Deciding some Maltsev conditions in finite idempotent algebras

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1 Introduction

In this paper we investigate the computational complexity of deciding if a given finite algebraic structure satisfies a certain type of existential condition on its set of term operations. These conditions, known as *Maltsev conditions*, have played a central role in the classification, study, and applications of general algebraic structures. Several well studied properties of equationally defined classes of algebras (also known as varieties of algebras), such as congruence permutability, distributivity, and modularity are equivalent to particular Maltsev conditions.

The general set up for the decision problems that we consider in this paper is as follows: for a fixed Maltsev condition Σ, an instance of Σ-testing is a finite algebraic structure (or just algebra for short) $\mathbf{A}$. The question to decide is if the variety of algebras generated by $\mathbf{A}$ satisfies the Maltsev condition Σ. This is a natural computational problem in universal algebra (especially in the case of important Maltsev conditions such as having a majority operation) that finds practical applications in the UACalc [8] software system for studying algebras on a computer. Moreover, some Maltsev conditions are also known to correspond to complexity classes of the Constraint Satisfaction Problem (see the recent survey [2] for an overview) and deciding these Maltsev conditions for algebras of polymorphisms of relational structures is an important meta-problem (which, however, is beyond the scope of this paper).

It turns out that deciding many common Maltsev conditions for finite algebras is often EXPTIME-complete [9, 14] and so some of the recent work in this area has focussed on the restriction of these decision problems to finite idempotent algebras. Idempotent algebras display rich behavior, while also offering relative comfort when it comes to composing operations. As two examples of the importance of idempotency, we note that classical Maltsev conditions that characterize properties of congruence lattices in varieties are idempotent [13], and the algebraic approach to the complexity of the Constraint Satisfaction Problem studies finite idempotent algebras (although the theory gradually moves towards general algebras, see [3]).

Our goal in this paper is to show that Σ-testing can be accomplished in polynomial time when the algebras tested are idempotent and the (strong) Maltsev condition Σ can be described using paths. While not all Maltsev conditions are of this form, path Maltsev conditions include many important conditions such as the Maltsev term, ternary majority, Jónsson terms or Gumm terms. While the efficient decidability of having the Maltsev or majority term was known before [9] as was the efficient decidability of a chain of n Hagemann-Mitschke terms [21], our framework of path Maltsev conditions unifies these earlier results and shows that it is similarly tractable to decide if an algebra has a fixed length chain of (classical or directed) Jónsson or Gumm terms. See Corollary 8 for a summary of the Maltsev conditions we can work with.

While our framework is quite general, it does not cover all previously known positive results. For example, in [14] it is shown that testing for the presence of a k-edge term, for some fixed $k > 1$, can be done in polynomial time for finite idempotent algebras. In [9] it is shown that testing if a finite idempotent algebra generates a congruence distributive or congruence modular variety can also be carried out by a polynomial time algorithm. In contrast, the same paper also proves that testing for either of these conditions in general algebras is an EXPTIME-complete problem.

2 Preliminaries

An *algebra* is a structure $\mathbf{A} = \langle A, f_i (i \in I)\rangle$ consisting of a non-empty set A, called the *universe* or *domain* of $\mathbf{A}$, and a list of finitary operations on A, f_i, for $i \in I$, for some index set I, called the *basic operations* of $\mathbf{A}$. The *type* of $\mathbf{A}$ is the I-indexed sequence $\langle n_i \mid i \in I\rangle$, where the arity of f_i is n_i. A subset B of A is a *subuniverse* of $\mathbf{A}$ if it is closed under the basic operations of $\mathbf{A}$. For any subset X of A there is a smallest subuniverse of $\mathbf{A}$, with respect to inclusion, that contains X. This subuniverse is called the *subuniverse of* $\mathbf{A}$ *generated by* X and is denoted by $\mathrm{Sg}^{\mathbf{A}}(X)$.

A *variety* of algebras of type τ is a class of algebras of type τ that is closed under taking homomorphisms, subalgebras, and direct products. The *variety generated by* $\mathbf{A}$, denoted by $V(\mathbf{A})$, is the smallest variety of algebras having the same type as $\mathbf{A}$ that contains $\mathbf{A}$. For background material on algebras and varieties, the reader may consult one of [5, 4, 20].

As noted in the Introduction, we will primarily be concerned with algebras that are finite (their domains are finite and their lists of basic operations are finite) and idempotent. An operation t on a set A is *idempotent* if the equation

$$t(x, x, \dots, x) \approx x.$$

holds. An algebra $\mathbf{A}$ is *idempotent* if each of its basic operations is idempotent.

We will often be working with tuples. The usual notation for an n-tuple of elements of A will be $\overline{a} \in A^n$. To concatenate two tuples (or, more often, an n-tuple and a single element of A) we will use the notation $\overline{a}\overline{b}$ (resp. $\overline{a}b$ if b is an element of A). We will use the notation $\hat{a}$ for a tuple of tuples, i.e.. $\hat{c} = (\overline{c}_1, \dots, \overline{c}_n)$ where each $\overline{c}_i$ is a tuple.

An *n-ary relation* R over a set A is nothing more than a subset of A^n. We will usually write elements of relations in columns, as this allows us to apply operations of $\mathbf{A}$ to n-tuples of elements of A: Given $\overline{r}_1, \dots, \overline{r}_k \in A^n$ and a k-ary operation $f\colon A^k \to A$, the n-tuple $f(\overline{r}_1, \dots, \overline{r}_k)$ is the result of applying f to the rows of the matrix with columns $\overline{r}_1, \dots, \overline{r}_k$. A relation R is *invariant* under $\mathbf{A}$ if whenever $k \in \mathbb{N}$, f is a k-ary operation of $\mathbf{A}$ and $\overline{r}_1, \overline{r}_2, \dots, \overline{r}_k \in R$ we have $f(\overline{r}_1, \dots, \overline{r}_k) \in R$. A relation $R \leq \prod_{i=1}^{n} B_i$ is *subdirect* in the product if the i-th projection of R is equal to B_i for each i.

A good part of our exposition concerns directed graphs, or digraphs. A *digraph* G on the set of vertices $V(G)$ is a relational structure with one binary relation $E(G)$. We allow loops in our digraphs and, because our digraphs' edges will have labels on them, we allow multiple edges between a given pair of vertices.

Graphically, to denote that $(u, v) \in E(G)$ for vertices u and v, we draw an arrow (oriented edge) from u to v. We say that there is an *oriented walk* from $u \in V(G)$ to $v \in V(G)$ if there is a sequence of vertices $u = w_0, w_1, w_2, \dots, w_n = v$ such that for all i there is an edge from w_i to w_{i+1} or from w_{i+1} to w_i. A walk is *directed* if for all i the edge is always from w_i to

w_{i+1}. An oriented walk where the vertices $w_0, \dots, w_n$ are pairwise distinct is an *oriented path*. If P is an oriented path with vertices $p_0, \dots, p_n$, then a *prefix* of P is any oriented path $p_0, \dots, p_k$ and a *suffix* of P is any oriented path $p_k, \dots, p_n$ where $0 \leq k \leq n$. A *cycle of length* n is a sequence of vertices $w_0, w_1, \dots, w_n$ such that $w_0 = w_n$ and for each i we have an edge from w_i to w_{i+1}.

For the purposes of this paper, a *strong Maltsev condition* Σ is a condition of the form: there are some operations $d_1, d_2, \dots, d_k$ that satisfy some set of equations Σ involving the d_i. For example

$$p(p(x, y), r(x)) \approx y \tag{1}$$

is a strong Maltsev condition involving a binary operation $p(x, y)$ and a unary operation $r(x)$. An algebra $\mathbf{A}$ satisfies a strong Maltsev condition Σ if for each operation s that appears in Σ there is a term operation t_s of $\mathbf{A}$ having the same arity as s such that the collection of operations t_s on $\mathbf{A}$ satisfies the equations of Σ.

Any Abelian group $\mathbf{G} = \langle G, x \cdot y, x^{-1}, e \rangle$ satisfies the above strong Maltsev condition, since the term operations $p(x, y) = x \cdot y$ and $r(x) = x^{-1}$ satisfy (1). In contrast, it can be checked that no non-trivial semilattice can satisfy this condition.

A strong Maltsev condition is *linear* if none of its equations involve the composition of operations. It is said to be *idempotent* if its equations imply that each of the operations involved in it are idempotent. An example of a strong linear idempotent Maltsev condition is that of having a Maltsev term, i.e., a ternary term p that satisfies the equations $p(y, x, x) \approx y$ and $p(x, x, y) \approx y$. A more thorough discussion of Maltsev conditions can be found in [10].

3 Path Maltsev conditions

In this section, we will show how to express several classical Maltsev conditions using paths and how to efficiently decide them in finite idempotent algebras by checking that they hold locally. The proof that one can go from operations that locally satisfy a given path Maltsev condition in a finite idempotent algebra to operations that satisfy it globally will be presented in the next section.

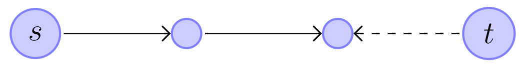

Figure 1: An example of a pattern digraph.

3.1 Pattern digraphs

Inspired by the work of Libor Barto and Marcin Kozik [1], we will represent some special Maltsev conditions as paths. To do this in a systematic way, we will need to introduce digraphs whose edges carry additional information.

Definition 1. A *pattern digraph* is a directed graph with two kinds of edges: Solid and dashed. Additionally, each pattern digraph has two special distinguished vertices: The initial vertex s and the terminal vertex t (see Figure 1). We allow multiple edges between pairs of vertices of pattern digraphs.

If G is a pattern digraph, then H is a *subgraph* of G if the set of vertices, dashed and solid edges of H is the subset of the set of vertices, dashed and solid edges, respectively of G. Additionally, any subgraph of G must have the same initial and terminal vertices as G.

A *pattern path* from s to t is a pattern digraph P with initial vertex s and terminal vertex t such that P viewed as a digraph is an oriented path with endpoints s and t. The length of a pattern path is the number of its edges.

If G and H are pattern digraphs, then a *pattern digraph homomorphism* (often shortened to just "homomorphism" in the rest of this paper) from G to H is a mapping $f\colon G \to H$ which is a digraph homomorphism that sends all solid edges of G to solid edges of H. (Note that dashed edges of G may be mapped to dashed or solid edges of H.) We do not require that a pattern digraph homomorphism maps the initial and terminal vertices of its domain to their counterparts in its co-domain. If G is a pattern digraph with initial vertex s and terminal vertex t, and H is a pattern digraph, then we say that there is a *G-shaped walk from u to v in H* if there is a pattern digraph homomorphism $f\colon G \to H$ such that $f(s) = u$ and $f(t) = v$.

