise from a product algebra is that the associated CSP instance is a $(k, k+1)$-instance.

In [8] Bergman asked whether the converse to Theorem 15 holds, namely, that if all $(k, k+1)$-instances over a variety are sensitive, must the variety have a $(k+1)$-ary near unanimity term? He provided examples that suggested that the answer is no, and we confirm this by proving that the condition is actually equivalent to the variety having a near unanimity term of arity $k+2$. The main result of this paper, viewed from the algebraic perspective (but stated in terms of the CSP), is the following:

▶ **Theorem 17.** *Let $k > 1$. A variety $\mathcal{V}$ has a $(k+2)$-ary near unanimity term if and only if each $(k, k+1)$-instance of the CSP over $\mathcal{V}$ is sensitive.*

The "if" direction of this theorem is proved in Section 3, while a sketch of a proof of the "only if" direction can be found in Section 5 (the complete reasoning can be found in Appendix C). We note that a novel and significant feature of this result is that it does not assume any finiteness or idempotency of the algebras involved.

1.3 Structure of the paper

The paper is structured as follows. In the next section we introduce local near unanimity operations and state Theorem 10 and Theorem 12 in their full power. In Section 3 we collect the proofs that establish the existence of (local) near unanimity operations. Section 4 contains a proof of a new loop lemma, which can be of independent interest, and is necessary in the proof in Appendix C. In Section 5 we provide a sketch of the proof showing that, in the presence of a near unanimity operation of arity $k + 2$, the $(k, k + 1)$-instances are sensitive. A complete proof of this fact, which is our main contribution, can be found in Appendix C. Finally, Section 6 contains conclusions.

Appendix A and Appendix B are provided for the convenience of the reader. They prove facts required for the classification, but known before, and facts which can be proved by minor adaptations of known reasoning. Finally, Appendix C contains, as already mentioned, the main technical contribution of the paper.

2 Details of the CSP viewpoint

In order to state our results in their full strength, we need to define local near unanimity operations. This special concept of local near unanimity operations is required, when considering infinite algebras.

▶ **Definition 18.** *Let $k > 1$. An algebra **A** has* local near unanimity term operations of arity $k + 1$ *if for every finite subset S of A there is some $(k + 1)$-ary term operation n_S of **A** such that*

$$n_S(b, a, \ldots, a, a) = n_S(a, b, a, \ldots, a) = \cdots = n_S(a, a, \ldots, b, a) = n_S(a, a, \ldots, a, b) = a.$$

for all $a, b \in S$.

It should be clear that, for finite algebras, having local near unanimity term operations of arity $k + 1$ and having a near unanimity term operation of arity $k + 1$ are equivalent, but for arbitrary algebras they are not. The following provides a characterization of when an idempotent algebra has local near unanimity term operations of some given arity; it will be used in the proofs of Theorems 20 and 21. It is similar to Theorem 14 and is proved in Appendix A.

▶ **Theorem 19.** *Let **A** be an idempotent algebra and $k > 1$. The following are equivalent:*
1. **A** *has local near unanimity term operations of arity $k + 1$;*
2. *for every $r > k$, every subalgebra of $\mathbf{A}^r$ is uniquely determined by its projections onto all k-element subsets of coordinates;*
3. *every subalgebra of $\mathbf{A}^{k+1}$ is uniquely determined by its projections onto all k-element subsets of coordinates.*

We are ready to state Theorem 10 in its full strength:

▶ **Theorem 20.** *Let **A** be an idempotent algebra and $k > 1$. The following are equivalent:*
1. **A** *(or equivalently $\mathbf{A}^2$) has local near unanimity term operations of arity $k + 1$;*
2. *in every $(k, k + 1)$-instance over $\mathbf{A}^2$, every partial solution extends to a solution;*
3. *in every $(k, k + 1)$-instance over $\mathbf{A}^2$ **on** $k + 2$ **variables**, every partial solution extends to a solution.*

Proof. Obviously condition 2 implies condition 3. A proof of condition 3 implying condition 1 can be found in Section 3. The implication from 1 to 2 is covered by Theorem 16. ◀

Analogously, the main result of the paper, for idempotent algebras, and the full version of
Theorem 12 states:

▶ **Theorem 21.** *Let* **A** *be an idempotent algebra and* $k > 1$*. The following are equivalent:*
1. **A** *(or equivalently* **A**^{2}*) has local near unanimity term operations of arity* $k + 2$*;*
2. *every* $(k, k + 1)$*-instance over* **A**^{2} *is sensitive;*
3. *every* $(k, k + 1)$*-instance over* **A**^{2} *on* $k + 2$ ***variables** is sensitive.*

Proof. Obviously condition 2 implies condition 3. For a proof that condition 3 implies
condition 1 see Section 3. A sketch of the proof of the remaining implication can be found in
Section 5 (see Appendix C for a complete proof).   ◀

The following examples show that in Theorems 19, 20, and 21 the assumption of idempotency
is necessary.

▶ **Example 22.** For $n > 2$, let $\mathbf{S}_n$ be the algebra with domain $[n] = \{1, 2, \ldots, n\}$ and with
basic operations consisting of all unary operations on $[n]$ and all non-surjective operations
on $[n]$ of arbitrary arity. The collection of such operations forms a finitely generated clone,
called the Słupecki clone. Relevant details of these algebras can be found in [17, Example
4.6] and [21]. It can be shown that for $m < n$, the subuniverses of $\mathbf{S}_n^m$ consist of all m-ary
relations R_θ over $[n]$ determined by a partition θ of $[m]$ by

$$R_\theta = \{(a_1, \ldots, a_m) \mid a_i = a_j \text{ whenever } (i, j) \in \theta\}.$$

These rather simple relations are preserved by any operation on $[n]$, in particular by any
majority operation or more generally, by any near unanimity operation.

It follows from Theorem 16 that if $k > 1$ and $\mathcal{I}$ is a $(k, k + 1)$-instance of $\mathsf{CSP}(\mathbf{S}_{2k+1}^2)$
then any partial solution of $\mathcal{I}$ extends to a solution. This also implies that $\mathcal{I}$ is sensitive.
Furthermore any subalgebra of $\mathbf{S}_{k+2}^{k+1}$ is determined by it projections onto all k-element sets
of coordinates. As noted in [17, Example 4.6], for $n > 2$, $\mathbf{S}_n$ does not have a near unanimity
term operation of any arity, since the algebra $\mathbf{S}_n^n$ has a quotient that is a 2-element essentially
unary algebra.

3   Constructing near unanimity operations

In this section we collect the proofs providing, under various assumptions, near unanimity or
local near unanimity operations. That is: the proofs of "3 implies 1" in Theorems 20 and
Theorem 21 as well as a proof of the "if" direction from Theorem 17.

In the following proposition we construct instances over **A**^{2} (for some algebra **A**). By
a minor abuse of notation, we allow in such instances two kinds of variables: variables
x evaluated in A and variables y evaluated in A^2. The former kind should be formally
considered as variables evaluated in A^2 where each constraint enforces that x is sent to
$\{(b, b) \mid b \in A\}$.

Moreover, dealing with k-uniform instances, we understand the condition "every set of
k variables is constrained by a single constraint" flexibly: in some cases we allow for more
constraints with the same set of variables, as long as the relations are proper permutations
so that every constraint imposes the same restriction.

▶ **Proposition 23.** *Let* $k > 1$ *and let* **A** *be an algebra such that, for every* $(k, k + 1)$*-instance*
$\mathcal{I}$ *over* **A**^{2} *on* $k + 2$ *variables every partial solution of* $\mathcal{I}$ *extends to a solution. Then each*
subalgebra of **A**$^{k+1}$ *is determined by its* k*-ary projections.*

Proof. Let $\mathbf{R} \leq \mathbf{A}^{k+1}$ and we will show that it is determined by the system of projections $\mathrm{proj}_I(R)$ as I ranges over all k elements subsets of coordinates. Using $\mathbf{R}$ we define the following instance $\mathcal{I}$ of $\mathsf{CSP}(\mathbf{A}^2)$. The variables of $\mathcal{I}$ will be the set $\{x_1, x_2, \ldots, x_{k+1}, y_{12}\}$ and the domain of each x_i is A, while the domain of y_{12} is A^2.

For $U \subseteq \{x_1, \ldots, x_{k+1}\}$ of size k, let C_U be the constraint with scope U and constraint relation $R_U = \mathrm{proj}_U(R)$. For U a $(k-1)$-element subset of $\{x_1, \ldots, x_{k+1}\}$, let $C_{U \cup \{y_{12}\}}$ be the constraint with scope $U \cup \{y_{12}\}$ and constraint relation $R_{U \cup \{y_{12}\}}$ that consists of all tuples $(b_v \mid v \in U \cup \{y_{12}\})$ such that there is some $(a_1, \ldots, a_{k+1}) \in R$ with $b_v = a_i$ if $v = x_i$ and with $b_{y_{12}} = (a_1, a_2)$.

The instance $\mathcal{I}$ is k-uniform and we will show that it is sensitive. Indeed every tuple in every constraining relation originates in some tuple $\mathbf{b} \in R$. Setting $x_i \mapsto b_i$ and $y_{12} \mapsto (b_1, b_2)$ defines a solution that extends such a tuple.

In particular $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ with $k + 2$ variables and so any partial solution of it can be extended to a solution. Let $\mathbf{b} \in A^{k+1}$ such that $\mathrm{proj}_I(\mathbf{b}) \in \mathrm{proj}_I(R)$ for all k element subsets I of $[k+1]$. Then $\mathbf{b}$ is a partial solution of $\mathcal{I}$ over the variables $\{x_1, \ldots, x_{k+1}\}$ and thus there is some extension of it to the variable y_{12} that produces a solution of $\mathcal{I}$. But there is only one consistent way to extend $\mathbf{b}$ to y_{12} namely by setting y_{12} to the value (b_1, b_2). By considering the constraint with scope $\{x_3, \ldots, x_{k+1}, y_{12}\}$ it follows that $\mathbf{b} \in R$, as required. ◀

Now we are ready to prove the first implication tackled in this section: 3 implies 1 in Theorem 20.

Proof of "3 implies 1" in Theorem 20. By Theorem 19 it suffices to show that each subalgebra of $\mathbf{A}^{k+1}$ is determined by its k-ary projections. Fortunately, Proposition 23 provides just that. ◀

We move on to proofs of "3 implies 1" in Theorem 21 and the "if" direction of Theorem 17. Similarly, as in the theorem just proved, we start with a proposition.

▶ **Proposition 24.** *Let $k > 1$ and let $\mathbf{A}$ be an algebra such that every $(k, k+1)$-instance $\mathcal{I}$ over $\mathbf{A}^2$ on $k + 2$ variables is sensitive. Then each subalgebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections.*

Proof. We will show that if $\mathbf{R}$ is a subalgebra of $\mathbf{A}^{k+2}$ then $R = R^*$ where

$$R^* = \{a \in A^{k+2} \mid \mathrm{proj}_I(a) \in \mathrm{proj}_I(R) \text{ whenever } |I| = k + 1\}.$$

In other words, we will show that the subalgebra $\mathbf{R}$ is determined by its projections into all $(k+1)$-element sets of coordinates.

We will use R and R^* from the previous paragraph to construct a $(k, k+2)$-instance $\mathcal{I} = (V, \mathcal{C})$ with $V = \{x_5, \ldots, x_{k+2}, y_{12}, y_{34}, y_{13}, y_{24}\}$ where each x_i is evaluated in A while all the y's are evaluated in A^2.

The set of constraints is more complicated. There is a *special constraint* on a *special variable set* $((y_{12}, y_{34}, x_5, \ldots, x_{k+2}), C)$ where

$$C = \{((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}) \mid (a_1, \ldots, a_{k+2}) \in R^*\}.$$

The remaining constraints are defined using the relation R. For each set of variables $S = \{v_1, \ldots, v_k\} \subseteq V$ (which is different than the set for the special constraint) we define a constraint $((v_1, \ldots, v_k), D_S)$ with $(b_1, \ldots, b_k) \in D_S$ if and only if there exists a tuple $(a_1, \ldots, a_{k+2}) \in R$ such that:

- if v_i is x_j then $b_i = a_j$, and
- if v_i is y_{lm} then $b_i = (a_l, a_m)$.

Note that the instance $\mathcal{I}$ is k-uniform.

$\triangleright$ Claim 25. $\mathcal{I}$ is a $(k, k+1)$-instance.

Let $S \subseteq V$ be a set of size k. If S is not the special variable set, then every tuple in the relation constraining S originates in some $(b_1, \ldots, b_{k+2}) \in R$ and, as in Proposition 23, sending $x_i \mapsto b_i$ and $y_{lm} \mapsto (b_l, b_m)$ defines a solution that extends such a tuple. We immediately conclude, that the potential failure of the $(k, k+1)$ condition must involve the special constraint.

