

MATH 4LT3/6LT3 Midterm Test

Midterm Test

Instructor: Matt Valeriote

Duration of test: 50 minutes

McMaster University

October 17, 2023

Last Name: _____

First Name: _____

Student Number: _____

Please answer all four questions. To receive full credit, provide justifications for your solutions. For all questions, write your answers in the answer booklet that has been provided. Please be sure to include your name and student number on all sheets of paper that you hand in.

NOTE: In your solutions you may make use of any theorems or results discussed in the lectures. You may not use other theorems or results, unless you fully justify them. This includes results from the homework assignments, unless otherwise stated.

No aids are allowed.

Each question is worth 5 points; the maximal number of marks is 20.

Score

Question	1	2	3	4	Total
Score					

The first six of Zermelo's axioms (The Axioms)

- (I) **Axiom of Extensionality.** For any two sets A and B ,

$$A = B \Leftrightarrow (\forall x)[x \in A \Leftrightarrow x \in B].$$

- (II) **Emptyset and Pairset Axiom.** (a) There is a special set \emptyset which has no members. (b) For any two sets x and y there is a set A whose only members are x and y .

- (III) **Separation Axiom.** For each set A and each unary definite condition P there exists a set B that satisfies

$$(\forall x)[x \in B \Leftrightarrow (x \in A \ \& \ P(x))].$$

- (IV) **Powerset Axiom.** For each set A , there exists a set B whose members are the subsets of A .

- (V) **Unionset Axiom.** For each set E , there exists a set B whose members are the members of the members of E .

- (VI) **Axiom of Infinity.** There exists a set I which contains the empty set \emptyset and the singleton of each of its members, i.e.,

$$\emptyset \in I \ \& \ (\forall x)[x \in I \Rightarrow \{x\} \in I].$$

1. (a) State the General Comprehension Principle.
 (b) Explain (i.e., prove) why this principle is not consistent with the Axioms of Set Theory that have been presented in the lectures. Be sure to reference any of the axioms that are used in your proof.
2. For each of the following collections, determine if they are sets or proper classes. Provide justifications for your answers, by either explaining how the axioms can be used to establish that a particular collection is a set, or by showing that the collection is a class and then explaining why it cannot be a set.

In both of the following, A is a nonempty set.

- (a) Let B be the collection $\{\mathcal{P}(X) \mid X \subseteq A\}$.
 - (b) Let C be the collection $\{X \mid X \text{ is a set and } X \leq_c A\}$.
3. (a) State the Recursion Theorem.
 (b) Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f(n) = 1 + 2^1 + 2^2 + \cdots + 2^n.$$

Use the Recursion Theorem, or some variant of it from the notes or text, to show that f , as a function, is guaranteed to exist by the Axioms. You may use, without having to justify it, that the addition, multiplication, and exponentiation operations on \mathbb{N} are functions.

4. Assume that $A \rightarrow |A|$ is a definite operation on sets that is a cardinal assignment (it could be weak or strong). Let κ and λ be cardinals under this assignment.
 - (a) Give the definition of the cardinal κ^λ .
 - (b) Suppose that $2 \leq_c \kappa$. Show that $2^\lambda \leq_c \kappa^\lambda$.
 Any properties of the relation \leq_c that you use in your solution should be carefully justified, but you don't need to refer to the Axioms.
 - (c) Suppose that in addition we know that $\kappa \leq_c \lambda$ and $\lambda \cdot \lambda =_c \lambda$. Show that $2^\lambda =_c \kappa^\lambda$.
 You may use, without justification that for cardinals α , β , and γ , $(\alpha^\beta)^\gamma =_c \alpha^{\beta \cdot \gamma}$. This was shown in homework assignment #2.