We define isomorphisms in the standard way: If G and H are two pattern digraphs, we say that G and H are isomorphic if and only if there is a bijection $b\colon G \to H$ such that both b and b^{-1} are pattern digraph homomorphisms that send initial vertices to initial vertices and terminal vertices to terminal vertices.

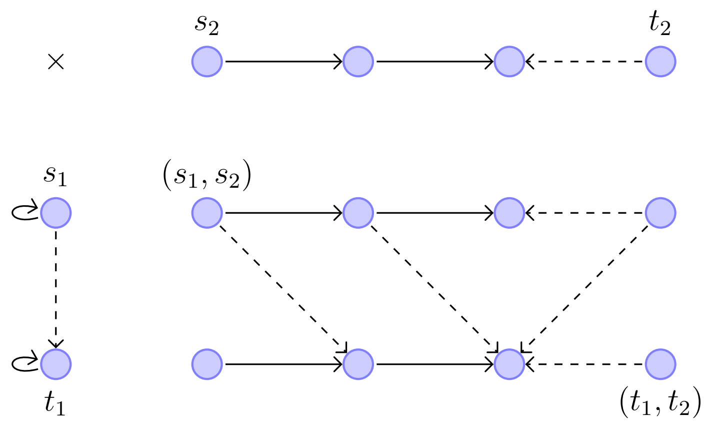

Figure 2: A product of two pattern digraphs.

If G and H are pattern digraphs with initial vertices s_1 and s_2 and terminal vertices t_1 and t_2 respectively, then their product $G \times H$ is the pattern digraph with the vertex set $V(G) \times V(H)$ (of which (s_1, s_2) is the initial and (t_1, t_2) the terminal vertex). $G \times H$ has an edge from (v_1, v_2) to (u_1, u_2) if and only if $(v_1, u_1) \in E(G)$ and $(v_2, u_2) \in E(H)$. The edge $((v_1, v_2), (u_1, u_2))$ is solid if and only if both edges (v_1, u_1) and (v_2, u_2) are solid; otherwise it is dashed (see Figure 2).

Let $\mathbf{A}$ be an algebra and let $\mathbf{F}$ be the 2-generated free algebra in $V(\mathbf{A})$ with generators x, y. The elements of $\mathbf{F}$ can be represented as binary term operations $b(x, y)$ of the algebra $\mathbf{A}$ in the variables x and y, and from this perspective can be regarded as members of $A^{|A|^2}$, i.e., as $|A|^2$-tuples over A. Under this representation the generators x and y correspond to the two binary projection functions $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$, respectively, and the universe of $\mathbf{F}$ is the subuniverse of $\mathbf{A}^{|A|^2}$ generated by π_0 and π_1.

Consider the following subalgebra of $\mathbf{F}^3$:

$$K(\mathbf{A}) = \mathrm{Sg}^{\mathbf{F}^3}\left(\left\{\begin{pmatrix} x \\ x \\ x \end{pmatrix}, \begin{pmatrix} y \\ x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \\ y \end{pmatrix}\right\}\right)$$

Most of the time, we shall view $K(\mathbf{A})$ as a pattern digraph on the set $\mathbf{F}$ by treating each $(a, b, c) \in K(\mathbf{A})$ as an edge from b to c. If $a = x$, we declare the edge (b, c) to be solid, otherwise it will be dashed. The initial and terminal vertices of the pattern digraph $K(\mathbf{A})$ are x and y, respectively.

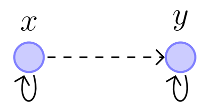

Figure 3: The pattern digraph J.

If we look at the generators of $K(\mathbf{A})$ only and view them as a pattern digraph with initial vertex x and terminal vertex y, we will get a digraph that has two vertices, two solid loops and one dashed edge (see Figure 3). We will call this digraph J.

While most of the time we only have to distinguish solid and dashed edges, sometimes this approach is too coarse. For example, when we view $K(\mathbf{A})$ as a pattern digraph, we are losing information about the first coordinates of members of $K(\mathbf{A})$. This is why we will sometimes be talking about labelled digraphs instead of pattern digraphs. A *labelled digraph* is a directed graph G together with a function $f\colon E(G) \to L$ that assigns a label from the set L to each edge of G. We will again allow multiple edges between two vertices.

3.2 From paths to Maltsev conditions

Let P be a pattern path from 0 to n with vertex set $\{0, 1, 2, \ldots, n\}$, and let $\mathbf{A}$ be an algebra. When does there exist a P-shaped walk from x to y in the pattern digraph $K(\mathbf{A})$? We will show that such a path exists if and only if $\mathbf{A}$ satisfies a specific strong linear Maltsev condition $M(P)$.

Given P, we will construct $M(P)$ as follows: For each $i \in \{0, 1, \ldots, n\}$, take a binary operation symbol $s_i(x, y)$, and for each $i \in [n] = \{1, 2, \ldots, n\}$, take a ternary operation symbol $t_i(x, y, z)$. Start with the equations $s_0(x, y) \approx x$ and $s_n(x, y) \approx y$. The equations of $M(P)$ that connect the operation symbols s_i and t_i depend on the nature of the edge between $i-1$ and i in P: If the edge has the form $(i-1, i)$ (what we call a forward edge), add to $M(P)$ the pair of equations

$$t_i(x, x, y) \approx s_{i-1}(x, y)$$
$$t_i(x, y, y) \approx s_i(x, y),$$

while if the i-th edge of P is a backward edge $(i, i-1)$, we add to $M(P)$ the pair

$$
\begin{aligned}
t_i(x,x,y) &\approx s_i(x,y) \\
t_i(x,y,y) &\approx s_{i-1}(x,y).
\end{aligned}
$$

Moreover, whenever the i-th edge of P is solid, we add to $M(P)$ the equation $t_i(x,y,x) \approx x$.

Looking at $M(P)$, we see that it is a strong, linear Maltsev condition. Observe also that all the terms t_i and s_i have to be idempotent, since if we set $x = y$, the chain gives us that all of the $t_i(x,x,x)$ and $s_i(x,x)$ are equal to each other and to x.

Observation 2. *Let* $\mathbf{A}$ *be an algebra and* P *be a pattern path. Then the idempotent, strong, linear Maltsev condition* $M(P)$ *is satisfied by* $\mathbf{A}$ *if and only if there is a* P*-shaped walk from* x *to* y *in* $K(\mathbf{A})$.

Proof. Suppose that $\mathbf{A}$ satisfies $M(P)$, as witnessed by the terms $s_i(x,y)$, for $0 \le i \le n$ and $t_i(x,y,z)$ for $1 \le i \le n$. We claim that then $s_0, s_1, \ldots, s_n$ is a P-shaped walk from x to y in $K(\mathbf{A})$. Obviously, $s_0(x,y) = x$ and $s_n(x,y) = y$. To see that the edge between s_{i-1} and s_i is of the right type, one needs to consider four cases, of which we will only do one in detail: Assume that $(i-1,i)$ is a solid forward edge of P. Then

$$
t_i\left(\begin{pmatrix} x \\ x \\ x \end{pmatrix}, \begin{pmatrix} y \\ x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \\ y \end{pmatrix}\right) = \begin{pmatrix} t_i(x,y,x) \\ t_i(x,x,y) \\ t_i(x,y,y) \end{pmatrix}
$$

lies in $K(\mathbf{A})$. Using the equations of $M(P)$ involving t_i, s_{i-1}, and s_i, we immediately see that $(s_{i-1}(x,y), s_i(x,y))$ is a solid edge of $K(\mathbf{A})$ and we are done.

Conversely, suppose that $f_0, f_1, \ldots, f_n$ is a P-shaped walk from x to y in $K(\mathbf{A})$. We will show how to get the terms t_i, s_i from forward edges of P; the construction for backward edges is similar. If $(i-1,i)$ is an edge of P then (f_{i-1}, f_i) is an edge of $K(\mathbf{A})$ and so there is some ternary term $g_i(x,y,z)$ of $\mathbf{A}$ such that $g_i(x,x,y) = f_{i-1}$ and $g_i(x,y,y) = f_i$. The way to satisfy $M(P)$ is then to let $t_i(x,y,z)$ to be $g_i(x,y,z)$, $s_{i-1}(x,y)$ to be f_{i-1}, and $s_i(x,y)$ to be f_i. Since $f_0 = x$ and $f_n = y$, the equations $s_0 \approx x$ and $s_n \approx y$ are satisfied. Finally, if $(i-1,i)$ is a solid edge, then (f_{i-1}, f_i) must be solid as well, and thus we can demand that $g_i(x,y,x) = x$, satisfying the corresponding condition in $M(P)$. $\square$

Figure 4: A path P yielding trivial $M(P)$.

Figure 5: Maltsev term.

3.3 Example gallery

To illustrate the connection between a pattern path P and the associated Maltsev condition $M(P)$, we present some well known Maltsev conditions as $M(P)$ for some pattern paths P. Compare the paths in the pictures with the set of generators of $K(\mathbf{A})$ as shown in Figure 3. To save space, we will replace s_0 by x and s_n by y in the equations.

3.3.1 The trivial case

If P contains any dashed forward edges $(i, i+1)$ (see Figure 4) then the condition $M(P)$ will be trivially satisfied by all algebras; one needs only to set $t_j(x,y,z) = x$ for $j < i$, $t_i(x,y,z) = y$, and $t_j(x,y,z) = z$ for $j > i$.

3.3.2 Maltsev term

By the classic result due to Maltsev [19], a variety is congruence permutable if and only if it has a ternary term $t_1(x,y,z)$ that satisfies the equations

$$\begin{aligned} x &\approx t_1(x,y,y) \\ t_1(x,x,y) &\approx y \end{aligned}$$

This strong Maltsev condition is equivalent to $M(P)$ for the path P pictured in Figure 5 that consists of a single dashed backward edge $(1,0)$.

3.3.3 Majority

The path P that gives rise to a majority term consists of a single solid forward edge (see Figure 6). The equations that define $M(P)$ are:

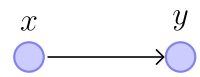

Figure 6: Majority.