Thus $S = \{y_{12}, y_{34}, x_5, \ldots, x_{k+2}\}$ and if $\mathbf{b}$ is a tuple from the special constraint C then there is some $(a_1, \ldots, a_{k+2}) \in R^*$ with

$$\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}).$$

The extra variable that we want to extend the tuple $\mathbf{b}$ to is either y_{13} or y_{24}. Both cases are similar and we will only work through the details when it is y_{13}. In this case, assigning the value (a_1, a_3) to the variable y_{13} will produce an extension $\mathbf{b}'$ of $\mathbf{b}$ to a tuple over $S \cup \{y_{13}\}$ that is consistent with all constraints of $\mathcal{I}$ whose scopes are subsets of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$.

To see this, consider a k element subset S' of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$ that excludes some variable x_j. Then, by the definition of R^* there exists some tuple of the form $(a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_{k+2}) \in R$. This tuple from R can be used to witness that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$ since the scope of this constraint does not include the variable x_j.

Suppose that S' is a k element subset of $\{y_{12}, y_{34}, x_5, \ldots, x_{k+2}, y_{13}\}$ that excludes y_{12}. By the definition of R^* there is some tuple of the form $(a_1, a'_2, a_3, \ldots, a_{k+2}) \in R$. Using this tuple it follows that the restriction of $\mathbf{b}'$ to S' satisfies the constraint $D_{S'}$. This is because neither of the variables y_{12} and y_{24} are in S' and so the value $a'_2 \in A_2$ does not matter. A similar argument works when S' is assumed to exclude y_{34} and the claim is proved.

Since $\mathcal{I}$ is a $(k, k+1)$-instance over $\mathbf{A}^2$ and it has $k+2$ variables then by assumption, $\mathcal{I}$ is sensitive. We can use this to show that $R^* \subseteq R$ to complete the proof of this proposition. Let $(a_1, \ldots, a_{k+2}) \in R^*$ and consider the associated tuple $\mathbf{b} = ((a_1, a_2), (a_3, a_4), a_5, \ldots, a_{k+2}) \in C$. Since $\mathcal{I}$ is sensitive then this k-tuple can be extended to a solution $\mathbf{b}'$ of $\mathcal{I}$. Using any constraints of $\mathcal{I}$ whose scopes include combinations of y_{12} or y_{34} with y_{13} or y_{24} it follows that the value of $\mathbf{b}'$ on the variables y_{13} and y_{24} are (a_1, a_3) and (a_2, a_4) respectively. Then considering the restriction of $\mathbf{b}'$ to $S = \{x_5, \ldots, x_{k+2}, y_{13}, y_{24}\}$ it follows that $(a_1, \ldots, a_{k+2}) \in R$ since this restriction lies in the constraint relation D_S. $\blacktriangleleft$

We are in a position to provide the two final proofs in this section.

Proof of "3 implies 1" in Theorem 21. By Theorem 19 it suffices to show that each sub-algebra of $\mathbf{A}^{k+2}$ is determined by its $(k+1)$-ary projections. Fortunately Propositions 24 provides just that. $\blacktriangleleft$

Proof of the "if" direction in Theorem 17. For this direction we apply Proposition 24 to a special member of $\mathcal{V}$, namely the $\mathcal{V}$-free algebra freely generated by $\mathbf{x}$ and $\mathbf{y}$, which we will denote by $\mathbf{F}$. Up to isomorphism, this algebra is unique and its defining property is that $\mathbf{F} \in \mathcal{V}$ and for any algebra $\mathbf{A} \in \mathcal{V}$, any map $f : \{\mathbf{x}, \mathbf{y}\} \to A$ extends uniquely to a homomorphism from $\mathbf{F}$ to $\mathbf{A}$. Consequently, for any two terms $s(x, y)$ and $t(x, y)$ in the signature of $\mathcal{V}$ if $s^{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = t^{\mathbf{F}}(\mathbf{x}, \mathbf{y})$ then the equation $s(x, y) \approx t(x, y)$ holds in $\mathcal{V}$.

Let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^{k+2}$ generated by the tuples $(\mathbf{y}, \mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$, $(\mathbf{x}, \mathbf{y}, \mathbf{x}, \ldots, \mathbf{x})$, $\ldots, (\mathbf{x}, \ldots, \mathbf{x}, \mathbf{y})$. By Proposition 24, the algebra $\mathbf{R}$ is determined by its $(k+1)$-ary projections and so the constant tuple $(\mathbf{x}, \ldots, \mathbf{x})$ belongs to R. The term generating this tuple from the given generators of $\mathbf{R}$ defines the required $(k+2)$-ary near unanimity operation. ◀

4   New loop lemmata

A *loop lemma* is a theorem stating that a binary relation satisfying certain structural and algebraic requirements necessarily contains a *loop* – a pair (a, a). In this section we provide two new loop lemmata, Theorem 31 and Theorem 32, which generalize an "infinite loop lemma" of Olšák [19] and may be of independent interest. Theorem 32 is a crucial tool for the proof sketched in Section 5 and presented in Appendix C.

The algebraic assumptions in the new loop lemmata concern absorption, a concept that has proven to be useful in the algebraic theory of CSPs and in universal algebra [6]. We adjust the standard definition to our specific purposes. We begin with a very elementary definition.

▶ **Definition 26.** *Let R and S be sets. We call a tuple $(a_1, \ldots, a_n)$ a one-S-in-R tuple if for exactly one i we have $a_i \in S$ and all the other a_i's are in R.*

Next we proceed to define a relaxation of the standard absorbing notion. We follow a standard notation, silently extending operations of an algebra to powers (by computing them coordinate-wise).

▶ **Definition 27.** *Let $\mathbf{A}$ be an algebra, $\mathbf{R} \leq \mathbf{A}^k$ and $S \subseteq A^k$. We say that R locally n-absorbs S if, for every finite set $\mathcal{C}$ of one-S-in-R tuples of length n, there is a term operation t of $\mathbf{A}$ such that $t(\mathbf{a^1}, \ldots, \mathbf{a^n}) \in R$ whenever $(\mathbf{a^1}, \ldots, \mathbf{a^n}) \in \mathcal{C}$. We will say that R locally absorbs S, if R locally n-absorbs S for some n.*

Absorption, even in this form, is stable under various constructions. The following lemma lists some of them and we leave it without a proof (the reasoning is identical to the one in e.g. Proposition 2 in [6]).
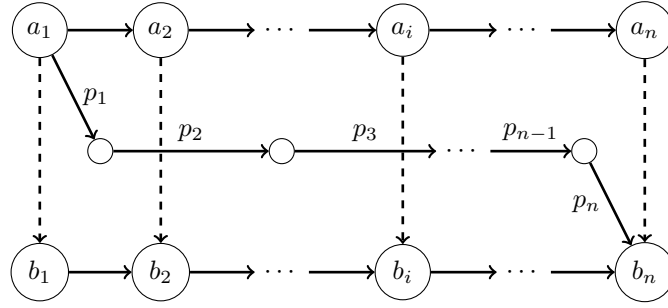
▶ **Lemma 28.** *Let $\mathbf{A}$ be an algebra and $\mathbf{R} \leq \mathbf{A}^2$ such that R locally n-absorbs S. Then R^{-1} locally n-absorbs S^{-1}; and $R \circ R$ locally n-absorbs $S \circ S$, and $R \circ R \circ R$ locally n-absorbs $S \circ S \circ S$ etc.*

Let us prove a first basic property of local absorption.

▶ **Lemma 29.** *Let $\mathbf{A}$ be an idempotent algebra and $\mathbf{R} \leq \mathbf{A}^2$ such that R locally n-absorbs S. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be directed walks in R, and let $(a_i, b_i) \in S$ for each i (see Figure 1). Then there exists a directed walk from a_1 to b_n of length n in R.*

Proof. We will show that there is a term operation t of the algebra $\mathbf{A}$ such that the following $(n+1)$-tuple of elements of A is a walk of length n in R from a_1 to b_n.

$$
\begin{aligned}
(a_1 = {}& t(a_1, a_1, a_1, \ldots, a_1), \\
& t(b_1, a_2, a_2, \ldots, a_2), \\
& t(b_2, b_2, a_3, \ldots, a_3), \\
& \vdots \\
& t(b_{n-1}, b_{n-1}, \ldots, b_{n-1}, a_n), \\
b_n = {}& t(b_n, b_n, b_n, \ldots, b_n)).
\end{aligned}
$$

Figure 1 Solid arrows represent tuples from R and dashed arrows represent tuples from S.

In order to choose a proper t we apply the definition of local absorption to the set of $(n+1)$ one-S-in-R tuples corresponding to the steps in the path. ◀

The loop lemma of Olšák concerns symmetric relations absorbing the equality relation $\{(a,a) \mid a \in A\}$, which is denoted $=_A$. The original result, stated in a slightly different language, does not cover the case of local absorption. However, a typographical modification of a proof mentioned in [19] shows that the theorem holds. For completeness sake, we present this proof in Appendix B.

▶ **Theorem 30** ([19]). *Let* **A** *be an **idempotent** algebra and* $\mathbf{R} \le \mathbf{A}^2$ *be nonempty and symmetric. If* R *locally absorbs* $=_A$*, then* R *contains a loop.*

In order to apply this theorem in the case of sensitive instances, we need to generalize it. In the following two theorems we will gradually relax the requirement that R is symmetric. In the first step, we substitute it with a condition requiring a closed, directed walk in the graph (i.e., a sequence of possibly repeating vertices, with consecutive vertices connected by forward edges and the first and last vertex identical). Recall that R^{-1} is the inverse relation to R and let us denote by $R^{\circ l}$ the l-fold relational composition of R with itself.

▶ **Theorem 31.** *Let* **A** *be an **idempotent** algebra and* $\mathbf{R} \le \mathbf{A}^2$ *contain a directed closed walk. If* R *locally absorbs* $=_A$*, then* R *contains a loop.*

Proof. Let n denote the arity of the absorbing operations. The proof is by induction on $l \ge 0$, where l is a number such that there exists a directed closed walk from a_1 to a_1 of length 2^l.

We start by verifying that such an l exists. Take a directed walk $(a_1, \ldots, a_{k-1}, a_k = a_1)$ in R. We may assume that its length k is at least n, since we can, if necessary, traverse the walk multiple times. An application of Lemma 29 to the relations $R, =_A$ and tuples $(a_1, \ldots, a_n), (a_1, \ldots, a_n)$ gives us a directed walk from a_1 to a_n of length n. Appending this walk with the walk $(a_n, a_{n+1}, \ldots, a_k = a_1)$ yields a directed walk from a_1 to a_1 of length $k+1$. In this way, we can get a directed walk from a_1 to a_1 of any length greater than k.

Now we return to the inductive proof and start with the base of induction for $l = 0$ or $l = 1$. If $l = 0$, then we have found a loop. If $l = 1$ we have a closed walk of length 2, that is, a pair (a, b) which belongs to both R and R^{-1}. We set $R' = R \cap R^{-1}$ and observe that R' is nonempty and symmetric, and it is not hard to verify that R' locally absorbs $=_A$. Olšák's loop lemma, in the form of Theorem 30, gives us a loop in R.

Finally, we make the induction step from $l-1$ to l. Take a closed walk $(a_1, a_2, \ldots)$ of length 2^l and consider $R' = R^{\circ 2}$. Observe that R' contains a directed closed walk of length 2^{l-1} (namely $(a_1, a_3, \ldots)$), and that R' locally absorbs $=_A$ (by Lemma 28), so, by the

inductive hypothesis, R' has a loop. In other words, R has a directed closed walk of length 2 and we are done by the case $l = 1$. ◄

Note that we cannot further relax the assumption on the graph by requiring that, for example, it has an infinite directed walk. Indeed the natural order of the rationals (taken for R) locally 2-absorbs the equality relation by the binary arithmetic mean operation $(a + b)/2$ (i.e., all the absorbing evaluations are realized by a single operation). The same relation locally 4-absorbs equality with the near unanimity operation $n(x, y, z, w)$ which, when applied to $a \leq b \leq c \leq d$, in any order, returns $(b + c)/2$.

Nevertheless, we can strengthen the algebraic assumption and still provide a loop; the following theorem is one of the key components in the proof sketch provided in Section 5 and the full proof found in Appendix C (albeit applied there with $l = 1$).

▶ **Theorem 32.** *Let* $\mathbf{A}$ *be an* ***idempotent*** *algebra and* $\mathbf{R} \leq \mathbf{A}^2$ *contain a directed walk of length* $n - 1$*. If* R *locally* n*-absorbs* $=_A$ *and* $R^{\circ l}$ *locally* n*-absorbs* R^{-1} *for some* $l \in \mathbb{N}$ *then* R *contains a loop.*

Proof. By applying Lemma 29 similarly as in the proof of Theorem 31, we can get, from a directed walk of length $n - 1$, a directed walk $(a_1, a_2, \ldots)$ of an arbitrary length. Moreover, by the same reasoning, for each i and j with $j \geq i + n - 1$, there is a directed walk from a_i to a_j of any length greater than or equal to $j - i$.

