MATH 4LT3/6LT3 Midterm Test Solutions

Midterm Test   Instructor: Matt Valeriote
Duration of test: 50 minutes
McMaster University
October 17, 2023

Last Name:_____   First Name:_____

Student Number:_____

Please answer all four questions. To receive full credit, provide justifications
for your solutions. For all questions, write your answers in the answer booklet
that has been provided. Please be sure to include your name and student
number on all sheets of paper that you hand in.

NOTE: In your solutions you may make use of any theorems or results
discussed in the lectures. You may not use other theorems or results, unless
you fully justify them. This includes results from the homework assignments,
unless otherwise stated.

No aids are allowed.

Each question is worth 5 points; the maximal number of marks is 20.

Score

Question	1	2	3	4	Total
Score					

The first six of Zermelo's axioms (The Axioms)

(I) **Axiom of Extensionality.** For any two sets A and B,

$$A = B \Leftrightarrow (\forall x)[x \in A \Leftrightarrow x \in B].$$

(II) **Emptyset and Pairset Axiom.** (a) There is a special set $\emptyset$ which has no members. (b) For any two sets x and y there is a set A whose only members are x and y.

(III) **Separation Axiom.** For each set A and each unary definite condition P there exists a set B that satisfies

$$(\forall x)[x \in B \Leftrightarrow (x \in A \ \& \ P(x))].$$

(IV) **Powerset Axiom.** For each set A, there exists a set B whose members are the subsets of A.

(V) **Unionset Axiom.** For each set E, there exists a set B whose members are the members of the members of E.

(VI) **Axiom of Infinity.** There exists a set I which contains the empty set $\emptyset$ and the singleton of each of its members, i.e.,

$$\emptyset \in I \ \& \ (\forall x)[x \in I \Rightarrow \{x\} \in I].$$

1. (a) State the General Comprehension Principle.

   (b) Explain (i.e., prove) why this principle is not consistent with the Axioms of Set Theory that have been presented in the lectures. Be sure to reference any of the axioms that are used in your proof.

   Solution: If we assume that this principle is valid, then the collection

   $$R = \{x \mid x \text{ is a set and } x \notin x\}$$

   is a set. Here the definite condition $P(x)$ is: x is a set and $x \notin x$. Then consider whether or not $R \in R$. We see that $R \in R$ if and only if $R \notin R$. This is a contradiction (aka Russell's Paradox) and so the General Comprehension Principle is not consistent with the Axioms.

2. For each of the following collections, determine if they are sets or proper classes. Provide justifications for your answers, by either explaining how the axioms can be used to establish that a particular collection is a set, or by showing that the collection is a class and then explaining why it cannot be a set.

   In both of the following, A is a nonempty set.

   (a) Let B be the collection $\{\mathcal{P}(X) \mid X \subseteq A\}$.

   Solution: B is a set. This can be seen by applying the Separation Axiom to the set $D = \mathcal{P}(\mathcal{P}(A))$ as follows:

   $$B = \{y \in D \mid (\exists x)[x \in \mathcal{P}(A) \wedge y = \mathcal{P}(x)]\}.$$

   Using the Powerset Axiom (twice) we see that D is a set, and so from the Separation Axiom we conclude that B is also a set. If y is of the form $\mathcal{P}(X)$ for some $X \subseteq A$, then $y = \mathcal{P}(X) \in \mathcal{P}(\mathcal{P}(A)) = D$.

   (b) Let C be the collection $\{X \mid X \text{ is a set and } X \leq_c A\}$.

   Solution: C is not a set, and so is a proper class. Since A is nonempty, then for every set x, $\{x\} \leq_c A$, and so for every set x, $\{x\} \in C$. If C is a set, then so is $\cup C$. But $\cup C$ contains all sets, and so cannot be a set. Thus C isn't a set.

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3. (a) State the Recursion Theorem.

   Solution: Consult the textbook.

   (b) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

   $$f(n) = 1 + 2^1 + 2^2 + \cdots + 2^n.$$

   Use the Recursion Theorem, or some variant of it from the notes or text, to show that f, as a function, is guaranteed to exist by the Axioms. You may use, without having to justify it, that the addition, multiplication, and exponentiation operations on $\mathbb{N}$ are functions.

   Solution: The following is a recursive definition of f: $f(0) = 1$, and $f(n+1) = 2 \times f(n)+1$. So with $a = 1$ and $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = 2 \times n + 1$, we conclude that f is a function.

4. Assume that $A \to |A|$ is a definite operation on sets that is a cardinal assignment (it could be weak or strong). Let κ and λ be cardinals under this assignment.

   (a) Give the definition of the cardinal κ^λ.

   Solution: $\kappa^\lambda = |(\lambda \to \kappa)|$

   (b) Suppose that $2 \leq_c \kappa$. Show that $2^\lambda \leq_c \kappa^\lambda$.

   Any properties of the relation $\leq_c$ that you use in your solution should be carefully justified, but you don't need to refer to the Axioms.

   Solution: Since $2 \leq_c \kappa$ then there is an injection $f : \{0,1\} \to \kappa$. We use this to define an injection π from $(\lambda \to 2)$ to $(\lambda \to \kappa)$ as follows: for $h \in (\lambda \to 2)$, $\pi(h)$ is the function $f \circ h$ in $(\lambda \to \kappa)$. It is not hard to show that π is an injection, which shows that $2^\lambda \leq_c \kappa^\lambda$.

   (c) Suppose that in addition we know that $\kappa \leq_c \lambda$ and $\lambda \cdot \lambda =_c \lambda$. Show that $2^\lambda =_c \kappa^\lambda$.

   You may use, without justification that for cardinals α, β, and γ, $(\alpha^\beta)^\gamma =_c \alpha^{\beta \cdot \gamma}$. This was shown in homework assignment #2.

   Solution: By the Schröder-Bernstein Theorem and part (b), it suffices to show that $\kappa^\lambda \leq_c 2^\lambda$. But

   $$\kappa^\lambda \leq_c \lambda^\lambda \leq_c (2^\lambda)^\lambda =_c 2^{(\lambda \cdot \lambda)} =_c 2^\lambda.$$