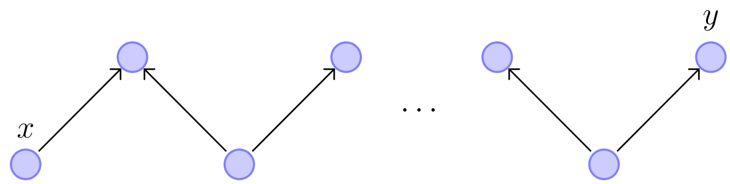

Figure 7: A chain of an odd number of Jónsson terms.

$$\begin{aligned} x &\approx t_1(x,x,y) \\ t_1(x,y,y) &\approx y \\ t_1(x,y,x) &\approx x. \end{aligned}$$

3.3.4 Chain of n Jónsson terms

A fence is a sequence of vertices $v_0, \dots, v_i$ such that $(v_i, v_{i+1}) \in E(G)$ for i even and $(v_{i+1}, v_i) \in E(G)$ for i odd. Classic Jónsson terms for congruence distributivity [15] arise from a fence P of n solid edges, for some $n \geq 1$, from x to y that starts with a forward edge. The final edge's direction depends on the parity of n; in the picture (Figure 7), n is taken to be odd.

The corresponding condition $M(P)$ is:

$$\begin{aligned} x &\approx t_1(x,x,y) \\ t_i(x,y,y) &\approx t_{i+1}(x,y,y) \quad \text{for } 1 \leq i < n \text{ odd}, \\ t_i(x,x,y) &\approx t_{i+1}(x,x,y) \quad \text{for } 1 \leq i < n \text{ even}, \\ t_n(x,y,y) &\approx y \quad \text{for } n \text{ odd}, \\ t_n(x,x,y) &\approx y \quad \text{for } n \text{ even}, \\ t_i(x,y,x) &\approx x \quad \text{for all } i. \end{aligned}$$

Note that having a single Jónsson term is the same thing as having a majority term.

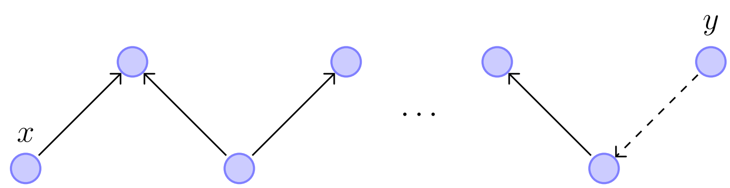

Figure 8: CM($2k$)

3.3.5 Chain of $n+1$ Gumm terms

A variety will be congruence modular if and only if it has a finite sequence of Gumm terms [11]. These terms are similar to Jónsson terms, except that the last edge ($(n+1)$-st in our counting) is a backward dashed edge. We note that our formalism does not capture Day terms [7], another chain of terms that captures congruence modularity, because Day terms have arity four.

Gumm terms are given by the condition $M(P)$, for P the path pictured in Figure 8. We denote this condition by CM(n).

$$\begin{aligned}
x &\approx t_1(x,x,y) \\
t_i(x,y,y) &\approx t_{i+1}(x,y,y) \quad \text{for } 1 \leq i < n \text{ odd}, \\
t_i(x,x,y) &\approx t_{i+1}(x,x,y) \quad \text{for } 1 \leq i < n \text{ even}, \\
t_n(x,y,y) &\approx t_{n+1}(x,y,y) \quad \text{for } n \text{ odd}, \\
t_n(x,x,y) &\approx t_{n+1}(x,y,y) \quad \text{for } n \text{ even}, \\
t_{n+1}(x,x,y) &\approx y \\
t_i(x,y,x) &\approx x \quad \text{for all } i \leq n.
\end{aligned}$$

3.3.6 Chain of n directed Jónsson terms

Directed Jónsson terms are a variation of those presented in Subsection 3.3.4 and also can be used to characterize congruence distributivity for varieties. See [16] for more details about these terms. The condition $M(P)$ that arises from the path pictured in Figure 9 provides a sequence of directed Jónsson

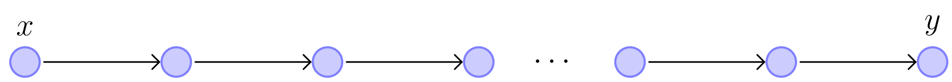

Figure 9: Directed Jónsson chain of length n

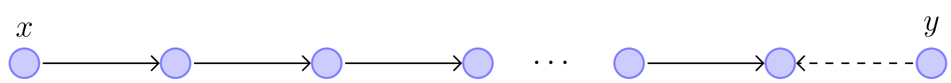

Figure 10: Directed Gumm chain of length $n+1$

terms:

$$\begin{aligned}
x &\approx t_1(x,x,y)\\
t_i(x,y,y) &\approx t_{i+1}(x,x,y) \quad \text{for } 1 \leq i < n\\
t_n(x,y,y) &\approx y\\
t_i(x,y,x) &\approx x \quad \text{for all } i.
\end{aligned}$$

3.3.7 Chain of directed Gumm terms of length $n+1$

In a similar manner, one can consider the directed version of Gumm terms (see [16]). The corresponding path P is pictured in Figure 10 and the Maltsev condition $M(P)$ is given by:

$$\begin{aligned}
x &\approx t_1(x,x,y)\\
t_i(x,y,y) &\approx t_{i+1}(x,x,y) \quad \text{for } 1 \leq i < n\\
t_n(x,y,y) &\approx t_{n+1}(x,y,y)\\
t_{n+1}(x,x,y) &\approx y\\
t_i(x,y,x) &\approx x \quad \text{for all } i.
\end{aligned}$$

3.3.8 Chain of n Hagemann-Mitschke terms

Hagemann-Mitschke terms can be used to characterize varieties that are congruence $(n{+}1)$-permutable, for a given natural number $n \geq 1$ [12]. The strong

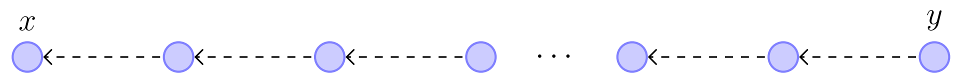

Figure 11: Hagemann-Mitschke terms (n-permutability)

Maltsev condition corresponding to this property is given by $M(P)$ for the path P pictured in Figure 11:

$$\begin{aligned} x &\approx t_1(x, y, y) \\ t_i(x, x, y) &\approx t_{i+1}(x, y, y) \quad \text{for } 1 \leq i < n \\ t_n(x, x, y) &\approx y \end{aligned}$$

3.4 The $\mathrm{HasPath}_P$ property

For a given finite algebra $\mathbf{A}$, we would like to decide if there is a P-shaped walk from x to y in $K(\mathbf{A})$ and hence if $\mathbf{A}$ satisfies the Maltsev condition $M(P)$. It turns out that for each fixed P there is a polynomial time algorithm that decides this question as long as $\mathbf{A}$ is idempotent. In the rest of this paper, we will assume that P is a pattern path from 0 to n with vertex set $\{0, 1, \ldots, n\}$ and that P has no dashed forward edges (for else $M(P)$ would be trivial).

As noted earlier, we regard the free algebra $\mathbf{F}$ as a subuniverse of $\mathbf{A}^{|A|^2}$. Therefore, the labelled graph $K(\mathbf{A})$ is a subuniverse of the $3|A|^2$-th power of $\mathbf{A}$ and most likely too large to be searched directly. Our goal in the following is to approximate P-shaped walks in $K(\mathbf{A})$ using lower powers of $\mathbf{A}$.

One issue with our approach is that we need to approximate the images of different members of P by different subpowers of $\mathbf{A}$. To facilitate this, we will use products.

Consider the pattern digraph $K(\mathbf{A}) \times P$. The vertex set of this pattern digraph consists of $n+1$ sets of the form $K(\mathbf{A}) \times \{i\}$, each of which will be easy to approximate using subpowers of $\mathbf{A}$. Moreover, it is elementary to show that there is a P-shaped walk from x to y in $K(\mathbf{A})$ if and only if there exists a P-shaped walk from $(x, 0)$ to (y, n) in $K(\mathbf{A}) \times P$ (and that each such walk necessarily sends the i-th vertex of P to a pair in $K(\mathbf{A}) \times \{i\}$).

As an example, consider Figure 2. The product in this picture is in fact $J \times P$ for the pattern path P that corresponds to a chain of $2 + 1$ directed Gumm terms and J the pattern digraph that records the generators of $K(\mathbf{A})$.

To obtain $K(\mathbf{A}) \times P$, one has to close the set of edges between $J \times \{i\}$ and $J \times \{i+1\}$ under the operations of $\mathbf{A}$. This gives us a hint on how to approximate $K(\mathbf{A}) \times P$ by smaller digraphs and we formalize this idea in the notion of testing pattern digraphs:

Definition 3. Let $\mathbf{A}$ be an algebra and P a pattern path of length n. Let $(m_0, \ldots, m_n; p_1, \ldots, p_n)$ be a tuple of natural numbers, let

$$(\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) = \hat{a}, \qquad (\overline{c}_1, \ldots, \overline{c}_n) = \hat{c},$$
$$(\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_n) = \hat{b}, \qquad (\overline{d}_1, \ldots, \overline{d}_n) = \hat{d},$$

be tuples of tuples of members of A such that $\overline{a}_i, \overline{b}_i$ are m_i-ary and $\overline{c}_i, \overline{d}_i$ are p_i-ary for all applicable values of i.

The *testing* $(m_0, \ldots, m_n; p_1, \ldots, p_n)$*-ary pattern digraph for* $\mathbf{A}$ *and* P *generated by* $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, denoted by $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, has vertex set consisting of a disjoint union of sets (which, after Barto and Kozik [1], we will call potatoes) $B_i = \mathrm{Sg}^{\mathbf{A}^{m_i}}(\{\overline{a}_i, \overline{b}_i\})$, where $i = 0, 1, \ldots, n$. The vertices $\overline{a}_0$ and $\overline{b}_n$ are the initial and terminal vertices, respectively, of $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$.