Consider the relations $R' = R^{\circ l n^2}$ and $S = (R^{-1})^{\circ n^2}$, and tuples

$$\mathbf{c} = (c_1, \ldots, c_n) := (a_{n^2}, a_{(n+1)n}, \ldots a_{(2n-1)n}), \text{ and}$$
$$\mathbf{d} = (d_1, \ldots, d_n) := (a_n, a_{2n} \ldots, a_{n^2})$$

By the previous paragraph and the definitions, both $\mathbf{c}$ and $\mathbf{d}$ are directed walks in R', and $(c_i, d_i) \in S$ for each i. Moreover, since $R^{\circ l}$ locally n-absorbs R^{-1}, Lemma 28 implies that R' locally absorbs S. We can thus apply Lemma 29 to the relations R', S and the tuples $\mathbf{c}, \mathbf{d}$ and obtain a directed walk from $c_1 = a_{n^2}$ to $d_{n-1} = a_{n^2}$ in R'. This closed walk in turn gives a closed directed walk in R and we are in a position to finish the proof by applying Theorem 31. ◄
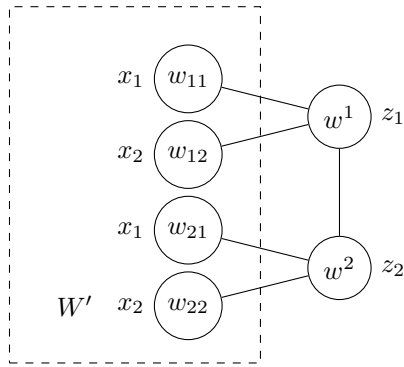
5   Consistent instances are sensitive (sketch of a proof)

In this section we present the main ideas that are used to prove the "only if" direction in Theorem 17 and "1 implies 2" in Theorem 21. These ideas are shown in a very simplified situation, in particular, only the case that $k = 2$ and $\mathbf{A}$ is finite is considered. In the end of this section we briefly discuss the necessary adjustments in the general situation. A complete proof is given in Appendix C.
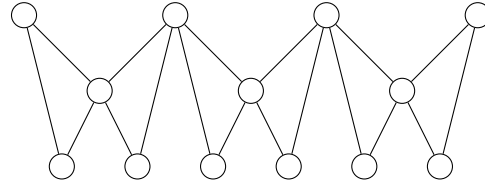
Consider a finite idempotent algebra $\mathbf{A}$ with a 4-ary near unanimity term operation and a $(2, 3)$-instance $\mathcal{I} = (V, \mathcal{C})$ over $\mathbf{A}$. Each pair $\{x, y\}$ of variables is constrained by a unique constraint $((x, y), R_{xy})$ or $((y, x), R_{yx})$. For convenience we also define $R_{yx} = R_{yx}^{-1}$ (or $R_{xy} = R_{yx}^{-1}$ in the latter case) and R_{xx} to be the equality relation on A. Our aim is to show that every pair in every constraint relation extends to a solution. The overall structure of the proof is by induction on the number of variables of $\mathcal{I}$.

We fix a pair of variables $\{x_1, x_2\}$ and a pair $(a_1, a_2) \in R_{x_1 x_2}$ that we want to extend. The strategy is to consider the instance $\mathcal{J}$ obtained by removing x_1 and x_2 from the set of variables and shrinking the constraint relations R_{uv} to R'_{uv} so that only the pairs consistent with the fixed choice remain, that is,

$$R'_{uv} = \{(b, c) \in R_{uv} \mid (a_1, b) \in R_{x_1 u}, (a_2, b) \in R_{x_2 u}, (a_1, c) \in R_{x_1 v}, (a_2, c) \in R_{x_2 v}\}.$$

Figure 2 Pattern $\mathbb{P}$ in Lemma 35.

Figure 3 Path of three bow ties.

We will show that $\mathcal{J}$ contains a nonempty $(2,3)$-subinstance, that is, an instance whose constraint relations are nonempty subsets of the original ones. The induction hypothesis then gives us a solution to $\mathcal{J}$ which, in turn, yields a solution to $\mathcal{I}$ that extends the fixed choice.

Having a nonempty $(2,3)$-subinstance can be characterized by the solvability of certain relaxed instances. The following concepts will be useful for working with relaxations of $\mathcal{I}$ and $\mathcal{J}$.

▶ **Definition 33.** *A* pattern *is a triple* $\mathbb{P} = (W; \mathcal{F}, l)$, *where* $(W; \mathcal{F})$ *is an undirected graph, and* l *is a mapping* $l : W \to V$. *The variable* $l(i)$ *is referred to as the* label *of* i.

A realization *(*strong realization*, respectively) of* $\mathbb{P}$ *is a mapping* $\alpha : W \to A$, *which satisfies* *every edge* $\{w_1, w_2\} \in \mathcal{F}$, *that is,* $(\alpha(w_1), \alpha(w_2)) \in R_{l(w_1),l(w_2)}$ *(*$(\alpha(w_1), \alpha(w_2)) \in R'_{l(w_1),l(w_2)}$, *respectively). (Strong realization only makes sense if* $l(W) \subseteq V \setminus \{x_1, x_2\}$.*)*

*A pattern is (*strongly*) realizable* *if it has a (strong) realization.*

The most important patterns for our purposes are *2-trees*, these are patterns obtained from the empty pattern by gradually adding triangles (patterns whose underlying graph is the complete graph on 3 vertices) and merging them along a vertex or an edge to the already constructed pattern. Their significance stems from the following well known fact.

▶ **Lemma 34.** *An instance (over a finite domain) contains a nonempty (2,3)-subinstance if and only if every 2-tree is realizable in it.*

The "only if" direction of the lemma applied to the instance $\mathcal{I}$ implies that every 2-tree is realizable. The "if" direction applied to the instance $\mathcal{J}$ tells us that our aim boils down to proving that every 2-tree is strongly realizable. This is achieved by an induction on a suitable measure of complexity of the tree using several constructions. We will not go into full technical details here, we rather present several lemmata whose proofs contain essentially all the ideas that are necessary for the complete proof.

▶ **Lemma 35.** *Every edge (i.e., a pattern whose underlying graph is a single edge) is strongly realizable.*

Proof sketch. Let $\mathbb{Q}$ be the pattern formed by an undirected edge with vertices w^1 and w^2 labeled z_1 and z_2, respectively. Let $\mathbb{P}$ be the pattern obtained from $\mathbb{Q}$ by adding a set of four fresh vertices $W' = \{w_{11}, w_{12}, w_{21}, w_{22}\}$ labeled x_1, x_2, x_1, x_2, respectively, and adding the edges $\{w^i, w_{i1}\}$ and $\{w^i, w_{i2}\}$ for $i = 1, 2$, see Figure 2. Observe that the restriction of a

realization β of $\mathbb{P}$, such that $\beta(w_{ij}) = a_j$ for each $i, j \in \{1, 2\}$, to the set $\{w^1, w^2\}$ is a strong realization of $\mathbb{Q}$.

We consider the set T of restrictions of realizations of $\mathbb{P}$ to the set W'. Since constraint relations are subuniverses of $\mathbf{A}^2$, it follows that T is a subuniverse of $\mathbf{A}^4$.

$$T = \{(\beta(w_{11}), \beta(w_{12}), \beta(w_{21}), \beta(w_{22})) \mid \beta \text{ realizes } \mathbb{P}\} \le \mathbf{A}^4$$

We need to prove that the tuple $\mathbf{a} = (a_1, a_2, a_1, a_2)$ is in T. By the Baker-Pixley theorem, Theorem 14, it is enough to show that for any 3-element set of coordinates, the relation T contains a tuple that agrees with $\mathbf{a}$ on this set. This is now our aim.

For simplicity, consider the set of the first three coordinates. We will build a realization β of $\mathbb{P}$ in three steps. After each step, β will satisfy all the edges where it is defined. First, since $(a_1, a_2) \in R_{x_1 x_2}$ and $\mathcal{I}$ is a (2,3)-instance, we can find $b_1 \in A$ such that $(a_1, b_1) \in R_{x_1 z_1}$ and $(a_2, b_1) \in R_{x_2 z_1}$, and we set $\beta(w_{11}) = a_1$, $\beta(w_{12}) = a_2$, and $\beta(w^1) = b_1$. Second, we find $b_2 \in A$ such that $(a_1, b_2) \in R_{x_1 z_2}$ and $(b_1, b_2) \in R_{z_1 z_2}$ (here we use $(a_1, b_1) \in R_{x_1 z_1}$ and that $\mathcal{I}$ is a (2,3)-instance), and set $\beta(w_{21}) = a_1$, $\beta(w^2) = b_2$. Third, using $(a_1, b_2) \in R_{x_1 z_2}$ we find a'_2 such that $(b_2, a'_2) \in R_{z_2 x_2}$ and set $\beta(w_{22}) = a'_2$. By construction, β is a realization of $\mathbb{P}$ and $(\beta(w_{11}), \beta(w_{12}), \beta(w_{21})) = (a_1, a_2, a_1)$, so our aim has been achieved. ◄

Using Lemma 35, one can go a step further and prove that every pattern built on a graph which is a triangle is strongly realizable. We are not going to prove this fact here.

▶ **Lemma 36.** *Every bow tie (a pattern whose underlying graph is formed by two triangles with a single common vertex) is strongly realizable.*

Proof sketch. Let $\mathbb{W}'_1$ and $\mathbb{W}'_2$ be two triangles (viewed as undirected graphs) with a single common vertex w. Let $\mathbb{Q}'$ be any pattern over $W'_1 \cup W'_2$ with labelling l' sending $W'_1 \cup W'_2$ to $V \setminus \{x_1, x_2\}$. Similarly as in the proof of Lemma 35 we form a pattern $\mathbb{Q}$ by adding to $\mathbb{Q}'$ ten additional vertices (five of them labeled x_1, the other five x_2) and edges so that the restriction of a realization α of $\mathbb{Q}$ to the set $W'_1 \cup W'_2$ is a strong realization of $\mathbb{Q}'$ whenever the additional vertices have proper values (that is, value a_i for vertices labeled x_i).

We will gradually construct a realization α of $\mathbb{Q}$, which sends all the vertices labeled by x_1 to a_1, and all the vertices labeled by x_2 and adjacent to a vertex in W'_1 to a_2. First use the discussion after Lemma 35 to find a strong realization of $\mathbb{Q}'$ restricted to W'_1. This defines α on W'_1 and its adjacent vertices labeled by x_1 and x_2.

Next, we want to use Lemma 35 for assigning values to the two remaining vertices of W'_2. However, in order to accomplish that, we need to shift the perspective: the role of x_1 is played by x_1, but the role of x_2 is played by $l'(w)$; and the role of (a_1, a_2) is played by $(a_1, \alpha(w))$. In this new context, we use Lemma 35 to find a strong realization of the edge-pattern formed by the two remaining vertices of W'_2 (with a proper restriction of l'). This defines α on all the vertices of $\mathbb{Q}$, except for the two vertices adjacent to $W'_2 \setminus \{w\}$ and labeled by x_2. Finally, similarly as in the third step in the proof of Lemma 35, we define α on the remaining two vertices (labeled x_2) to get a sought after realization of $\mathbb{Q}$.

Now α assigns proper values (a_1 or a_2) to all additional vertices, except those two coming from the non-central vertices of W'_2 and labeled by x_2. We apply the 4-ary near unanimity term operation to the realization α and its 3 variants obtained by exchanging the roles of W'_1 and W'_2 and x_1 and x_2. The result of this application is a realization of $\mathbb{Q}$ which defines a strong realization of $\mathbb{Q}'$. ◄

In the same way it is possible to prove strong realizability of further patterns, such as those in the following corollary.

▶ **Corollary 37.** *Every "path of 3 bow ties" (i.e., a pattern whose underlying graph is as in Figure 3) is strongly realizable.*

The application of the loop lemma is illustrated by the final lemma in this section.

▶ **Lemma 38.** *Every diamond (i.e., a pattern whose underlying graph is formed by two triangles with a single common edge) is strongly realizable.*

Proof sketch. The idea is to merge two vertices in a bow tie using the loop lemma. Let $\mathbb{Q}'$ be a pattern over a graph which is a bow tie on two triangles W_1' and W_2' (just like in the proof of Lemma 36). Let $w_1 \in W_1' \setminus W_2'$ and $w_2 \in W_2' \setminus W_1'$ be such that $l(w_1) = l(w_2)$.

Let $\mathbb{Q}$ be obtained from $\mathbb{Q}'$ exactly as in the proof of Lemma 36 and notice that a proper realization α of $\mathbb{Q}$ with $\alpha(w_1) = \alpha(w_2)$ gives us a strong realization of a diamond. Let $\mathbb{Q}^3$ be the pattern obtained by taking the disjoint union of 3 copies of $\mathbb{Q}$ and identifying the vertex w_2 in the i-th copy with the vertex w_1 in the $(i+1)$-first copy, for each $i \in \{1, 2\}$ (Figure 3 shows $\mathbb{Q}^3$ without the additional vertices).

