MATH 3GR3 Midterm Test #1 Solutions

Midterm Test
Duration of test: 50 minutes
McMaster University
October 17, 2023

Instructor: Matt Valeriote

Last Name:____________________________ First Name:_______________

Student Number:____________________________________

Please answer all five questions. To receive full credit, provide justifications for your solutions. For all questions, write your answers in the answer booklet that has been provided. Please be sure to include your name and student number on all sheets of paper that you hand in.

NOTE: In your solutions you may make use of any theorems or results discussed in the lectures. You may not use other theorems or results, unless you fully justify them. This includes results from the homework assignments.

No aids are allowed.

Each question is worth 5 points; the maximal number of marks is 25.

Score

Question	1	2	3	4	5	Total
Score						

1. (a) Give the definition of a group.

[5] Solution: Consult the textbook.

(b) Let $\mathbb{R}' = \{r \in \mathbb{R} : r \geq 0\}$. Let $\diamond$ be the binary operation on $\mathbb{R}'$ defined by $r \diamond s = |r - s|$. Is $\mathbb{R}'$ with the operation $\diamond$ a group? Justify your answer.

Solution: We see that the element 0 is an identity element for this operation, since $0 \diamond x = |0 - x| = x = |x - 0| = x \diamond 0$ for all $x \in \mathbb{R}'$. Further, for any $x \in \mathbb{R}'$, $x \diamond x = |x - x| = 0$, and so the inverse of x is just x. This operation is not associative since, for example, $1 \diamond (1 \diamond 2) = 1 \diamond (|1 - 2|) = 1 \diamond 1 = 0$, but $(1 \diamond 1) \diamond 2 = 0 \diamond 2 = 2$. So, $\mathbb{R}'$ with the operation $\diamond$ is not a group.

2. Let $k > 1$ be an integer and let G be a group with identity element e. Define H_k to be the following subset of G:

$$H_k = \{g \in G : g^k = e\}.$$

[5]

(a) Show that if G is abelian, then H_k is a subgroup of G.

Solution: We need to show that H_k contains the identity element and is closed under products and taking inverses. Since $e^k = e$, then $e \in H_k$. If $a, b \in H_k$, then $a^k = b^k = e$. Then $(ab)^k = a^k b^k = ee = e$, since G is abelian. Similarly, $(a^{-1})^k = (a^k)^{-1} = e^{-1} = e$ and so both ab and a^{-1} are in H_k. Thus H_k is a subgroup of G.

(b) Find an example of a non-abelian group G such that the subset H_2 of G is **not** a subgroup of G.

To receive full credit, explain why the subset H_2 is not a subgroup of the group G that you provide.

Solution: Consider the group S_3. In this group

$$H_2 = \{id, (1,2), (1,3), (2,3)\}.$$

Since H_2 is not closed under products, for example $(1,2)(2,3) = (1,2,3) \notin H_2$, then H_2 is not a subgroup of S_3. Note that H_2 contains the identity element and is closed under taking inverses.

[5] 3. Let $\sigma = (1,\ 7,\ 2,\ 5,\ 4)(2,\ 5,\ 3,\ 4)(1,\ 5,\ 6,\ 4)$, a member of the group S_7.

(a) Express σ as a product of **disjoint** cycles.

(b) What is the order of σ?

(c) Is σ an even permutation? Justify your answer to receive credit.

Solution: $\sigma = (1,3)(2,4,7)(5,6)$. Since σ is the product of two disjoint 2-cycles and one 3-cycle, then the order of σ is equal to 6, the least common multiple of 2 and 3. Since any 3-cycle is an even permutation (it can be written as a product of two 2-cycles), then σ is an even permutation (and can be written as a product of four transpositions).

[5] 4. (a) List the elements of $U(14)$, the group of units in $\mathbb{Z}_{14}$.

(b) Is $U(14)$ a cyclic group?

(c) List **all** of the subgroups of $U(14)$.

Solution: (a) The elements of $U(14)$ are those integers k with $1 \leq k < 14$ and with $\gcd(k, 14) = 1$. So, $U(14) = \{1, 3, 5, 9, 11, 13\}$.

(b) $U(14)$ will be cyclic if and only if it contains an element of order 6, the order of $U(14)$. We see that the element 3 has order 6 since in $U(14)$, $3^2 = 9$, $3^3 = 13$, $3^4 = 11$, $3^5 = 5$, and $3^6 = 1$.

(c) Since $U(14)$ is cyclic, then all of its subgroups are as well. It will have one subgroup order 1, $\{1\}$, one of order 2, $\{1, 13\}$, one of order 3, $\{1, 9, 11\}$, and one of order 6, $U(14)$.

5. Let G be a group that has at least two elements and that has no proper non-trivial subgroups.

[5] (a) Show that G must be a cyclic group.

Solution: Let $a \in G$ with a not equal to the identity element e of G. Then $H = \langle a \rangle$ is a non-trivial subgroup of G, since it contains at least 2 elements, e and a. By assumption, H cannot be a proper subgroup, so $H = G$. Thus $G = \langle a \rangle$ is a cyclic group.

(b) Show that G must be a finite group and that $|G|$ is a prime number.

Solution: If G is an infinite cyclic group that is generated by the element a, then $G = \langle a \rangle = \{a^i : i \in \mathbb{Z}\}$, and for $i \neq j$, $a^i \neq a^j$.

But then the subset of all even powers of a, $\{a^{2i} : i \in \mathbb{Z}\}$ will be a non-trivial proper subgroup of G and so G cannot be infinite.

If $|G| = n$ and $G = \langle a \rangle$, then the element a has order n. If n is not prime, say $n = m \cdot k$ for natural numbers m, $k > 1$, then the element a^m has order k in G and $\langle a^m \rangle$ is a non-trivial proper subgroup of G. Since this can't happen, n cannot be composite and so must be prime.