To obtain the edge set of $G = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, first generate, for $i = 1, \ldots, n$, the labeled edge sets E_i as follows:

- If the i-th edge of P is a forward edge, then

$$E_i = \mathrm{Sg}^{\mathbf{A}^{p_i+m_{i-1}+m_i}}\left(\left\{\begin{pmatrix}\overline{c}_i\\ \overline{a}_{i-1}\\ \overline{a}_i\end{pmatrix}, \begin{pmatrix}\overline{d}_i\\ \overline{a}_{i-1}\\ \overline{b}_i\end{pmatrix}, \begin{pmatrix}\overline{c}_i\\ \overline{b}_{i-1}\\ \overline{b}_i\end{pmatrix}\right\}\right) \leq \mathbf{A}^{p_i} \times B_{i-1} \times B_i$$

- If the i-th edge of P is a backward edge, then swap B_{i-1} and B_i (as well as the corresponding generators) in the above definition, i.e.

$$E_i = \mathrm{Sg}^{\mathbf{A}^{p_i+m_i+m_{i-1}}}\left(\left\{\begin{pmatrix}\overline{c}_i\\ \overline{a}_i\\ \overline{a}_{i-1}\end{pmatrix}, \begin{pmatrix}\overline{d}_i\\ \overline{a}_i\\ \overline{b}_{i-1}\end{pmatrix}, \begin{pmatrix}\overline{c}_i\\ \overline{b}_i\\ \overline{b}_{i-1}\end{pmatrix}\right\}\right) \leq \mathbf{A}^{p_i} \times B_i \times B_{i-1}$$

To obtain the edges of G, we translate all members of $\bigcup_{i=1}^n E_i$ into either solid or dashed edges: Given $(\overline{e}, \overline{f}, \overline{g}) \in E_i$, we place into G an edge from $\overline{f}$ to $\overline{g}$. If $\overline{e} = \overline{c}_i$ and the i-th edge of P is solid, the new edge of G is solid, otherwise the new edge is dashed. (As a consequence, the values of $p_i, \overline{c}_i, \overline{d}_i$ only matter when the i-th edge of P is solid. When the i-th edge is dashed, we will nonetheless keep these dummy parameters to make the notation simpler.)

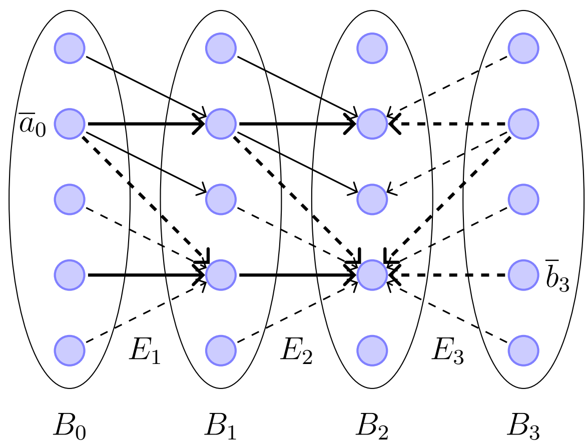

Figure 12: An example of a testing pattern digraph G with the generating set drawn thick. The path P is the same as in Figure 2.

See Figure 12 for an example of a testing pattern digraph G (and its generating set) in the case when P encodes a chain of $2+1$ Gumm terms.

Note that in $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ the generators of all E_i's form a pattern digraph isomorphic to $J \times P$ (where J is the pattern digraph in Figure 3). This will be important later in Lemma 13.

Observation 4. *Let* $\mathbf{A}$ *be an idempotent algebra,* $\overline{m}, \overline{p}$ *be tuples of positive integers and* $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *be an* $(\overline{m}, \overline{p})$*-ary testing pattern digraph. Then:*

(a) the set of edges between B_{i-1} *and* B_i *forms a subdirect relation,*

(b) if the i*-th edge of* P *is solid then the set of solid edges between* B_{i-1} *and* B_i *is also a subdirect relation,*

(c) if the i*-th edge of* P *is a forward edge then* $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *contains an edge from any* $\overline{p} \in B_{i-1}$ *to* $\overline{b}_i$*, and*

(d) if the i*-th edge of* P *is a backward edge then* $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *contains an edge from any* $\overline{p} \in B_i$ *to* $\overline{b}_{i-1}$*.*

Proof. To see the first two claims, observe that the projection of each E_i to $A^{p_i} \times B_i$ contains the tuples $(\overline{c}_i, \overline{a}_i), (\overline{c}_i, \overline{b}_i)$ and $B_i = \mathrm{Sg}(\{\overline{a}_i, \overline{b}_i\})$ (the situation for B_{i-1} is similar).

To prove the third claim, observe that the projection of E_i to $B_{i-1} \times B_i$ contains the tuples $(\overline{a}_{i-1}, \overline{b}_i)$ and $(\overline{b}_{i-1}, \overline{b}_i)$. Since $\mathbf{A}$ is idempotent and $B_{i-1} = \mathrm{Sg}(\{\overline{a}_{i-1}, \overline{b}_{i-1}\})$, the projection of E_i to $B_{i-1} \times B_i$ contains $B_{i-1} \times \{\overline{b}_i\}$. The proof of the last claim is similar. $\square$

A notable example of a testing pattern digraph is $K(\mathbf{A}) \times P$ itself. Continuing with our representation of $\mathbf{F}$ as a subuniverse of $\mathbf{A}^{|A|^2}$, with the free generators x and y equal to the binary projection maps π_0 and π_1 respectively, we have that $K(\mathbf{A}) \times P$ is isomorphic to the $(\overline{|A|^2}; \overline{|A|^2})$-ary testing pattern digraph $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, where $\hat{a} = (x, x, \ldots, x)$, $\hat{b} = (y, y, \ldots, y)$, $\hat{c} = (x, \ldots, x)$, and $\hat{d} = (y, \ldots, y)$. Here, and in the following, $\overline{|A|^2}$ denotes a constant tuple or sequence of an appropriate length with value $|A|^2$ at each entry. To see this claim, note that all of the potatoes $B_0, B_1, \ldots, B_n$ will be equal to F and each E_i is either $K(\mathbf{A})$ or $K(\mathbf{A})$ with its second and third coordinates swapped.

Definition 5. We say that an algebra $\mathbf{A}$ satisfies the condition $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ if whenever G is an $(\overline{m}; \overline{p})$-ary testing pattern digraph for $\mathbf{A}$ and P, there is a P-shaped walk from the initial to the terminal vertex in G.

For example, the G in Figure 12 fails to have a P-shaped walk from $\overline{a}_0$ to $\overline{b}_3$.

The next observation connects the $\mathrm{HasPath}_P$ property with the Maltsev condition $M(P)$.

Observation 6. *Let P be a pattern path of length n. Then the following are equivalent:*

1. *Algebra $\mathbf{A}$ satisfies $M(P)$*
2. *There is a P-shaped walk from x to y in $K(\mathbf{A})$*
3. *There is a P-shaped walk from $(x, 0)$ to (y, n) in $K(\mathbf{A}) \times P$*
4. *Algebra $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ for all choices of tuples $\overline{m}$ and $\overline{p}$.*
5. *Algebra $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(\overline{|A|^2}; \overline{|A|^2})$.*

Proof. The proof is easy once we unpack the definitions. We already know that the first three items are equivalent. Trivially, (4) $\Rightarrow$ (5).

We show that (1) $\Rightarrow$ (4) as follows: Take a testing pattern digraph $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$. Since $\mathbf{A}$ satisfies $M(P)$, there are binary and ternary terms

$s_0, \ldots, s_n$ and $t_1, \ldots, t_n$ that satisfy the equations in $M(P)$. Comparing the equations in $M(P)$ with the generators of E_i, it is straightforward to verify that the sequence $h_0, h_1, \ldots, h_n$ defined by $h_i = s_i(\overline{a}_i, \overline{b}_i)$ is a P-shaped walk from $\overline{a}_0$ to $\overline{b}_n$.

To see that (5) $\Rightarrow$ (3), recall that $K(\mathbf{A}) \times P$ can be regarded as an $(\overline{|A|^2}; \overline{|A|^2})$-ary testing pattern digraph. The property $\mathrm{HasPath}_P(\overline{|A|^2}; \overline{|A|^2})$ then immediately gives us a P-shaped walk from $(x, 0)$ to (y, n) in $K(\mathbf{A}) \times P$. $\square$

It turns out that if $\mathbf{A}$ is idempotent, then the minimum arity of instances that we need to check to determine if $\mathbf{A}$ satisfies $M(P)$ is merely $(1, \ldots, 1; 1, \ldots, 1)$ rather than $(\overline{|A|^2}; \overline{|A|^2})$.

Theorem 7. *For a finite idempotent algebra* $\mathbf{A}$ *and pattern path* P *there is a* P*-shaped walk from* x *to* y *in the pattern digraph* $K(\mathbf{A})$ *if and only if* $\mathbf{A}$ *satisfies* $\mathrm{HasPath}_P(1, \ldots, 1; 1, \ldots, 1)$.

The proof of this theorem will be the goal of the next section. Before we start proving it, though, let us remark on its significance. The condition $\mathrm{HasPath}_P(1, \ldots, 1; 1, \ldots, 1)$ asserts that we can always satisfy the Maltsev condition $M(P)$ locally, and this local satisfiability is something that one can verify in time polynomial in $\|\mathbf{A}\|$, where $\|\mathbf{A}\|$ is a measure of the size of the algebra $\mathbf{A}$. To make this definite, we use the measure from [9]:

$$\|\mathbf{A}\| = \sum_{i=0}^{r} k_i |A|^i,$$

where r is the largest arity of the basic operations of $\mathbf{A}$ and k_i is the number of basic operations of $\mathbf{A}$ of arity i, for $0 \leq i \leq r$. In our analysis below, we will assume that $\mathbf{A}$ has at least one at least unary operation (nontrivial idempotent algebras can't have constant operations) and hence $\|\mathbf{A}\| \geq |A|$.

Let us now fix a pattern path P of length n with k solid edges (and $n-k$ dashed edges). To test if an algebra $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(1, \ldots, 1; 1, \ldots, 1)$, we need to examine all $(1, \ldots, 1; 1, \ldots, 1)$-ary testing pattern digraphs and check them for P-shaped walks from the initial to the terminal vertex.

Definition 3 gives us an algorithmic procedure to generate these digraphs: For each possible combination of values $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, generate the sets B_i, E_i and translate them into edges of $G = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$. Moreover, for the $n-k$ dashed edges of P, the choice of labels c_i, d_i has no effect on G and we only need to calculate the second and third coordinates of E_i. Omitting these dummy labels, there are $|A|^{2(n+1)+2k} = |A|^{2n+2k+2}$ many tuples $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ to consider.

Using Proposition 6.1 of [9] it follows that, given $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, the graph $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ can be constructed in time

$$O((n+1)r\|\mathbf{A}\| + kr\|\mathbf{A}\|^3 + (n-k)r\|\mathbf{A}\|^2) = O(kr\|\mathbf{A}\|^3 + (n-k)r\|\mathbf{A}\|^2).$$

Since k is fixed, if $k = 0$ the asymptotic simplifies to $O(r\|\mathbf{A}\|^2)$ and if $k > 0$ the asymptotic is $O(r\|\mathbf{A}\|^3)$.