Denote by T the set of all the realizations β of $\mathbb{Q}$ and denote by $S \subseteq T$ the set of those $\beta \in T$ that are proper. By a straightforward argument, both T and S are subuniverses of $\prod_{w \in Q} \mathbf{A}$. Using the near unanimity term operation of arity 4, S clearly 4-absorbs T.

The plan is to apply Theorem 32 to the binary relation $\mathrm{proj}_{w_1, w_2} S \subseteq A \times A$. As noted above, a loop in this relation gives us the desired strong realization of a diamond, so it only remains to verify the assumptions of Theorem 32. By Corollary 37, the pattern $\mathbb{Q}^3$ has a proper realization. The images of copies of vertices w_1 and w_2 in such a realization yield a directed walk in $\mathrm{proj}_{w_1, w_2}(S)$ of length 3. Next, since S 4-absorbs T, then $\mathrm{proj}_{w_1, w_2}(S)$ 4-absorbs $\mathrm{proj}_{w_1, w_2}(T)$, so it is enough to verify that the latter relation contains $=_A$ and $\mathrm{proj}_{w_1, w_2}(S)^{-1}$. We only look at the latter property. Consider any $(b_1, b_2) \in \mathrm{proj}_{w_1, w_2}(S)^{-1}$. By the definition of S, the pattern $\mathbb{Q}$ has a realization α such that $\alpha(w_1) = b_2$ and $\alpha(w_2) = b_1$. We flip the values $\alpha(w_1)$ and $\alpha(w_2)$, restrict α to $\{w_1, w_2\}$ together with the middle vertex of the bow tie, and then extend this assignment to a realization of $\mathbb{Q}$, giving us $(b_1, b_2) \in \mathrm{proj}_{w_1, w_2}(T)$. ◀

There are two major adjustments needed for the general case. First, the "if" direction of Lemma 34 (and its analogue for a general k) is no longer true over infinite domains. This is resolved by working directly with the realizability of k-trees and proving a more general claim by induction: instead of "a $(k, k+1)$-instance is sensitive" we prove, roughly, that any evaluation, which extends to a sufficiently deep k-tree, extends to a solution. Second, for higher values of k than 2 we do not prove strong realizability in one step as in, e.g., Lemma 35, but rather go through a sequence of intermediate steps between realizability and strong realizability.

6   Conclusion

We have characterized varieties that have sensitive $(k, k+1)$-instances of the CSP as those that possess a near unanimity term of arity $k+2$. From the computational perspective, the following corollary is perhaps the most interesting consequence of our results.

▶ **Corollary 39.** *Let $\mathbb{A}$ be a finite CSP template whose relations all have arity at most k and which has a near unanimity polymorphism of arity $k+2$. Then every instance of the CSP over $\mathbb{A}$, after enforcing $(k, k+1)$-consistency, is sensitive.*

Therefore not only is the $(k, k+1)$-consistency algorithm sufficient to detect global inconsistency, we also additionally get the sensitivity property. Let us compare this result to

some previous results as follows. Consider a template $\mathbb{A}$ that, for simplicity, has only unary and binary relations and that has a near unanimity polymorphism of arity $k + 2 \geq 4$. Then any instance of the CSP over $\mathbb{A}$ satisfies the following.

1. After enforcing $(2,3)$-consistency, if no contradiction is detected, then the instance has a solution [4] (this is the bounded width property).
2. After enforcing $(k, k+1)$-consistency, every partial solution on k variables extends to a solution (this is the sensitivity property).
3. After enforcing $(k+1, k+2)$-consistency, every partial solution extends to a solution [13] (this is the bounded strict width property).

For $k + 2 > 4$ there is a gap between the first and the second item. Are there natural conditions that can be placed there?

The properties of a template $\mathbb{A}$ from the first and the third item (holding for every instance) can be characterized by the existence of certain polymorphisms: a near unanimity polymorphism of arity $k + 2$ for the third item [13] and weak near unanimity polymorphisms of all arities greater than 2 for the first item [5, 11, 18]. This paper does not give such a direct characterization for the second item (essentially, since Theorem 21 involves a square). Is there any? Moreover, there are characterizations for natural extensions of the first and the third to relational structures with higher arity relations [13, 3]. This remains open for the second item as well.

In parallel with the flurry of activity around the CSP over finite templates, there has been much work done on the CSP over infinite ω-categorical templates [9, 20]. These templates cover a much larger class of computational problems but, on the other hand, share some pleasant properties with the finite ones. In particular, the $(k, k+1)$-consistency of an instance can still be enforced in polynomial time. Corollary 39 can be extended to this setting as follows.

▶ **Corollary 40.** *Let $\mathbb{A}$ be an ω-categorical CSP template whose relations all have arity at most k and which has local idempotent near unanimity polymorphisms of arity $k + 2$. Then every instance of the CSP over $\mathbb{A}$, after enforcing the $(k, k+1)$-consistency, is sensitive.*

Bounded strict width k of an ω-categorical template was characterized in [10] by the existence of a *quasi-near unanimity* polymorphism n of arity $k + 1$, i.e.,

$$n(y, x, \ldots, x) \approx n(x, y, \ldots, x) \approx \cdots \approx n(x, x, \ldots, y) \approx n(x, x, \ldots, x),$$

which is, additionally, *oligopotent*, i.e., the unary operation $x \mapsto n(x, x, \ldots, x)$ is equal to an automorphism on every finite set. This result extends the characterization of Feder and Vardi since an oligopotent quasi-near unanimity polymorphism generates a near unanimity polymorphism as soon as the domain is finite. On an infinite domain, however, oligopotent quasi-near unanimity polymorphisms generate local near unanimity polymorphisms which, unfortunately, do not need to be idempotent on the whole domain. Our results thus fall short of proving the following natural generalization of Corollary 39 to the infinite.

▶ **Conjecture 41.** *Let $\mathbb{A}$ be an ω-categorical CSP template whose relations all have arity at most k and which has an oligopotent quasi-near unanimity polymorphism of arity $k + 2$. Then every instance of the CSP over $\mathbb{A}$, after enforcing $(k, k+1)$-consistency, is sensitive.*

To confirm the conjecture, a new approach, that does not use a loop lemma, will be needed since there are examples of ω-categorical structures having oligopotent quasi-near unanimity polymorphisms for which the counterpart to Theorem 30 does not hold. Indeed, one such an example is the infinite clique.

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A   Proofs of Theorems 16 and 19

The first result is due to Bergman [8], we provide a short proof for the convenience of the reader.

▶ **Theorem 16.** *Let $k > 1$ and $\mathcal{V}$ be a variety. The following are equivalent:*
1. *$\mathcal{V}$ has a $(k+1)$-ary near unanimity term;*
2. *any partial solution of a $(k, k+1)$-instance over $\mathcal{V}$ extends to a solution.*

Proof of Theorem 16. A straightforward modification of the "if" direction of the proof of Theorem 17, using Proposition 23 in place of Proposition 24 shows that the second condition implies the existence of a $(k+1)$-ary near unanimity term (also see [8, Lemma 11]). For the converse, suppose that $\mathcal{V}$ has a $(k+1)$-ary near unanimity term $n(x_1, \ldots, x_{k+1})$ and let $\mathcal{I} = (V, \mathcal{C})$ be a $(k, k+1)$-instance of $\mathsf{CSP}(\mathcal{V})$.

Let $n = |V|$. We will show by induction on $r < n$ that if $W \subseteq V$ with $|W| = r$ then any solution of $\mathcal{I}_{|W}$ can be extended to a solution of $\mathcal{I}_{|W \cup \{v\}}$ for any $v \in V \setminus W$. From this, the implication will follow. By the assumption that $\mathcal{I}$ is a $(k, k+1)$-instance it follows that this property holds for $r = k$. So, assume that $k < r < n$ and suppose that $W \subseteq V$ with $|W| = r$. Let $v \in V \setminus W$ and let f be a solution of $\mathcal{I}_{|W}$.

Fix some listing of the elements of W, say $W = \{v_1, v_2, \ldots, v_r\}$ and for $1 \leq i \leq r$ let $W_i = (W \setminus \{v_i\}) \cup \{v\}$. By induction, there is a solution f_i of $\mathcal{I}_{|W_i}$ that extends the restriction of f to $W \setminus \{v_i\}$, for $1 \leq i \leq k+1$. We claim that the extension of f to $W \cup \{v\}$ by setting $f(v) = n(f_1(v), f_2(v), \ldots, f_{k+1}(v))$ produces a solution of $\mathcal{I}_{|W \cup \{v\}}$.

We need to show that if $U \subseteq W \cup \{v\}$ with $|U| = k$ then $(f(u) \mid u \in U)$ satisfies the unique constraint (U, R) of $\mathcal{I}$ with scope U. When $U \subseteq W$, this is immediate, so assume that $v \in U$. For $1 \leq i \leq k+1$, let g_i be the restriction of f_i to U, if $v_i \notin U$ and otherwise let g_i be some partial solution of $\mathcal{I}_{|U}$ that extends the restriction of f_i to $U \setminus \{v_i\}$. Since each g_i satisfies the constraint (U, R) then so does $n(g_1, g_2, \ldots, g_{k+1})$. Using that n is a near unanimity term it can be shown that this element is equal to $f_{|U}$, as required. ◀

The next theorem is a variation of the Baker-Pixley [1] result for idempotent, not necessarily finite, algebras.

▶ **Theorem 19.** *Let $\mathbf{A}$ be an idempotent algebra and $k > 1$. The following are equivalent:*
1. *$\mathbf{A}$ has local near unanimity term operations of arity $k + 1$;*
2. *for every $r > k$, every subalgebra of $\mathbf{A}^r$ is uniquely determined by its projections onto all k-element subsets of coordinates;*
3. *every subalgebra of $\mathbf{A}^{k+1}$ is uniquely determined by its projections onto all k-element subsets of coordinates.*

Proof of Theorem 19. To show that Condition 1 implies Condition 2, suppose that $\mathbf{A}$ has local near unanimity term operations of arity $k+1$ and let $\mathbf{R}$ be a subalgebra of $\mathbf{A}^r$ for some $r > k$. Let $\mathbf{a} = (a_1, \ldots, a_r) \in A^r$ be a tuple such that for every subset I of $[r]$ of size k, there is some element $\mathbf{b} \in R$ with $\mathrm{proj}_I(\mathbf{a}) = \mathrm{proj}_I(\mathbf{b})$. We will show by induction on $n \geq k$ that if $n \leq r$ then for every subset J of $[r]$ of size n there is some $\mathbf{b} \in R$ with $\mathrm{proj}_J(\mathbf{a}) = \mathrm{proj}_J(\mathbf{b})$. With $n = r$ we conclude that $\mathbf{a} \in R$, as required.

By assumption, this property holds when $n = k$. Suppose that it has been established for some n with $k \leq n < r$ and let J be a subset of $[r]$ of size $n + 1$. By symmetry it suffices to consider the case when $J = \{1, 2, \ldots, n+1\}$. For each i, with $1 \leq i \leq k+1$, let $\mathbf{b}_i \in R$ be such that $\mathbf{a}$ and $\mathbf{b}_i$ agree on the set $J \setminus \{i\}$. Let $n(x_1, \ldots, x_{k+1})$ be a $(k+1)$-ary local near unanimity term operation of $\mathbf{A}$ for the subset of A consisting of all of the

components of the tuples $\mathbf{b}_i$, for $1 \leq i \leq k+1$. A straightforward calculation shows that $\mathbf{b} = n(\mathbf{b}_1, \ldots, \mathbf{b}_{k+1}) \in R$ has the desired property.

Clearly Condtion 2 implies Condition 3. For the remaining implication, we use Corollary 2.7 from [14] that shows that if $\mathbf{A}$ is finite (and idempotent) then it will have a $(k+1)$-ary near unanimity term operation if and only if for every a_i, $b_i \in A$, for $1 \leq i \leq k+1$, there is some term operation t of $\mathbf{A}$ such that

$$t(b_1, a_1, a_1, \ldots, a_1) = a_1$$
$$t(a_2, b_2, a_2, \ldots, a_2) = a_2$$
$$\vdots$$
$$t(a_{k+1}, a_{k+1}, a_{k+1}, \ldots, b_{k+1}) = a_{k+1}.$$

It can be seen from the proof of this result that if $\mathbf{A}$ is not assumed to be finite, then one can conclude that it has local near unanimity term operations of arity $k+1$ if and only if this condition holds for all a_i and b_i.

This local term condition can be translated into a statement about subalgebras of $\mathbf{A}^{k+1}$, namely that for every a_i, $b_i \in A$, for $1 \leq i \leq k+1$, the $(k+1)$-tuple $\mathbf{a} = (a_1, \ldots, a_{k+1})$ belongs to the subalgebra $\mathbf{R}$ of $\mathbf{A}^{k+1}$ generated by the set of $k+1$ tuples

$$\{(b_1, a_2, a_3, \ldots, a_{k+1}), (a_1, b_2, a_3, \ldots, a_{k+1}), \ldots, (a_1, a_2, a_3, \ldots, b_{k+1})\}.$$

Our assumption on $\mathbf{A}$ guarantees that $\mathbf{a}$ belongs to R since any projection of R onto k coordinates will contain the corresponding projection of $\mathbf{a}$. Thus $\mathbf{A}$ will have local near unanimity term operations of arity $k+1$. ◄

B   Proof of Theorem 30

In this section we present a proof of Theorem 30. The proof is a trivial adaptation of reasoning attributed to Ralph McKenzie in [19].

▶ **Theorem 30.** *Let $\mathbf{A}$ be an idempotent algebra and $\mathbf{R} \leq \mathbf{A}^2$ be nonempty and symmetric. If R locally absorbs $=_A$, then R contains a loop.*
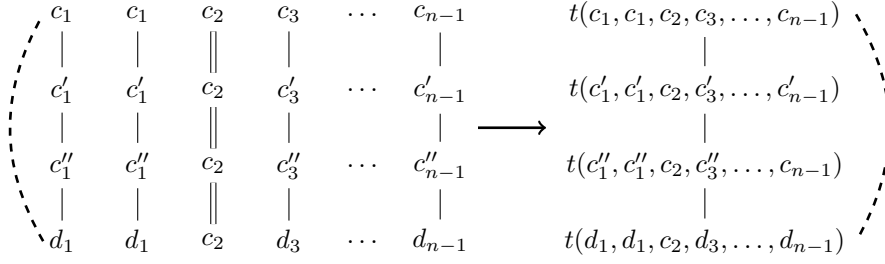
The remaining part of this section is devoted to a proof of Theorem 30 by the way of contradiction.

Let n denote the arity of the absorbing operations. We choose a counterexample to the theorem minimal with respect to n. Then, we fix an algebra $\mathbf{A}$ and will call an $\mathbf{R} \leq \mathbf{A}^2$ a *counterexample candidate* if it is non-empty, symmetric, locally n-absorbs $=_A$ and has no loop.

▷ Claim 42.   Every counterexample candidate has a closed walk of odd length.

Proof. Since R is nonempty and symmetric we have $(a, b), (b, a) \in R$. Apply Lemma 29 to the walk $(a, b, a, b, \ldots, a/b)$ of length $n-1$ (i.e., n vertices, $n-1$ steps) taken twice (where the last element is either a or b depending on the parity of n). The lemma provides a directed walk of length n connecting the first and last elements. Since R is symmetric all the edges are undirected and we obtained a closed walk of odd length. ◄

▷ Claim 43.   There exists a counterexample candidate containing a 3-element clique.

$$
\begin{pmatrix}
c_1 & c_1 & c_2 & c_3 & \cdots & c_{n-1} & t(c_1, c_1, c_2, c_3, \ldots, c_{n-1}) \\
| & | & \| & | & & | & | \\
c_1' & c_1' & c_2 & c_3' & \cdots & c_{n-1}' & t(c_1', c_1', c_2, c_3', \ldots, c_{n-1}') \\
| & | & \| & | & & | & | \\
c_1'' & c_1'' & c_2 & c_3'' & \cdots & c_{n-1}'' & t(c_1'', c_1'', c_2, c_3'', \ldots, c_{n-1}) \\
| & | & \| & | & & | & | \\
d_1 & d_1 & c_2 & d_3 & \cdots & d_{n-1} & t(d_1, d_1, c_2, d_3, \ldots, d_{n-1})
\end{pmatrix}
$$

◾ Figure 4 Solid lines are are S-related and dashed lines are T-related.

Proof. Take a counterexample candidate R; it has an odd cycle, and if it has a triangle we are done. Thus the length of a shortest odd cycle is greater than 3. In this case, however $R \circ R \circ R$ is a counterexample candidate (we use Lemma 28 to provide local absorption) with shorter odd cycle. We proceed this way and, in the end, find a counterexample candidate with a 3-cycle (which is a 3-clique). ◀

▷ Claim 44. No counterexample candidate contains an n-element clique.

Proof. Suppose $(a_1, \ldots, a_n)$ is such a clique. We can choose, using the definition of local absorption, t such that $(t(a_1, \ldots, a_n), t(a_i, \ldots, a_i)) \in R$ for all i. We use this fact, and the fact that $\mathbf{R} \leq \mathbf{A}^2$, to conclude that

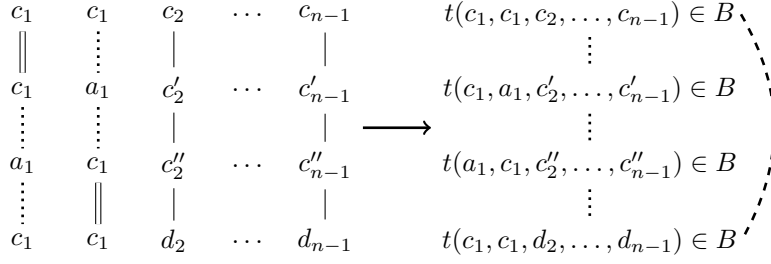$$\big(t(t(a_1, \ldots, a_n), \ldots, t(a_1, \ldots, a_n)), t(a_1, \ldots, a_n)\big) \in R,$$

but, by the idempotency of t, the two elements are equal and we have obtained a loop — a contradiction. ◀

In order to finish the proof we fix R to be a counterexample candidate with a 3-element clique and let $a_1, \ldots, a_m$ be distinct, forming a maximal clique in R (such a clique exists by the last claim). Let B be the subset of A containing vertices with edges to each of $a_1, \ldots, a_{m-2}$. Note that B is a subuniverse (since $\mathbf{A}$ is idempotent) and $S = B^2 \cap R$ is nonempty as $(a_{m-1}, a_m), (a_m, a_{m-1}) \in S$.

Note, that $\mathbf{S} \leq \mathbf{B}^2$ is symmetric, nonempty, has no 3-clique and it locally n-absorbs $=_B$. We obtain a contradiction by showing that $\mathbf{T} = \mathbf{S} \circ \mathbf{S} \circ \mathbf{S}$ locally $n-1$ absorbs $=_B$. The graph T is non-empty, symmetric, has no loop and $\mathbf{T} \leq \mathbf{B}^2$. We will fix a one-$=_B$-in-$T$ tuple, and construct a finite set of one-$=_A$-in-R tuples such that if $t(x_1, \ldots, x_n)$ is an operation of $\mathbf{A}$ producing elements of R on the tuples from the last set then $t(x_1, x_1, x_2, x_3, \ldots, x_{n-1})$ produces an element of T on the original tuple. The theorem we are working to prove clearly follows from this fact.

Let $(c_1, d_1), (c_2, d_2), \ldots, (c_{n-1}, d_{n-1})$ be a one-$=_B$-in-T tuple. We consider two cases: in case one $c_i = d_i$ for some $i > 1$ and in case two $c_1 = d_1$. In case one (see Figure 4), we assume, wlog that $i = 2$, and find for all $j \neq 2$ elements c_j', c_j'' such that $(c_j, c_j'), (c_j', c_j''), (c_j'', d_j) \in S$. It suffices to take care of the three following one-$=_A$-in-R evaluations:

$(c_1, c_1'), (c_1, c_1'), (c_2, c_2), (c_3, c_3'), \ldots, (c_{n-1}, c_{n-1}'),$
$(c_1', c_1''), (c_1', c_1''), (c_2, c_2), (c_3', c_3''), \ldots, (c_{n-1}', c_{n-1}'')$ and
$(c_1'', d_1), (c_1'', d_1), (c_2, c_2), (c_3'', d_3), \ldots, (c_{n-1}'', d_{n-1}).$

$$
\begin{array}{cccccc}
c_1 & c_1 & c_2 & \cdots & c_{n-1} & t(c_1, c_1, c_2, \ldots, c_{n-1}) \in B \\
\| & \vdots & | & & | & \vdots \\
c_1 & a_1 & c_2' & \cdots & c_{n-1}' & t(c_1, a_1, c_2', \ldots, c_{n-1}') \in B \\
\vdots & \vdots & | & \longrightarrow & | & \vdots \\
a_1 & c_1 & c_2'' & \cdots & c_{n-1}'' & t(a_1, c_1, c_2'', \ldots, c_{n-1}'') \in B \\
\vdots & \| & | & & | & \vdots \\
c_1 & c_1 & d_2 & \cdots & d_{n-1} & t(c_1, c_1, d_2, \ldots, d_{n-1}) \in B
\end{array}
$$

Figure 5 Solid lines are are S-related, dashed lines are T-related, and dotted lines are R-related.

In case two (see Figure 5) the situation is a bit more involved, we define c_i', c_i'' for all $i > 1$ but need 4 evaluations:

$(c_1, c_1), (c_1, a_1), (c_2, c_2') \ldots, (c_{n-1}, c_{n-1}'),$

$(c_1, a_1), (a_1, c_1), (c_2', c_2''), \ldots, (c_{n-1}', c_{n-1}''),$

$(a_1, c_1), (c_1, c_1), (c_2'', d_2), \ldots, (c_{n-1}'', d_{n-1})$ and two new ones

$(c_1, a_1), (a_1, a_1), (c_2', a_1), \ldots, (c_{n-1}', a_1),$

$(a_1, a_1), (c_1, a_1), (c_2'', a_1), \ldots, (c_{n-1}'', a_1).$

The list contains 5 evaluations, but the second one (included for simplicity) is in fact not a one-$=_A$-in-R evaluation, but a usual application of the term to elements of R. Any term, putting all these evaluations in R puts (by idempotency and the fact that all considered elements are adjacent to a_i if $1 < i < m - 1$) $t(c_1, a_1, c_2', \ldots, c_{n-1}'), t(a_1, c_1, c_2'', \ldots, c_{n-1}'') \in B$. These elements witness the path required to put the pair $(t(c_1, c_1, c_2, c_3, \ldots, c_{n-1}), t(c_1, c_1, d_2, d_3 \ldots, d_{n-1}))$ in T.

C   Consistent instances are sensitive

In this section we provide a proof for the "only if" direction in Theorem 17 and "1 implies 2" in Theorem 21. We will proceed with the two proofs in parallel; in one case we fix an algebra $\mathbf{A}$ and in the other a variety $\mathcal{V}$. We will assume, without loss of generality, that the only operation symbol of $\mathcal{V}$ is $(k + 2)$-ary and is a near unanimity operation for all members of $\mathcal{V}$. So, all members of $\mathcal{V}$ are idempotent. Formally, in the case of Theorem 21, we should be working with instances over $\mathbf{A}^2$, but if $\mathbf{A}$ has local $(k + 2)$-ary near unanimity term operations, then so does $\mathbf{A}^2$ and so we can work directly with an algebra possessing local near unanimity term operations and denote it by $\mathbf{A}$. We will remark on the differences between these two cases only in the places where we apply near unanimity operations.

For the purpose of this section we modify the definition of an instance slightly: an instance is a triple $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$, where $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \le k\}$ and $R_S \le \prod_{x \in S} \mathbf{A}_x$. Note that the definition of a CSP instance is, formally, different than our standard definition: the variables involved in a constraint are a set and not a tuple. This minor modification will allow us to present the proofs more succinctly. In order for the interpretation of a constraint to be unique we assume, without loss of generality, that the algebras $\mathbf{A}_x$ are disjoint. When applying the results of this section in Theorem 17 we will set each $\mathbf{A}_x$ to be an isomorphic copy of $\mathbf{A}$, and in case of Theorem 21 we will choose isomorphic copies from the variety, so that their domains are disjoint.

The rough idea of the proof is to fix, in a $(k, k + 1)$-instance, a tuple from the relation constraining set of variables Y and consider the instance obtained by removing Y from the set of variables and shrinking the constraint relations so that only the tuples extending the fixed choice of values for the variables in Y remain. If we were able to show that the obtained instance contains a $(k, k + 1)$-subinstance, both theorems would then follow by induction on the number of variables of the instance. It is well known that for instances **with finite domains**, the latter property is equivalent to the solvability of certain relaxed instances, here called k-trees. Our strategy for the proof is, in fact, to prove the solvability of k-trees, by induction on a measure of complexity of k-trees. Unfortunately, for infinite domains, the solvability of k-trees is in general weaker than having a $(k, k+1)$-subinstance, and this brings several technical complications into our proof. In particular, we will be working with CSP instances, that **won't necessarily be $(k, k + 1)$-instances**, or even k-uniform.

The remaining parts of this section are organized as follows. In the first subsection we introduce concepts that are useful for working with instances and their solutions – patterns and realizations. The next subsection studies solvability with a fixed evaluation for k variables and provides two core technical claims for the inductive proof of the solvability of k-trees; the proof is then assembled in the third subsection and the missing parts of Theorems 17 and 21 are derived as a consequence.