Any given testing pattern digraph G has $O(n|A|)$ vertices and $O(n|A|^2)$ edges (for a given pair of vertices, we need to only remember the "best" edge, where solid is better than dashed is better than none), testing for a P-shaped walk from a_0 to b_n can be done by standard methods in time $O(n|A|^2)$ which is negligible compared to the time needed to generate G. All in all, deciding if $\mathrm{HasPath}_P(1, \ldots, 1; 1, \ldots, 1)$ holds can be carried out by an algorithm whose run-time is $O(r|A|^{2n+2}\|\mathbf{A}\|^2)$ for $k = 0$ and $O(r|A|^{2n+2k+2}\|\mathbf{A}\|^3)$ for $k > 0$.

Corollary 8. *Let P be a fixed pattern path. The associated idempotent, strong, linear Maltsev condition $M(P)$ can be decided for a finite idempotent algebra $\mathbf{A}$ by an algorithm whose run-time can be bounded by a polynomial in $\|\mathbf{A}\|$.*

Using Theorem 7 and referring to the list of examples from the example gallery, we immediately obtain polynomial-time algorithms for deciding some well known strong Maltsev conditions for finite idempotent algebras.

Corollary 9. *Let $n \geq 1$. Each of the following strong Maltsev conditions can be decided for a finite idempotent algebra $\mathbf{A}$ by a polynomial-time algorithm with the prescribed run-time.*

1. *Having a sequence of n Jónsson terms, directed or not, can be decided in time $O(r|A|^{4n+2}\|\mathbf{A}\|^3)$.*
2. *Having a sequence of $n+1$ Gumm terms, directed or not, can be decided in time $O(r|A|^{4n+2}\|\mathbf{A}\|^3)$.*
3. *[21] Having a sequence of n Hagemann-Mitschke terms can be decided in time $O(r|A|^{2n+2}\|\mathbf{A}\|^2)$.*
4. *[9] Having a Maltsev term can be decided in time $O(r|A|^4\|\mathbf{A}\|^2)$.*
5. *[9] Having a majority term can be decided in time $O(r|A|^6\|\mathbf{A}\|^3)$.*

4 From local to global

Let $\mathbf{A}$ be a finite idempotent algebra that satisfies the condition

$$\mathrm{HasPath}_P(1,1,\dots,1;1,\dots,1).$$

Our goal is to increase all of the parameters of $\mathrm{HasPath}_P$, eventually establishing that $\mathrm{HasPath}_P(\overline{|A|^2};\overline{|A|^2})$ holds for $\mathbf{A}$ and thereby proving, using Observation 6, that $\mathbf{A}$ satisfies $M(P)$.

First observe that by adding dummy coordinates, we can always decrease the parameters of $\mathrm{HasPath}_P$:

Observation 10. *If $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(\overline{m};\overline{p})$, $m'_i \leq m_i$, for $0 \leq i \leq n$, and $p'_i \leq p_i$, for $1 \leq i \leq n$, then $\mathbf{A}$ also satisfies $\mathrm{HasPath}_P(\overline{m}';\overline{p}')$.*

Definition 11. Let $\mathbf{A}$ be an algebra, P a pattern path and $\overline{m},\overline{p}$ be tuples of positive integers. The $(\overline{m};\overline{p})$-ary testing pattern digraph $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ is *minimal* if no proper subgraph of $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ is isomorphic to another $(\overline{m};\overline{p})$-ary testing pattern digraph for $\mathbf{A}$ and P.

It follows that every testing pattern digraph contains a subgraph isomorphic to a minimal testing pattern digraph. Since isomorphism respects initial and terminal vertices and removing edges or vertices can't create new P-shaped walks, it is enough to look at minimal testing pattern digraphs.

Observation 12. *Let $\mathbf{A}$ be an algebra and $\overline{m},\overline{p}$ be tuples of values. If every minimal $(\overline{m};\overline{p})$-ary testing pattern digraph $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ has a P-shaped walk from the initial to the terminal vertex, then $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(\overline{m};\overline{p})$.*

The following two lemmas allow us to comfortably handle minimal testing pattern digraphs.

Lemma 13. *If $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ is a minimal testing pattern digraph and g is a $J \times P$-shaped walk from $\overline{a}_0$ to $\overline{b}_n$ in $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ then $\mathrm{TPG}(\hat{a},\hat{b},\hat{c},\hat{d})$ is equal to the pattern digraph $\mathrm{TPG}(\hat{a}',\hat{b}',\hat{c},\hat{d}')$ where we retain the labels $\hat{c}$, let $\overline{a}'_i = g(x,i), \overline{b}'_i = g(y,i)$, and choose $\overline{d}'_i$ so that $(\overline{d}'_i, g(x,i-1), g(y,i)) \in E_i$ (if the i-th edge of P is a forward edge) or $(\overline{d}'_i, g(x,i), g(y,i-1)) \in E_i$ (if the i-th edge of P is a backward edge).*

Proof. Let E'_i be the i-th labelled edge set of $\mathrm{TPG}(\hat{a}',\hat{b}',\hat{c},\hat{d}')$. By the choice of generators, we immediately have $E'_i \subseteq E_i$ and hence $\mathrm{TPG}(\hat{a}',\hat{b}',\hat{c},\hat{d}')$ is a subgraph of $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$. By construction, the testing pattern digraph $\mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$ has the same initial and terminal vertices as $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and so by the minimality of $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ we conclude that these two testing patterns are equal. $\square$

The following Lemma will help us to prove Lemmas 16, 17, and 18 by induction on the arity of testing pattern digraphs. The hypothesis of the Lemma is a bit long, but it describes something quite natural: Given a testing pattern digraph G^0, we can expand the tuples generating G^0 to get a more complicated testing pattern digraph G. It now turns out that we can lift any P-shaped path in G^0 to an "almost path" in G.

Lemma 14 (Partial lifting)**.** *Let $\mathbf{A}$ be an algebra and P be a pattern path of length n. Let $G^0 = \mathrm{TPG}(\hat{a}^0, \hat{b}^0, \hat{c}^0, \hat{d}^0)$ be an $(\overline{m}; \overline{p})$-ary testing pattern digraph. Let $(\overline{m}'; \overline{p}')$ be such that $m_i' \geq m_i$ for each $i \in \{0, 1, \ldots, n\}$ and $p_i' \geq p_i$ for each $i = 1, \ldots, n$. Let $G = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ be any $(\overline{m}'; \overline{p}')$-ary testing pattern digraph so that for each applicable i the tuple $\overline{a}_i^0$ is a prefix of $\overline{a}_i$, $\overline{b}_i^0$ is a prefix of $\overline{b}_i$, $\overline{c}_i^0$ is a prefix of $\overline{c}_i$, and $\overline{d}_i^0$ is a prefix of $\overline{d}_i$.*

Let p be a P-shaped path from $\overline{a}_0^0$ to $\overline{b}_n^0$ in G^0. Denote by $\overline{e}_1, \ldots, \overline{e}_n$ the labels of the edges of p. Then there exist tuples $\overline{u}_0, \ldots, \overline{u}_n$, $\overline{w}_0, \ldots, \overline{w}_n$ and $\overline{f}_1, \ldots, \overline{f}_n$ so that

1. *for each $i = 0, 1, \ldots, n$, the tuple $p(i)$ is a prefix of both $\overline{u}_i$ and $\overline{w}_i$,*
2. *for each $i = 1, 2, \ldots, n$, the tuple $\overline{e}_i$ is a prefix of $\overline{f}_i$,*
3. *$\overline{w}_0 = \overline{a}_0$ and $\overline{u}_n = \overline{b}_n$,*
4. *if the i-th edge of P is a forward edge, then $(\overline{f}_i, \overline{u}_{i-1}, \overline{w}_i) \in E_i$ (where E_i is the i-th edge relation of G); if the i-the edge of P is a backward edge then $(\overline{f}_i, \overline{w}_i, \overline{u}_{i-1}) \in E_i$,*
5. *if $i \in \{0, 1, \ldots, n\}$ is such that the last $m_i' - m_i$ coordinates of $\overline{a}_i$ and $\overline{b}_i$ agree, then $\overline{u}_i = \overline{w}_i$,*
6. *if $i \in [n]$ is such that the last $p_i' - p_i$ coordinates of $\overline{c}_i$ and $\overline{d}_i$ agree and the i-th edge is solid then $\overline{f}_i = \overline{c}_i$.*

Proof. Let us examine the i-th edge of p. Without loss of generality assume that the i-th edge is a forward edge (the case of backward edges is similar).

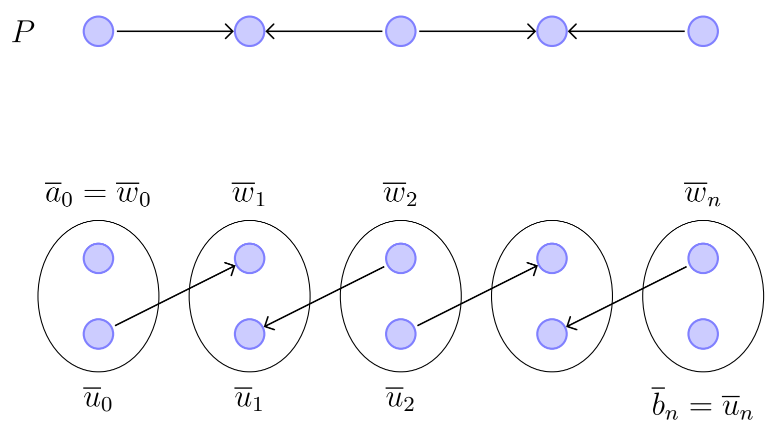

Figure 13: The "almost path" we obtain by lifting the a P-shaped path for the P above. Edge labels omitted for simplicity.

In order for p to be a P-shaped path in G, we have $(\overline{e}_i, p(i-1), p(i)) \in E_i^0$ (where E_i^0 is the i-th edge relation of G^0). Since E_i^0 is a subpower of $\mathbf{A}$ generated by the three tuples

$$(\overline{c}_i^0, \overline{a}_{i-1}^0, \overline{a}_i^0),\ (\overline{d}_i^0, \overline{a}_{i-1}^0, \overline{b}_i^0),\ (\overline{c}_i^0, \overline{b}_{i-1}^0, \overline{b}_i^0),$$

there exists a ternary term operation t_i of $\mathbf{A}$ such that

$$\begin{aligned} \overline{e}_i &= t_i(\overline{c}_i^0, \overline{d}_i^0, \overline{c}_i^0) \\ p(i-1) &= t_i(\overline{a}_{i-1}^0, \overline{a}_{i-1}^0, \overline{b}_{i-1}^0) \\ p(i) &= t_i(\overline{a}_i^0, \overline{b}_i^0, \overline{b}_i^0). \end{aligned}$$

We obtain $\overline{f}_i$, $\overline{u}_{i-1}$, and $\overline{w}_i$ by extending the input tuples (i.e. removing the zero superscripts):

$$\begin{aligned} \overline{f}_i &= t_i(\overline{c}_i, \overline{d}_i, \overline{c}_i) \\ \overline{u}_{i-1} &= t_i(\overline{a}_{i-1}, \overline{a}_{i-1}, \overline{b}_{i-1}) \\ \overline{w}_i &= t_i(\overline{a}_i, \overline{b}_i, \overline{b}_i). \end{aligned}$$

This procedure works for $i = 1, 2, \ldots, n$, so it remains to define $\overline{w}_0 = \overline{a}_0$ and $\overline{u}_n = \overline{b}_n$ (see Figure 13).