Until Theorem 59 in the last section we fix

- an integer $k \geq 2$,
- a variety $\mathcal{V}$ with a $(k + 2)$-ary near unanimity term in case of Theorem 17 or an algebra **A** with local near unanimity term operations of arity $k + 2$ in case of Theorem 21;
- an instance $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$, where $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \leq k\}$ and $R_S \leq \prod_{x \in S} \mathbf{A}_x$, such that, for any $S' \subseteq S$ with $|S| \leq k$, the projection of R_S onto S' is contained in $R_{S'}$ (here either every $\mathbf{A}_x$ is in a variety $\mathcal{V}$ in case of Theorem 17, or $\mathbf{A}_x$ is an isomorphic copy of **A** in case of Theorem 21).

A $(k, k+1)$-instance can naturally be expanded to meet the condition in the last item by adding the constraints $(S', R_{S'})$ for $|S'| < k$, where $R_{S'}$ is defined as the projection of R_S onto S' for an arbitrary k-element superset S of S'. It is an easy exercise, and we leave it to the reader, to verify that this definition does not depend on the choice of S.

For a tuple of (not necessarily distinct) variables $x_1, \ldots, x_l$ with $l \leq k$ we denote $R_{x_1, \ldots, x_l} = \{(r_{x_1}, \ldots, r_{x_l}) \mid \mathbf{r} \in R_{\{x_1, \ldots, x_l\}}\} \leq \prod_{i=1}^{l} \mathbf{A}_{x_i}$. Finally, we set $A = \bigcup_{x \in V} A_x$.

C.1 Patterns

A pattern is a hypergraph whose vertices are labeled by variables and hyperedges indicate that constraints should be satisfied. It will be convenient to have the set of hyperedges closed under taking subsets.

▶ **Definition 45.** *A* pattern *is a triple* $\mathbb{P} = (P; \mathcal{F}, v)$, *where* P *is a set of* vertices, $\mathcal{F}$ *is a family of at most k-element subsets of P closed under taking subsets, and v is a mapping* $v : P \to V$. *Members of $\mathcal{F}$ are called* faces *and the variable $v(i)$ is referred to as the* label *of* i.

A realization *of $\mathbb{P}$ is a mapping $\alpha : P \to A$, which is* consistent *with v, that is, $\alpha(i) \in A_{v(i)}$ for every $i \in P$, and* satisfies *every face $\{f_1, \ldots, f_l\} \in \mathcal{F}$, that is, $(\alpha(f_1), \ldots, \alpha(f_l)) \in R_{v(f_1), \ldots, v(f_l)}$.*

For clarity, we will always call a mapping from a set of vertices to A (which is not necessarily a realization of a pattern) an *assignment* (denoted α, β, ...), a mapping from

a set of variables to A an *evaluation* (denoted ϕ, ρ, ...), and a mapping from a set of vertices to V a *labeling* (denoted v). We say that an assignment α *extends* an evaluation ϕ if $\alpha(p) = \phi(v(p))$ for any p in the domain of α such that $v(p)$ is in the domain of ϕ.

Since we assume that the A_x's are disjoint, any assignment uniquely determines a consistent labeling and it makes sense to say that an assignment satisfies a set of vertices F, provided $|F| \leq k$. Also note that, by the assumptions on $\mathcal{I}$, if an assignment α satisfies F, then it satisfies every subset of F. Finally, note that in the same situation $\alpha(i) = \alpha(i')$ whenever $v(i) = v(i')$.

A pattern $\mathbb{P}' = (P'; \mathcal{F}', v')$ is a *subpattern* of $\mathbb{P}$ if $P' \subseteq P$, $\mathcal{F}' \subseteq \mathcal{F}$, and v' is the restriction of v to P'. By a union of two patterns we mean the set-theoretical union of the vertex sets, face sets, and labelings. It can only be formed if there are no collisions among labels (that is, each vertex belonging to both patterns must have the same label in both patterns).

The richest patterns are the *complete patterns*, whose faces are all the subsets of the vertex set of size at most k. Note that a realization of a complete pattern with $l \leq k$ vertices is essentially the same as a tuple in the corresponding constraint relation. The most important patterns for our purposes are l-trees with $l \leq k$. These are, informally, patterns obtained from the empty pattern by gradually adding complete patterns with at most $l + 1$ vertices and merging them along a face to the already constructed pattern.

▶ **Definition 46.** *Let $l \leq k$ and let F be a set of labeled vertices of size at most l. The complete l-tree with base F of depth 1 is the complete pattern with vertex set F. The complete l-tree with base F of depth $d + 1$ is obtained from the complete l-tree $\mathbb{P}$ with base F of depth d by adding to $\mathbb{P}$, for every face E of $\mathbb{P}$ and every $(l + 1 - |E|)$-element set of variables U, a set G of $|U|$ fresh vertices labeled by all elements of U and all the at most k-element subsets of $E \cup G$ as faces.*

An l-tree is a subpattern of a complete l-tree.

The significance of l-trees is apparent from the following observation.

▶ **Lemma 47.** *Assume that $\mathcal{I}$ is a $(k, k + 1)$-instance (with small arity constraints added). Let $l \leq k$, let $\mathbb{P}$ be an l-tree, and let F be a face of $\mathbb{P}$. Then any assignment $\alpha : F \to A$ that satisfies F can be extended to a realization of $\mathbb{P}$. In particular, every l-tree is realizable.*

Proof. If $\mathbb{P}$ is a complete l-tree with base F, then α can be gradually extended to a realization of $\mathbb{P}$ by a straightforward application of the definition of $(k, k + 1)$ instance. It remains to observe that every l-tree with a face F is a subpattern of a complete l-tree with base F. ◀

As noted above, realizability of k-trees in some sense even characterizes $(k, k + 1)$-instances for finite domains. From this perspective it makes sense to use k-trees to measure the consistency level (called the quality) of a tuple in a constraint relation and, more generally, the consistency level of a realization.

▶ **Definition 48.** *Let F be a labeled set of vertices of size at most k. We say that an assignment α, whose domain includes F and which is consistent with the labeling, satisfies F with quality d if $\alpha_{|F}$ can be extended to a realization of the complete k-tree with base F of depth d. A realization α of a pattern $\mathbb{P}$ has quality d (or α satisfies $\mathbb{P}$ with quality d) if α satisfies each face of the pattern with quality d.*

Similarly, we say that an evaluation $\phi : W \to A$ (where $|W| \leq k$) has quality d if the corresponding assignment for a $|W|$-element set of vertices labeled by all the elements of W has quality d.

Informally, an evaluation ϕ has quality d if it survives d steps in a certain naturally defined consistency procedure. Note that a realization of a pattern is the same as a realization of quality 1 and a realization of quality d is also a realization of quality d' for any $d' \le d$. Finally, observe that if an assignment α satisfies F with quality d, then it satisfies every subset of F with quality d.

We finish this subsection with two observations.

▶ **Lemma 49.** *The set of quality-d realizations of a pattern $\mathbb{P}$ is a subuniverse of $\prod_{i \in P} \mathbf{A}_{v(i)}$.*

Proof. For $d=1$ the claim is a straightforward consequence of the fact that constraint relations are subuniverses of products of $\mathbf{A}_x$'s. Otherwise we observe that the set of quality-d realizations of $\mathbb{P}$ is the projection of the set of quality-1 realizations of a larger pattern $\mathbb{Q}$ to P. Indeed, $\mathbb{Q}$ can be taken as the pattern obtained from $\mathbb{P}$ by appending to every face F the complete k-tree with base F of depth d. ◀

▶ **Lemma 50.** *Let $E \subseteq F$ be labeled sets of vertices, $E \le k$, $|F| \le k+1$, and let $\alpha : E \to A$ be an assignment which is consistent with the labeling and satisfies E with quality $d+1$. Then α can be extended to an assignment $\beta : F \to A$ which is consistent with the labeling and satisfies each at most k element subset of F with quality d.*

More generally, for any k-tree $\mathbb{P}$, any face F, and any d, there exists d' such that every assignment $\alpha : F \to A$ which satisfies F with quality d' can be extended to a realization of $\mathbb{P}$ of quality d.

Proof. The first observation follows from the definitions while the second one is proved by induction from the first one. ◀
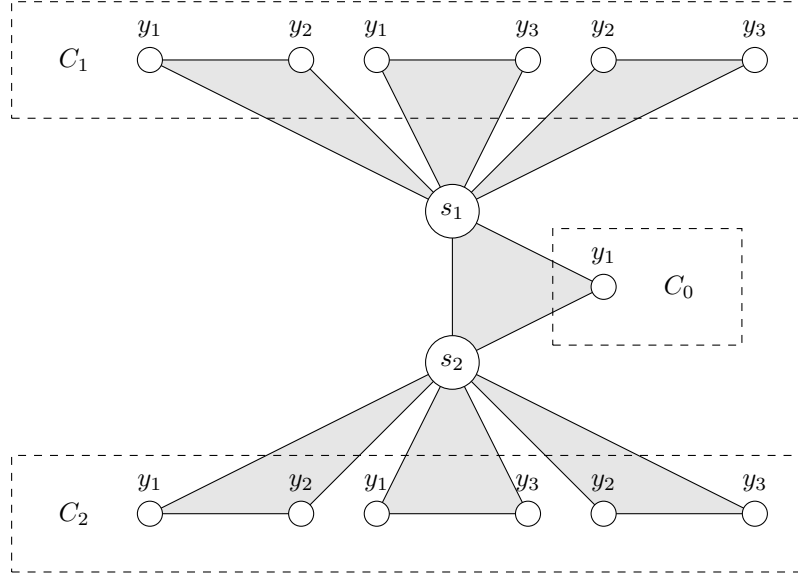
C.2 Fixing patterns

A fixing pattern is a pattern together with a specified set Y of fixing variables. The idea is to require that any consistent evaluation of Y can be extended to a realization of the whole pattern. Since our instance isn't necessarily a $(k, k+1)$-instance the following modification is needed.

▶ **Definition 51.** *A* fixing pattern *is a pair $(\mathbb{P}, Y)$, where $\mathbb{P}$ is a pattern and Y is a set of variables of size at most k. The elements of Y are called* fixing *variables, the remaining variables from $v(P) \setminus Y$ are called* inner.

A fixing pattern $(\mathbb{P}, Y)$ is f-realizable *if for every d there exists $d' = z_{(\mathbb{P}, Y)}(d) \ge d$ such that every evaluation $\phi : Y \to A$ of quality d' can be extended to a realization of $\mathbb{P}$ of quality d.*

It will be a feature of the proofs in this subsection that the sufficient $d' = z_{(\mathbb{P}, Y)}(d)$ from the definition will actually depend only on the "shape" of the fixing pattern: it will not depend on the instance, or on the variety, or on the concrete choice of labeling (i.e., the same d' will work for a pattern obtained from $\mathbb{P}$ by changing v to $r(v)$ and Y to $r(Y)$, for any $r : V \to V$).

A vertex f of a fixing pattern $(\mathbb{P}, Y)$ is called fixing/inner if the variable $v(f)$ is. Faces consisting entirely of inner variables are called *inner*, the remaining faces are called *fixing*. A fixing face, whose set of inner vertices is F and whose set of labels of fixing vertices is Y', is denoted $[F, Y']$. Note that the definition of f-realization only depends on the "inner part" of the fixing pattern together with the list of those $[F, Y']$ that are present in the fixing pattern. It will often be convenient to choose $\mathbb{P}$ *free*, that is, the sets of fixing **vertices** of any two maximal fixing faces are disjoint.

Figure 6 Case $k = 3$, $l = 2$ in the proof of Lemma 52.

An inner face F is called *completely fixed* if $[F, Y']$ is a (fixing) face for every $(k - |F|)$-element set of variables $Y' \subseteq Y$. If $\mathbb{Q}$ is a pattern and Y a set of variables of size at most k, which is disjoint from $v(Q)$, then the *complete Y-fixing* (*complete vertex Y-fixing*, respectively) of $\mathbb{Q}$ is the free fixing pattern $(\mathbb{P}, Y)$, whose set of inner faces coincides with the set of faces of $\mathbb{Q}$ and each inner face (inner vertex, respectively) is completely fixed. Since complete fixings are chosen freely, a complete fixing of a k-tree is a k-tree.

We say that a pattern $\mathbb{Q}$ is *strongly realizable* if each complete fixing of $\mathbb{Q}$ is f-realizable.

Our aim, and the main technical contribution of this section is to prove that every k-tree is strongly realizable. We now present, in Lemma 52 and Lemma 55, two constructions that preserve f-realizability. A proof that the complete fixing of every k-tree can be obtained by these constructions is contained in the next subsection.