The conclusions of the lemma all follow from the way that the relations B_i and E_i are generated. The first two points are consequences of t_i acting coordinatewise. The third claim follows from the way we defined $\overline{w}_0$ and $\overline{u}_n$. Since E_i is invariant under t_i, we have $(\overline{f}_i, \overline{u}_{i-1}, \overline{w}_i) \in E_i$, proving the fourth point (the situation for backward edges is similar). Finally, the operation t_i is idempotent, so if, say, the last $m'_i - m_i$ coordinates of $\overline{a}_i$ and $\overline{b}_i$ are both equal to some tuple $\overline{q}$, then $\overline{u}_i = p(i)\overline{q} = \overline{w}_i$; the case of edge labels is similar. $\square$

Observe that Lemma 14 also holds if we replace prefixes by suffixes everywhere; we will sometimes use it in this way.

We will only need Lemma 14 in the case when $\overline{m}', \overline{p}'$ differ from $\overline{m}, \overline{p}$ in at most one position (either $m'_k > m_k$ or $p'_k > p_k$). Then $m'_i - m_i = 0$ and $p'_i - p_i = 0$ for almost all i. Therefore, the last two claims of Lemma 14 give us that $\overline{w}_0, \overline{w}_1, \dots, \overline{w}_k, \overline{u}_{k+1}, \dots, u_n$ fails to be a P-shaped path only because either $\overline{w}_k \neq \overline{u}_k$, or $\overline{f}_k \neq \overline{c}_k$ (if the k-th edge is solid).

Lemma 15. *Let* $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *be an* $(\overline{m}; \overline{p})$*-ary testing pattern digraph for* $\mathbf{A}$ *and* P *for some tuples* $\overline{m}$ *and* $\overline{p}$*. Let* Q *be a prefix of* P *and* S *a suffix of* P *(possibly overlapping with* Q*). Then:*

(a) Every Q*-shaped walk* $q\colon Q \to \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *from* $\overline{a}_0$ *to some* $\overline{e}_k \in B_k$ *can be extended to a* $J \times P$*-shaped walk from* $\overline{a}_0$ *to* $\overline{b}_n$*. (Extending* q *means that the image of* (x, i) *is* $q(i)$ *for all* $i = 0, 1, \dots, k$*.)*

(b) Every S*-shaped walk* $s\colon S \to \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *from some* $\overline{f}_\ell \in B_\ell$ *to* $\overline{b}_n$ *can be extended to a* $J \times P$*-shaped walk from* $\overline{a}_0$ *to* $\overline{b}_n$*. (Extending* s *means that the image of* (y, i) *is* $s(i)$ *for all* $i = \ell, \dots, n$*.)*

(c) If $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *is minimal, then for every* Q*-shaped walk* q *and* S*-shaped walk* s *as in the previous two parts there exists a* $J \times P$*-shaped walk from* $\overline{a}_0$ *to* $\overline{b}_n$ *in* $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ *that extends both* q *and* s *at the same time (see Figure 14).*

Proof. We begin with part (a). By Observation 4, we have that the set of (solid) edges between B_i and B_{i+1} is subdirect for each $i < n$. This allows the map q to be extended to a P-shaped walk q' from $\overline{a}_0$ to some $q'(n) \in B_n$. We claim that g defined by $g(x, i) = q'(i)$ and $g(y, i) = \overline{b}_i$ for all $0 \leq i \leq n$ is a $J \times P$-shaped walk from $\overline{a}_0$ to $\overline{b}_n$.

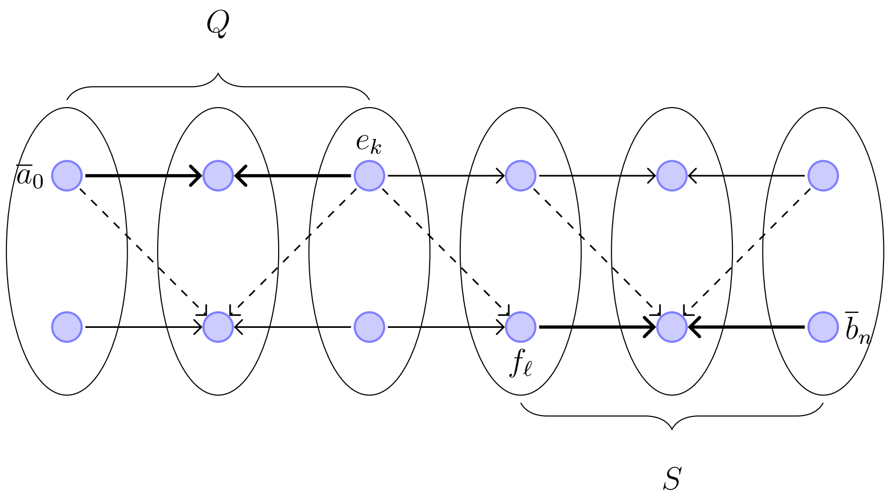

Figure 14: The situation in part (c) of Lemma 15. The arrows show the images of $J \times P$ under the common extension of q and s. Images of the paths Q and S are drawn in bold.

To prove this, let $0 \le i < n$ and consider the edge in P between i and $i+1$. We will examine in detail one of the three possibilities for this edge (recall that we disallow forward dashed edges) and leave the other two cases for the reader. Suppose that there is a solid edge from i to $i+1$ in P. We need to prove that $(g(x,i), g(x,i+1)) = (q'(i), q'(i+1))$, $(g(y,i), g(y,i+1)) = (\overline{b}_i, \overline{b}_{i+1})$, and $(g(x,i), g(y,i+1)) = (q'(i), \overline{b}_{i+1})$ are all edges in $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and that the first two are solid. The first two can be seen to be solid, since q' is a pattern digraph homomorphism and $(\overline{c}_i, \overline{b}_i, \overline{b}_{i+1}) \in E_i$, while the existence of the edge $(q'(i), \overline{b}_{i+1})$ follows from part (c) of Observation 4.

The proof of part (b) is similar. Note that our construction for this part is such that the image of each (x, i) in the resulting $J \times P$-shaped walk is exactly $\overline{a}_i$. We will use this in the next paragraph.

We prove part (c) by applying parts (a) and (b) in turn: First get a $J \times P$-shaped walk g from $\overline{a}_0$ to $\overline{b}_n$ that extends q. Using Lemma 13, we get that $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d}) = \mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$, where $\overline{a}'_i = g(x,i)$ for $i = 0, 1, \ldots, k$. Now apply part (b) that we have just proved to $\mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$ and s to obtain a $J \times P$-shaped walk from $\overline{a}_0$ to $\overline{b}_n$ that sends each (i, x) to $\overline{a}'_i = g(i, x)$ and hence extends q and s at the same time. $\square$

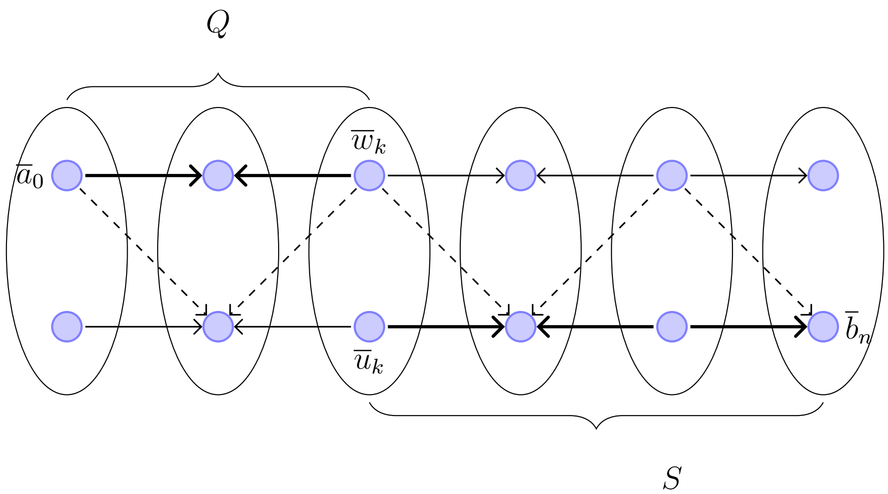

Figure 15: The situation in Lemma 16. The arrows show the image of u, while the thick arrows are images of q and s. Note that $\overline{w}_k$ and $\overline{u}_k$ differ in at most one coordinate.

Lemma 16. *Suppose that* $\mathbf{A}$ *satisfies* $\mathrm{HasPath}_P(\overline{m};\overline{p})$ *for some tuples* $\overline{m}$ *and* $\overline{p}$ *and let* $0 \le k \le n$. *Then* $\mathbf{A}$ *also satisfies* $\mathrm{HasPath}_P(\overline{m}^+;\overline{p})$, *where* $\overline{m}^+$ *is obtained from* $\overline{m}$ *by increasing* m_k *by 1.*

Proof. Suppose that $G = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ is an $(\overline{m}^+;\overline{p})$-ary testing pattern digraph for $\mathbf{A}$ and P (all the other testing pattern digraphs we will consider in this proof are also for $\mathbf{A}$ and P). We need to construct a P-shaped walk from $\overline{a}_0$ to $\overline{b}_n$. By Observation 12 we may assume that G is minimal.

Let Q be the prefix of P of length k and let S be the suffix of P of length $n-k$. The vertex sets for Q and S are $\{0, 1, \ldots, k\}$ and $\{k, k+1, \ldots, n\}$ respectively. By forgetting the (m_k+1)-th coordinate of B_k we obtain new label sets $\hat{a}^0$ and $\hat{b}^0$. Consider the $(\overline{m};\overline{p})$-ary testing pattern digraph $G^0 = \mathrm{TPG}(\hat{a}^0, \hat{b}^0, \hat{c}, \hat{d})$.