▶ **Lemma 52.** *Let $1 \le l \le k + 1$. Let $(\mathbb{P}, Y)$ be the complete vertex Y-fixing of a complete pattern $\mathbb{S}$ with l vertices and, if $l \le k - 1$, freely add to $\mathbb{P}$ an additional fixing face $[S, Y']$ (and its subfaces) for some $Y' \subseteq Y$ of size $k - l$.*

If each complete pattern with $l - 1$ vertices is strongly realizable, then $(\mathbb{P}, Y)$ is f-realizable.

Proof. The case $l = 1$ follows directly from Lemma 50 with the choice $d' = d + 1$ and we henceforth assume $l > 1$.

Fix an arbitrary d. We need to choose d' large enough so that the applications of the assumptions or Lemma 50, which will be used in the proof, do not decrease the quality of our assignments below d. Specifically, we first choose d'' so that $d'' \ge d + 2$ and, in case that $l = k + 1$, also $d'' \ge z_{(\mathbb{Q}, Z)}(d + 1)$ for each complete fixing $(\mathbb{Q}, Z)$ of a complete pattern with 2 vertices (note that $l = k + 1 \ge 3$ in this case); and then choose d' so that $d' \ge z_{(\mathbb{Q}, Z)}(d'')$ for each complete fixing $(\mathbb{Q}, Z)$ of a complete pattern with $l - 1$ vertices (we will actually only use $(\mathbb{Q}, Z)$ equal to $(\mathbb{P}, Y)$ take away one inner vertex).

Denote $S = \{s_1, \dots, s_l\}$ the set of inner vertices of $(\mathbb{P}, Y)$, C_i (where $i \in [l]$) the set of fixing vertices coming from the vertex-fixing faces $[\{s_i\}, \dots]$, and C_0 the set of fixing

vertices coming from the fixing face $[S, Y']$ (which is empty if $l \geq k$), see Figure 6. Let $C = C_0 \cup C_1 \cup \cdots \cup C_l$. Finally, let $\phi : Y \to A$ be an evaluation of quality d'.

We consider the set T of restrictions of quality-d realizations of $\mathbb{P}$ to the set C. Note that this set is a subuniverse of the product of the corresponding $\mathbf{A}_x$'s by Lemma 49.

$$T = \{\beta_{|C} : \beta \text{ satisfies } \mathbb{P} \text{ with quality } d\} \leq \prod_{c \in C} \mathbf{A}_{v(c)}$$

We need to prove that the tuple $\mathbf{a}$ defined by $\mathbf{a}(c) = \phi(v(c))$ for all $c \in C$ is in T. By the Baker-Pixley Theorem (Theorem 14 when proving Theorem 17 and Theorem 19 when proving Theorem 21) it is enough to show that for any $(k+1)$-element set of coordinates D, the relation T contains a tuple $\mathbf{b}$ that agrees with $\mathbf{a}$ on this set. This is now our aim.

Denote $D_i = C_i \cap D$ and assume that there exists $i \geq 1$ such that $|D_0 \cup D_i| \leq k - l + 1$. In this case we find a suitable tuple $\mathbf{b}$ in three steps as follows. First, by the choice of d', we can extend ϕ to an assignment $\gamma : P \setminus \{s_i\} \to A$ that satisfies every k-element subset of $P \setminus \{s_i\}$ with quality d'', and set $\beta(p) = \gamma(p)$ for each $p \in P \setminus (\{s_i\} \cup C_0 \cup C_i)$. Second, set $\beta(p) = \phi(v(p))$ for each $p \in D_0 \cup D_i$, let $F = (S \setminus \{s_i\}) \cup D_0 \cup D_i$, and note that F has size at most $(l-1) + (k-l+1) = k$ and that β satisfies F with quality d''. Therefore, by Lemma 50, $\beta_{|F}$ can be extended to $F \cup \{s_i\}$ so that β satisfies each at most k-element subset of $F \cup \{s_i\}$ with quality $d'' - 1 \geq d + 1$. Third, for each face E of $\mathbb{P}$ where β is not yet fully defined we again use Lemma 50 and extend $\beta_{|E \cap \mathrm{dom}(\beta)}$ to E so that β satisfies E with quality d. By construction, $\beta(c) = \phi(v(c))$ for every $c \in D$, and β satisfies every face of $\mathbb{P}$ with quality d: the fixing faces within $P \setminus (C_0 \cup C_i)$ because of the first step, the face S because of the second step, and the remaining fixing faces (within $S \cup C_0 \cup C_i$) because of the third step. Therefore $\mathbf{b} = \beta_{|C}$ is from T and agrees with $\mathbf{a}$ on D.

Let $i \geq 1$ be such that $|D_i|$ is minimal. If $l \leq k$, then simple arithmetic gives us that $|D_0 \cup D_i| \leq k - l + 1$ (so we are done in this case). Indeed, otherwise $|D_i| \geq k - l + 2 - |D_0|$ and $|D| \geq |D_0| + l|D_i| \geq |D_0| + l(k - l + 2 - |D_0|)$. For the maximum size of D_0, that is, $|D_0| = |C_0| = k - l$, the right hand side of the last inequality is equal to $k + l$, and if $|D_0|$ decreases it gets bigger. Then $|D| \geq k + l > k + 1$, a contradiction.
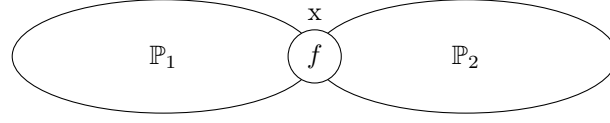
The remaining case is $l = k + 1$ (in particular, $C_0 = D_0 = \emptyset$) and $|D_i| > k - l + 1 = 0$ for each $1 \leq i \leq k+1$. Then, in fact, $D_i = \{d_i\}$ for each $i \geq 1$ (as $|D| \leq k+1$). By the pigeonhole principle, there are $i \neq j$ such that $v(d_i) = v(d_j)$. In this case we modify the three step procedure for finding $\mathbf{b}$ as follows. In the first step we define β only on $P \setminus (\{s_i, s_j\} \cup C_i \cup C_j)$, in the second step we set $\beta(d_i) = \beta(d_j) = \phi(v(d_i))$, define $F = (S \setminus \{s_i, s_j\}) \cup D_i \cup D_j$, and instead of Lemma 50 we use the choice of d'' (coming from complete fixings of 2-element complete patterns) to extend $\beta_{|F}$ to $F \cup \{s_i, s_j\}$. ◀

The next lemma provides the base case for the second construction. We remark that having a near unanimity term of arity $2k$, when proving Theorem 17, or local near unanimity term operations of arity $2k$, when proving Theorem 21, is sufficient for the proof.

▶ **Lemma 53.** *Let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be free fixing patterns with exactly one common vertex f, which is labeled by $x \notin Y$ and which is completely fixed in both patterns. For $i \in \{1, 2\}$ let $\mathbb{P}'_i$ be the pattern obtained from $\mathbb{P}_i$ by removing the fixing vertices and all the vertices labeled x (and all the incident faces). Let $\mathbb{Q}$ be the union of $\mathbb{P}_1$ and $\mathbb{P}_2$.*

If $(\mathbb{P}_i, Y)$, $i = 1, 2$ are f-realizable and $\mathbb{P}'_i$, $i = 1, 2$ are strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.

Proof. Fix d, choose d'' so that each complete fixing $(\mathbb{S}, Z)$ of $\mathbb{P}'_1$ or $\mathbb{P}'_2$, which we will use in the proof, satisfies $d'' \geq z_{(\mathbb{S},Z)}(d+1)$, and choose $d' \geq z_{(\mathbb{P}_i,Y)}(d'')$ for $i = 1, 2$.
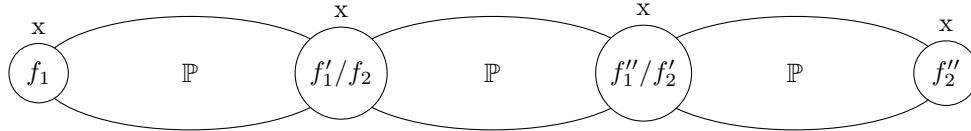
Figure 7 Pattern $\mathbb{Q}$ in Lemma 53.

Let $\phi : Y \to A$ be an evaluation of quality d' and denote $Y = \{y_1, \ldots, y_k\}$ (where variables can possibly repeat). For each $i \in \{1, 2\}$ and $j \in \{1, \ldots, k\}$ we construct a realization $\alpha_i^j : Q \to A$ of $\mathbb{Q}$ of quality d. The sought after quality-d extension α of ϕ will be obtained by applying a $2k$-ary (local) near unanimity operation to these realizations. In order to construct α_i^j we first extend ϕ to a realization β of $\mathbb{P}_i$ of quality d'' and define $\alpha_i^j(p) = \beta(p)$ for each $p \in \text{dom}(\beta) = P_i$. Next, we extend the evaluation $\rho : \{x\} \cup Y \setminus \{y_j\} \to A$, defined by $\rho(x) = \beta(f)$ and $\rho(y) = \phi(y)$ else, to a quality-$(d + 1)$ realization γ of the complete $(\{x\} \cup Y \setminus \{y_j\})$-fixing of $\mathbb{P}_{3-i}'$ and define $\alpha_i^j(c) = \gamma(c)$ for each $c \in \text{dom}(\gamma)$ (noting that ρ has quality d'' since β does and f is completely fixed in $\mathbb{P}_i$). Finally, for each face F of $\mathbb{Q}$ where α_i^j is not yet fully defined (this concerns fixing vertices of $\mathbb{P}_{3-i}$ labeled y_j) we use Lemma 50 and extend α_i^j so that it satisfies F with quality d. Now α_i^j satisfies all the faces of $\mathbb{Q}$ with quality d and agrees with ϕ on all of the fixing variables, except those from $\mathbb{P}_{3-i}$ labeled y_j. It follows that applying a $2k$-ary term operation to the α_i^j that satisfies the near unanimity condition for the set of components of the α_i^j gives an assignment of quality d (by Lemma 49) that extends ϕ, as required. ◀

▶ **Corollary 54.** *Let $(\mathbb{P}, Y)$ be a fixing pattern with two vertices $f_1 \neq f_2$ both labeled x and both completely fixed, and let n be a positive integer. Let $(\mathbb{Q}, Y)$ be the fixing pattern obtained from the disjoint union of n copies of $\mathbb{P}$ by identifying, for each $i \in \{1, \ldots, n-1\}$, the vertex f_2 in the i-th copy with the vertex f_1 in the $(i+1)$-st copy. Let $\mathbb{P}'$ be the pattern obtained from $\mathbb{P}$ by removing the fixing vertices and all the vertices labeled x.*

If $(\mathbb{P}, Y)$ is f-realizable and $\mathbb{P}'$ is strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.
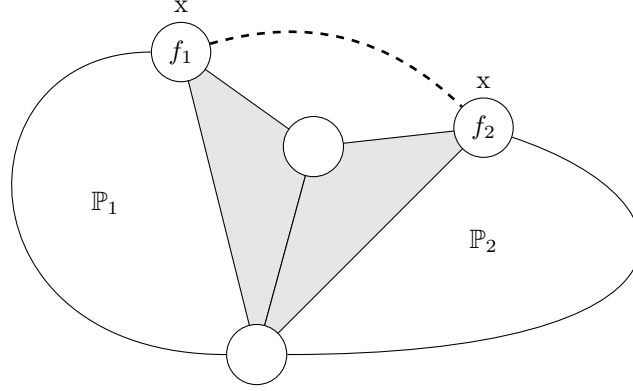

Figure 8 Pattern $\mathbb{Q}$ in Corollary 54.

Proof. The proof follows by induction from Lemma 53, noting that in each step if we remove vertices labeled x and fixing vertices from $\mathbb{Q}$, we get a pattern which is a disjoint union of strongly realizable patterns and is thus strongly realizable. ◀

The following lemma provides the second construction. The proof uses Corollary 54 (which requires a near unanimity term of arity $2k$ or local near unanimity term operations of arity $2k$) but the rest of the reasoning is based on the loop lemma stated in Theorem 30, for which a near unanimity term (or local near unanimity term operations) of any arity is sufficient.

▶ **Lemma 55.** *Let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be fixing patterns with a common inner face E and no other common vertices, such that both $\mathbb{P}_1$ and $\mathbb{P}_2$ are k-trees. For $i = 1, 2$ let f_i be a completely fixed inner vertex of $\mathbb{P}_i$ with label x such that $E \cup \{f_i\}$ is a face of $\mathbb{P}_i$. Let $\mathbb{Q}$ be*

■ **Figure 9** Patterns $\mathbb{P}_1 \cup \mathbb{P}_2$ and $\mathbb{Q}$ in Lemma 55

the pattern obtained from the union of $\mathbb{P}_1$ and $\mathbb{P}_2$ by identifying vertices f_1 and f_2, and let $\mathbb{Q}'$ be the pattern obtained from $\mathbb{Q}$ (or $\mathbb{P}_1 \cup \mathbb{P}_2$) by removing the fixing vertices and all the vertices labeled x.