Applying $\mathrm{HasPath}_P(\overline{m};\overline{p})$ to G^0 we obtain a P-shaped path p from the initial to the terminal vertex of G^0. Now apply Lemma 14 to G^0 and G. Since G^0 and G only differ in the k-th potato, we see that in G there exists a Q-shaped walk q from $\overline{a}_0$ to some $\overline{w}_k \in B_k$ and an S-shaped walk s from a suitable $\overline{u}_k \in B_k$ to $\overline{b}_n$, where $\overline{u}_k$ and $\overline{w}_k$ differ only in the last coordinate (see Figure 15).

Now, using Lemma 15 on G and paths p, q, and s, we obtain that we can extend both q and s to a $J \times P$-shaped walk α from $\overline{a}_0$ to $\overline{b}_n$. We now apply Lemma 13 to G and α to show that G is equal to the testing pattern digraph $G' = \mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$. Examining Lemma 13 in detail, we see that $\overline{a}'_k = \alpha(x, k) = \overline{w}_k$ and $\overline{b}'_k = \alpha(y, k) = \overline{u}_k$.

If we could show that there is a P-shaped path from $\overline{a}_0$ to $\overline{b}_n$ in G', we would be done. To that end, define $\overline{m}^-$ to be equal to $\overline{m}$ everywhere except at the k-th place, where $m^-_k = 1$, and consider the testing pattern digraph $G^- = \mathrm{TPG}(\hat{a}^-, \hat{b}^-, \hat{c}, \hat{d}')$ of arity $(\overline{m}^-, \overline{p})$ and with $\hat{a}^-, \hat{b}^-$ equal to $\hat{a}', \hat{b}'$ everywhere except at the k-th position, where we let $\overline{a}^-$ and $\overline{b}^-$ equal to the last entry of $\overline{w}_k$ and $\overline{u}_k$, respectively. Since $m^-_k \leq m_k$, we can apply $\mathrm{HasPath}_P(\overline{m}^-, \overline{p})$ to G^-. We next apply Lemma 14 to G^- and G', where we switch from prefixes to suffixes. Noting that the first m_k entries of $\overline{a}'_k$ and $\overline{b}'_k$ agree, we see that the sequence of vertices produced by Lemma 14 is in fact a P-shaped path from $\overline{a}'_0 = \overline{a}_0$ to $\overline{b}'_n = \overline{b}_n$ in $G' = G$, concluding the proof. □

Lemma 17. *Suppose that* $\mathbf{A}$ *satisfies* $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ *for some tuples* $\overline{m}$ *and* $\overline{p}$ *and let* $1 \leq k \leq n$. *If the* k*-th edge of* P *is a solid forward edge then* $\mathbf{A}$ *also satisfies* $\mathrm{HasPath}_P(\overline{m}; \overline{p}^+)$, *where* $\overline{p}^+$ *is obtained from* $\overline{p}$ *by increasing* p_k *by* 1.

Proof. The proof of this lemma is similar to that of Lemma 16. Again, all testing pattern digraphs will be for $\mathbf{A}$ and P. We start with a minimal $(\overline{m}; \overline{p}^+)$-ary testing pattern digraph $G = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and will prove that there is a P-shaped walk from $\overline{a}_0$ to $\overline{b}_n$ in G. We set Q to be the prefix of P with the vertex set $\{0, 1, \ldots, k-1\}$ and S to be the suffix of P with the vertex set $\{k, k+1, \ldots, n\}$. Take the $(\overline{m}; \overline{p})$-ary pattern digraph $G^0 = \mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}^0, \hat{d}^0)$ where we get $\hat{c}^0$ and $\hat{d}^0$ from $\hat{c}$ and $\hat{d}$ by forgetting the last entry of $\overline{c}_k$ and $\overline{d}_k$, respectively.

As before, $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ and Lemma 14 give us that there exists an "almost P-shaped" path from $\overline{a}_0$ to $\overline{b}_n$ in G. This time, we have a Q-shaped walk q from $\overline{a}_0$ to some $q(k-1)$ and an S-shaped walk s from $s(k)$ to $\overline{b}_n$ such that E_k contains an edge of the form $(\overline{f}_k, q(k-1), s(k))$, where $\overline{f}_k$ comes from Lemma 14. Were it $\overline{f}_k = \overline{c}_k$, we would have a solid edge from $q(k-1)$ to $s(k)$ and we would be done. Alas, it could happen that $\overline{f}_k$ and $\overline{c}_k$ differ in the last coordinate.

We again use part (c) of Lemma 15 to obtain a $J \times P$-shaped walk from

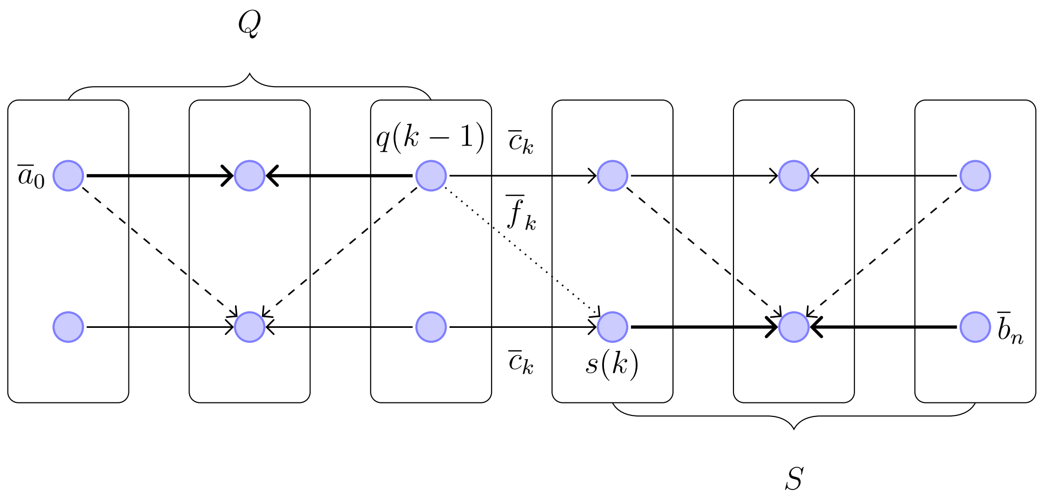

Figure 16: The situation in Lemma 17. As before, the images of q and s are in bold. The labels $\overline{e}f$ and $\overline{e}g$ refer to the labels of the middle edges.

$\overline{a}_0$ to $\overline{b}_n$ in G that extends q and s. Since we have $(\overline{f}_k, q(k-1), s(k)) \in E_k$, Lemma 13 gives us that G is equal to $G' = \mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$ for suitable $\hat{a}'$, $\hat{b}'$, and $\hat{d}'$, with $\overline{d}'_k = \overline{f}_k$. By forgetting the first p_k coordinates of the labels of E_k we obtain a $(\overline{m}; \overline{p}^-)$-ary testing pattern digraph G^-, where $\overline{p}^-$ is obtained from $\overline{p}$ by replacing p_k by 1. $\mathrm{HasPath}_P(\overline{m}; p_1, \ldots, p_{k-1}, 1, p_{k+1}, \ldots, p_n)$ can be applied to G^- to produce a P-shaped walk from the initial to the terminal vertex of G. Using Lemma 14 for suffixes instead of prefixes, we can lift this P-shaped walk to $G' = G$ and are done. $\square$

The case when the k-th edge is a backward solid edge will start out similarly to Lemma 17, but we will need an additional trick.

Lemma 18. *Suppose that* $\mathbf{A}$ *satisfies* $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ *for some tuples* $\overline{m}$ *and* $\overline{p}$ *and let* $1 \leq k \leq n$. *If the* k*-th edge of* P *is a solid backward edge then* $\mathbf{A}$ *also satisfies* $\mathrm{HasPath}_P(\overline{m}; \overline{p}^+)$, *where* $\overline{p}^+$ *is obtained from* $\overline{p}$ *by increasing* p_k *by 1.*

Proof. Take G, G^0, Q, and S as in the proof of Lemma 17. As before we apply $\mathrm{HasPath}_P(\overline{m}; \overline{p})$ to G^0 and lift the result to G via Lemma 14. We get a Q-shaped walk q from $\overline{a}_0$ and an S-shaped walk s to $\overline{b}_n$ such that

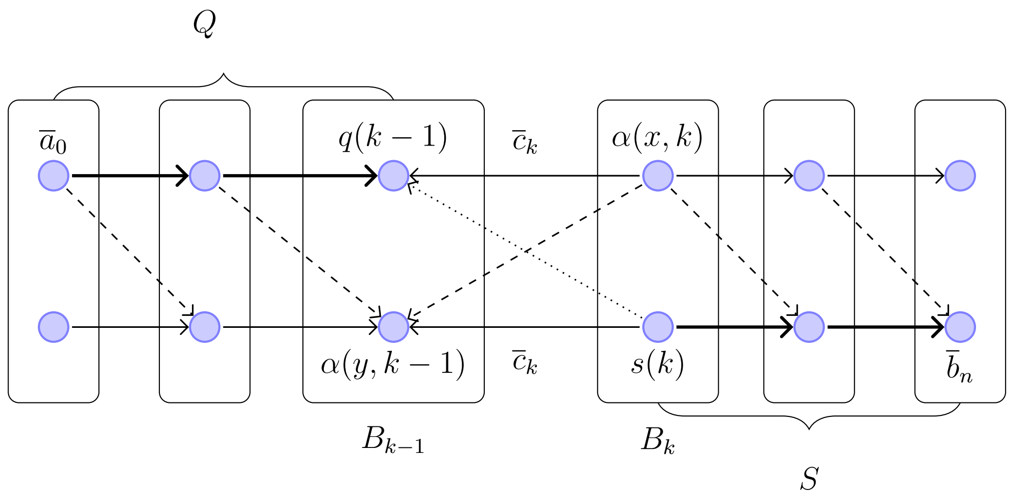

Figure 17: The situation in the first paragraph of the proof of Lemma 18. As before, the images of q and s are in bold. The label $\overline{f}_k$ is depicted as a dotted edge.

$(\overline{f}_k, s(k), q(k-1)) \in E_k$. Part (c) of Lemma 15 again gives us a $J \times P$-shaped walk α in $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ that extends q and s. Applying Lemma 13, we again show that G is equal to $G' = \mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}, \hat{d}')$ where $\overline{a}'_i = \alpha(x,i)$ and $\overline{b}'_i = \alpha(y,i)$ for each i. Unfortunately, copying the proof of Lemma 17 fails at this point. The problem is that the edge $(\overline{f}_k, s(k), q(k-1))$ has the wrong endpoints; we would need to have $(\overline{f}_k, \alpha(x,k), \alpha(y,k-1)) \in E_k$ to finish as we did in Lemma 17. We recover by studying an auxiliary relation.

Let us choose $\overline{e} \in A^{p_k}$ and $f, c \in A$ so that $\overline{f}_k = \overline{e}f$ and $\overline{c}_k = \overline{e}c$. Define the new relation $R \subseteq A \times B_k \times B_{k-1}$ as follows:

$$R = \{(x, \overline{y}, \overline{z}) \colon \exists \overline{t} \in B_{k-1}, \overline{v} \in B_k,\ (\overline{c}_k, \overline{y}, \overline{t}), (\overline{e}x, \overline{v}, \overline{t}), (\overline{c}_k, \overline{v}, \overline{z}) \in E_k\}.$$

Since $\mathbf{A}$ is idempotent, the relation R is a subuniverse of a power of $\mathbf{A}$. Moreover, R contains the tuples $(c, \alpha(x,k), \alpha(x,k-1))$, $(c, \alpha(y,k), \alpha(y,k-1))$, and $(f, \alpha(x,k), \alpha(y,k-1))$ (the last tuple is witnessed by $(\overline{c}_k, \alpha(x,k), \alpha(x,k-1))$, $(\overline{e}f, \alpha(y,k), \alpha(x,k-1))$, $(\overline{c}_k, \alpha(y,k), \alpha(y,k-1)) \in E_k$).

Consider now the $(\overline{m}; p_1, \ldots, 1, \ldots, p_n)$-ary testing pattern digraph $G^\star = \mathrm{TPG}(\hat{a}', \hat{b}', \hat{c}^\star, \hat{d}^\star)$ where $\hat{c}^\star$ is $\hat{c}$ with $\overline{c}_k$ replaced by c and $\hat{d}^\star$ is $\hat{d}'$ with $\overline{d}_k$ replaced by f. For the most part, this digraph is identical to $\mathrm{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$,

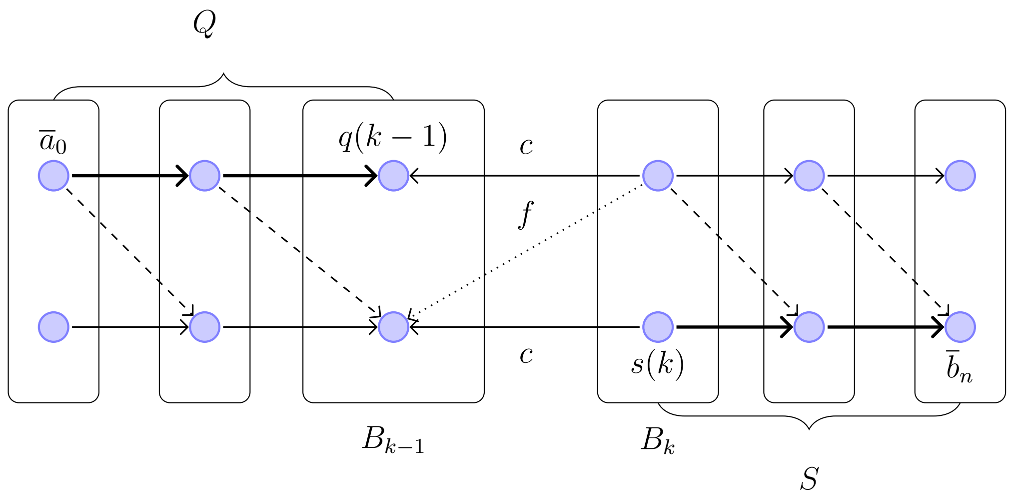

Figure 18: Replacing E_k by $E_k^\star$ in proof of Lemma 18.

only the edges from B_k to B_{k-1} have changed (see Figure 18). We have $E_k^\star \subseteq R$ because the three generators of $E_k^\star$ lie in R (see previous paragraph).

Using $\text{HasPath}(\overline{m}; p_1, \ldots, 1, \ldots, p_n)$ on $G^\star$, we get a P-shaped path $p^\star$ from $\overline{a}_0$ to $\overline{b}_n$. Since $E_k^\star \subseteq R$, there exist $\overline{t} \in B_{k-1}, \overline{v} \in B_k$ so that $(p^\star(k), \overline{t})$, $(\overline{v}, \overline{t})$, and $(\overline{v}, p^\star(k-1))$ are all solid edges in the original graph $\text{TPG}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$. One application of part (c) of Lemma 15 with Q^+ the prefix of P from 0 to k and S^+ the suffix of P from $k-1$ to n (i.e. one edge longer than the original Q and S) with $q^+(i) = p^\star(i)$ for $i \in \{0, \ldots, k-1\}$, $q^+(k) = \overline{v}$, and $s^+(i) = p^\star(i)$ for $i \in \{k, \ldots, n\}$, $s^+(k-1) = \overline{t}$ gives us that q^+ and s^+ can be extended to a $J \times P$-shaped walk β from $\overline{a}_0$ to $\overline{b}_n$ (see Figure 19) in the original G.

It remains to use β and Lemma 13 to conclude that G is equal to the testing pattern digraph $G^- = \text{TPG}(\hat{a}^-, \hat{b}^-, \hat{c}, \hat{d}^-)$ where $\overline{a_i^-} = \beta(x, i)$ and $\overline{b_i^-} = \beta(y, i)$ for each i. Crucially, with β in hand we can choose $\overline{d_k^-} = \overline{c}_k$. In other words, it turns out that all edges of the original G between B_{k-1} and B_k are solid. But then we can easily connect the walks q and s from the first paragraph of this proof into a P-shaped walk in G and are done. $\square$

Proof of Theorem 7. Start with the tuples $\overline{m} = (1, 1, \ldots, 1)$, $\overline{p} = (1, \ldots, 1)$, and the hypothesis that $\text{HasPath}_P(\overline{m}; \overline{p})$ holds. From the definition of testing

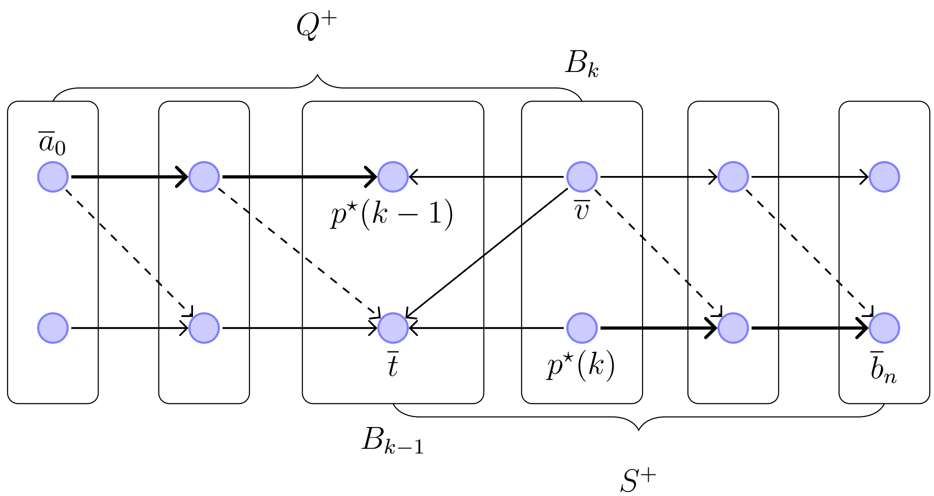

Figure 19: Going back to E_k and finishing the proof of Lemma 18. Note that we have three solid edges now.

pattern digraphs, we know that if the i-th edge of P is dashed, then the value of p_i does not affect G, so we can immediately set any such p_i to $|A|^2$ and keep the $\mathrm{HasPath}_P$ property. Next, repeatedly apply Lemmas 16, 17, and 18 to increase the entries of $\overline{m}$ and $\overline{p}$ until we get that $\mathbf{A}$ satisfies $\mathrm{HasPath}_P(\overline{|A|^2}; \overline{|A|^2})$. From this it follows that $\mathbf{A}$ satisfies $M(P)$. $\square$

5 Conclusion

It would be nice if the techniques introduced in proving Theorem 7 could be extended to handle conditions described by some graph (or structure) other than a path and so we ask whether this is the case. One prime example that has been considered is that of a ternary minority operation, i.e., an operation $m(x, y, z)$ that satisfies the equations

$$m(y, x, x) \approx m(x, y, x) \approx m(x, x, y) \approx y.$$

While this condition is quite similar to that of having a Maltsev term, it turns out that there is no chance of producing an efficient algorithm to decide if a finite idempotent algebra $\mathbf{A}$ has such a term operation that is based on some variant of the "local to global" term method. This follows from the construction, by Dmitriy Zhuk, of a sequence of finite idempotent algebras $\mathbf{A}_n$, for $n > 1$, such that for every subset of $\mathbf{A}_n$ of size n, $\mathbf{A}_n$ has a term operation that behaves as a minority operation on the subset, but $\mathbf{A}_n$ as a whole does not have a minority term operation [17]. The complexity of determining if a finite idempotent algebra has such a term operation remains open, but recently, the authors, in collaboration with J. Opršal, have shown that this problem is in NP [17].

One can also consider the Maltsev conditions of being congruence meet-semidistributive or congruence join-semidistributive. They are the unions of sequences of strong linear Maltsev conditions ([18]). We ask whether testing for these strong linear Maltsev conditions can be carried out by polynomial time algorithms for finite idempotent algebras. It was noted in the introduction that some Maltsev conditions that are not strong, such as being congruence distributive or congruence modular, can also be tested by polynomial time algorithms, for finite idempotent algebras. We ask whether our techniques can be extended to handle some interesting class of Maltsev conditions that are not strong.

All of the Maltsev conditions studied in this paper are strong, linear, and idempotent (these are called special Maltsev conditions in [13]). As far as we know, for any special Maltsev condition M, the problem of deciding if a given finite idempotent algebra satisfies M is in P, and we conjecture that this will always be the case. An earlier stronger version of this conjecture held that there would be a polynomial time algorithm based on a "local to global" term result along the lines of our Theorem 7, but the case of the minority term Maltsev condition falsified it.

Finally, one can also consider the related problems for finite relational structures. Namely, given a (strong/linear/idempotent) Maltsev condition M one can ask whether a given finite relational structure $\mathbf{B}$ has polymorphisms that witness the satisfaction of M. It is known that for strong linear idempotent Maltsev conditions M that imply congruence meet-semidistributivity, this problem can be solved by a polynomial time algorithm. Recently Hubie Chen and Benoit Larose have produced some significant results related to this class of problems [6].

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