If $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable and $\mathbb{Q}'$ is strongly realizable, then $(\mathbb{Q}, Y)$ is f-realizable.

Proof. Let $r > 2$ be such that, in the case of proving Theorem 17, $\mathcal{V}$ has an r-ary near unanimity term, and in the case of proving Theorem 21, $\mathbf{A}$ has local near unanimity term operations of arity r (so $r = k + 2$ works). Let $(\mathbb{Q}^{r-1}, Y)$ be the fixing pattern obtained by taking the disjoint union of $r - 1$ copies of $\mathbb{P}_1 \cup \mathbb{P}_2$ and identifying the vertex f_2 in the i-th copy with the vertex f_1 in the $(i + 1)$-first copy, for each $i \in \{1, \ldots, r - 2\}$. The pattern $(\mathbb{Q}^{r-1}, Y)$ is f-realizable by Corollary 54.

Fix d, choose d'' using Lemma 50 so that, for both $i \in \{1, 2\}$, every quality-d'' assignment $\alpha : E \cup \{f_i\} \to A$ extends to a quality-d realization of $\mathbb{P}_i$, and choose $d' \geq z_{(\mathbb{Q}^{r-1}, Y)}(d'' + 1)$.

Let $\phi : Y \to A$ be an evaluation of quality d'. Denote by B the set of all elements of $a \in A_x$ such that the evaluation $x \mapsto a$ has quality $d'' + 1$, denote by T the set of all the quality-d realizations β of $\mathbb{P}_1 \cup \mathbb{P}_2$ such that both $\{f_1\}$ and $\{f_2\}$ have quality $d'' + 1$ and both $E \cup \{f_1\}$ and $E \cup \{f_2\}$ have quality d'', and denote by $S \subseteq T$ the set of those $\beta \in T$ that extend ϕ. By a similar argument to that of Lemma 49, both T and S are subuniverses of $\prod_{p \in P_1 \cup P_2} \mathbf{A}_{v(p)}$. Using the near unanimity term of arity r (or local near unanimity term operations of arity r) S clearly locally r-absorbs T. The plan is to apply Theorem 32 to the binary relation $\text{proj}_{f_1, f_2} S \subseteq B \times B$. If this binary relation contains a loop, then the corresponding $\alpha \in S$ satisfies $\alpha(f_1) = \alpha(f_2)$ and, therefore, we actually obtain a realization of $\mathbb{Q}$ of quality d, as required.

It remains to verify the assumptions of Theorem 32. By the choice of d', the pattern $\mathbb{Q}^{r-1}$ has a quality-$(d'' + 1)$ realization that extends ϕ. The images of copies of vertices f_1 and f_2 in such a realization yield a directed walk in $\text{proj}_{f_1, f_2}(S)$ of length $r - 1$. Next, since S locally r-absorbs T, then $\text{proj}_{f_1, f_2}(S)$ locally r-absorbs $\text{proj}_{f_1, f_2}(T)$, so it is enough to verify that the latter relation contains $=_B$ and $\text{proj}_{f_1, f_2}(S)^{-1}$. For the first case, pick $b \in B$ and recall that the assignment $f_1 \mapsto b$ has quality $d'' + 1$ by the definition of B. We extend this assignment (using Lemma 50) to a quality d''-assignment $\alpha : E \cup \{f_1\} \to A$, define $\alpha(f_2) = \alpha(f_1)$, and extend α to a quality-d realization of $\mathbb{P}_1 \cup \mathbb{P}_2$. The obtained assignment witnesses $(b, b) \in \text{proj}_{f_1, f_2}(T)$. Finally, to show that $\text{proj}_{f_1, f_2}(T)$ contains $\text{proj}_{f_1, f_2}(S)^{-1}$, consider any $(a, b) \in \text{proj}_{f_1, f_2}(S)^{-1}$. By the definition of S, the pattern $\mathbb{P}_1 \cup \mathbb{P}_2$ has a realization α such that $\alpha(f_1) = b$, $\alpha(f_2) = a$, and both $E \cup \{f_1\}$ and $E \cup \{f_2\}$ have quality

d''. We flip the values $\alpha(f_1)$ and $\alpha(f_2)$, restrict α to $E \cup \{f_1, f_2\}$ and extend this assignment using the choice of d'' to a realization of $\mathbb{P}_1 \cup \mathbb{P}_2$ of quality d, giving us $(a, b) \in \mathrm{proj}_{f_1, f_2}(T)$ and concluding the proof.   ◀

C.3   Assembly

Lemma 52 and Lemma 55 enable us to prove that every k-tree is strongly realizable. We split the inductive proof of this fact into two lemmata.

▶ **Lemma 56.** *Let $1 < l \leq k + 1$ and assume, in case that $l > 2$, that every $(l - 2)$-tree is strongly realizable. Then every complete pattern with l vertices is strongly realizable.*

Proof. We start with a complete vertex Y-fixing of a complete pattern with l vertices, which is f-realizable by Lemma 52 (note that a complete pattern with $l - 1$ vertices is an $(l-2)$-tree), and add fixing faces one by one while preserving the f-realizability.

So, let $\mathbb{S}$ be an f-realizable Y-fixing of a complete pattern with l vertices and let $[E, Y']$ be such that $E = \{e_1, \ldots, e_{l'}\}$, $l' < k$, is an inner face of $\mathbb{S}$ and $Y' \subseteq Y$ is a $(k - |E|)$-element set of variables. Our aim is to show that $\mathbb{S}$ plus the fixing face $[E, Y']$ is f-realizable. Let $(\mathbb{C}, Y)$ be the complete vertex Y-fixing of a complete pattern with the set of inner vertices $G = \{g_1, \ldots, g_{l'}\}$ (where g_i's are fresh vertices) labeled according to E (i.e., $v(g_i) = v(e_i)$ for each $i \in [l']$) with an additional fixing face $[G, Y']$. By Lemma 52, this fixing pattern is realizable. Let $(\mathbb{C}^i, Y)$, $i \in \{0, \ldots, l'\}$ be the fixing pattern obtained by renaming the vertices $g_1, \ldots, g_i$ to $e_1, \ldots, e_i$, respectively. The aim, reformulated, is to show that $(\mathbb{S} \cup \mathbb{C}^i, Y)$ is f-realizable for $i = l'$. We prove this claim by induction on i.

For $i = 0$ the union $\mathbb{S} \cup \mathbb{C}^i$ is disjoint, therefore the claim follows from the f-realizability of $\mathbb{S}$ and $\mathbb{C}^0 = \mathbb{C}$. For the induction step from i to $i + 1$ we apply Lemma 55 with $\mathbb{P}_1 = \mathbb{S}$, $\mathbb{P}_2 = \mathbb{C}^i$, $f_1 = e_{i+1}$, and $f_2 = g_{i+1}$. Note that $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable by the induction hypothesis and $\mathbb{Q}'$ is strongly realizable since it is an $(l - 2)$-tree (or a single vertex in case that $l = 2$), so we can conclude that $(\mathbb{Q}, Y) = (\mathbb{S} \cup \mathbb{C}^{i+1}, Y)$ is f-realizable, finishing the proof.   ◀

▶ **Lemma 57.** *Let $1 < l \leq k + 1$ and assume that every complete pattern with l vertices is strongly realizable. Then every $(l - 1)$-tree is strongly realizable.*

Proof. It is enough to show that every complete $(l - 1)$-tree is strongly realizable. However, for an inductive proof of this claim, it will be convenient to use more general $(l - 1)$-trees, those that can be obtained from the empty pattern in n steps by taking the union of the already constructed pattern $\mathbb{S}$ with a complete pattern $\mathbb{C}$ on $l' \leq l$ vertices such that $S \cap C$ (where $0 \leq |S \cap C| < l'$) is a face in both patterns (with the same labelling in both patterns). The induction is primarily on n and secondarily on $|S \cap C|$. For $n = 1$ the claim follows from the assumption of the lemma. If $S \cap C = \emptyset$, then $\mathbb{S} \cup \mathbb{C}$ is a disjoint union and the claim follows by the inductive assumption and the assumption of the lemma.

Otherwise, take a fresh set Y of k variables and let $(\mathbb{Q}, Y)$ be a complete Y-fixing of $\mathbb{S} \cup \mathbb{C}$. Pick a vertex in $S \cap C$, say vertex f_1 labeled x, let $\mathbb{C}'$ be the pattern obtained from $\mathbb{C}$ by renaming vertex f_1 to a fresh vertex f_2, let $(\mathbb{P}_1, Y)$ and $(\mathbb{P}_2, Y)$ be complete Y-fixings of $\mathbb{S}$ and $\mathbb{C}'$, respectively, and let $E = (S \cap C) \setminus \{f_1\}$. Note that this notation is consistent with the statement of Lemma 55: $\mathbb{Q}$ can be obtained from $\mathbb{P}_1 \cup \mathbb{P}_2$ by identifying vertices f_1 and f_2. To conclude the proof, we observe that the assumptions of Lemma 55 are satisfied. Indeed, $(\mathbb{P}_1 \cup \mathbb{P}_2, Y)$ is f-realizable by the inductive assumption (since it is a complete fixing of $\mathbb{S} \cup \mathbb{C}'$ for which $|S \cap C'| < |S \cap C|$) and $\mathbb{Q}'$ is strongly realizable since it is a subpattern of $\mathbb{S} \cup \mathbb{C}'$.   ◀

The following corollary is the core technical contribution of this section. Its proof follows by induction from the previous two lemmata.

▶ **Corollary 58.** *Every k-tree is strongly realizable.*

Armed with Corollary 58, we are ready to execute the idea outlined in the beginning of this section. For the purpose of the following theorem, we call an instance $\mathcal{I} = (V, \{\mathbf{A}_x \mid x \in V\}, \mathcal{C})$ a *weak k-instance* if it satisfies the running assumption, that is, $\mathcal{C} = \{(S, R_S) \mid S \subseteq V, |S| \leq k\}$ and, for any $S' \subseteq S$ such that $|S| \leq k$, the projection of R_S onto S' is contained in $R_{S'}$.

▶ **Theorem 59.** *Let $k \geq 2$ and $n \geq 0$ be integers. Then there exists $d = z(n, k)$ such that for any variety $\mathcal{V}$ with a $(k+2)$-ary near unanimity term, or any idempotent algebra $\mathbf{A}$ with local near unanimity term operations of arity $k + 2$, any weak k-instance $\mathcal{I}$ of $\mathsf{CSP}(\mathcal{V})$ (or $\mathsf{CSP}(\mathbf{A})$) with at most n variables, and any at most k-element set of variables Y, every evaluation $\phi : Y \to A$ of quality d extends to a solution of $\mathcal{I}$.*

Proof. We prove the claim by induction on n. If $n \leq 1$, then the claim trivially holds with $d = 1$. Otherwise, we denote $d' = z(n - 1, k)$ and pick a d greater than or equal to $z_{(\mathbb{T}, Y)}(d')$ for every complete Y-fixing $(\mathbb{T}, Y)$ of a complete k-tree of depth d' (formed over the variables $V \setminus Y$).

Consider an instance $\mathcal{I}$ of $\mathsf{CSP}(\mathcal{V})$ (or $\mathsf{CSP}(\mathbf{A})$) and an evaluation $\phi : Y \to A$ of quality d. We define a new instance $\mathcal{I}' = (V', \{\mathbf{A}_x \mid x \in V'\}, \{(S, R'_S) \mid S \subseteq V, |S| \leq k\})$ by setting $V' = V \setminus Y$ and

$$R'_S = \{\rho_{|S} \mid \rho : Y \cup S \to A \text{ is a partial solution of } \mathcal{I} \text{ such that } \rho_{|Y} = \phi\}$$

Clearly, $\mathcal{I}'$ is a weak k-instance. We have chosen d so that, in the instance $\mathcal{I}$, the partial evaluation ϕ extends to a realization of the complete Y-fixing of a complete k-tree of depth d' over the set of variables $V \setminus Y$ (the base can be chosen arbitrarily for the argument). This realization witnesses that, in the instance $\mathcal{I}'$, there exists an evaluation of quality d'. By the choice of d', any such evaluation extends to a solution θ of $\mathcal{I}'$. Now $\phi \cup \theta$ is a solution of $\mathcal{I}$, finishing the proof. ◀

To conclude, we state the parts of Theorem 17 and Theorem 21 that we set out to prove in this section as the following corollary. It directly follows from Lemma 47 and the previous theorem.

▶ **Corollary 60.** **1.** *If $\mathcal{V}$ is a variety that has a $(k+2)$-ary near unanimity term then every $(k, k+1)$-instance of the CSP over $\mathcal{V}$ is sensitive.*
2. *If $\mathbf{A}$ is an idempotent algebra that has local near unanimity term operations of arity $k+2$ then every $(k, k+1)$-instance of $\mathsf{CSP}(\mathbf{A})$ is sensitive.*