The first isomorphism theorem

Theorem 11.10 (The first Isomorphism Theorem)

Let $\psi: G \to H$ be a homomorphism and let $K = \ker(\psi)$.
Let $\phi: G \to G/K$ be the canonical homomorphism ($g \mapsto gK$).
Then there is a unique isomorphism $\eta: G/K \to \psi(G)$ such that

$\psi = \eta \circ \phi$.

Proof: Define $\eta: G/K \to \psi(G)$

by: $\eta(aK) = \underline{\psi(a)}$

- Show: (1) η is well defined

(2) η is one-to-one,

(3) η is onto.

(4) $\eta((g_1K)(g_2K)) = \eta(g_1K)\cdot\eta(g_2K)$ for all $g_1, g_2 \in G$.

(5) For $g \in G$, $\psi(g) = \eta(\phi(g))$

(6) η is unique with the above properties.

(1) and (2):

For $g_1, g_2 \in G$,

$$\left.\begin{array}{l} \text{if } g_1K = g_2K \\ \text{then } m(g_1K) = m(g_2K), \end{array}\right\} m \text{ well defined}$$

$$\left.\begin{array}{l} \text{and if } m(g_1K) = m(g_2K), \\ \text{then } g_1K = g_2K \end{array}\right\} m \text{ is one-to-one.}$$

$g_1K = g_2K$

if and only if

$\psi(g_1) = \psi(g_2)$ (earlier lemma)

if and only if

$\eta(g_1K) = \eta(g_2K)$ (definition of η)

③ For $h \in \psi(G)$, $h = \psi(g)$ for some $g \in G$. Then $\eta(gK) = \psi(g) = h$.

So $\eta(G/K) = \psi(G)$.

④ $\eta((g_1K)(g_2K)) = \eta((g_1g_2)K) = \psi(g_1g_2) = \psi(g_1)\psi(g_2)$

$= \eta(g_1K)\eta(g_2K)$

(5) $\underline{\mu(\phi(g))} = \mu(gK) = \underline{\psi(g)}$

(6) Suppose $\mu' : G/K \to \psi(G)$ is an isomorphism such that $\psi = \mu' \circ \phi$

$\underline{\text{Show } \mu = \mu'}$

For $gK \in G/K$,

$\underline{\mu'(gK)} = \mu'(\underline{\phi(g)}) = \psi(g) = \underline{\mu(gK)}$

So $\mu = \mu'$.

(5) For $g \in G$, $\psi(g) \stackrel{?}{=} \mu(\phi(g))$

(6) μ is unique with the above properties.

example: Let $a, b \in \mathbb{N}$, $a, b > 0$

and $K = a\mathbb{Z} \times b\mathbb{Z}$.

Then K is a normal subgroup of $G = \mathbb{Z} \times \mathbb{Z}$

Show $G/K = (\mathbb{Z}\times\mathbb{Z})/(a\mathbb{Z}\times b\mathbb{Z})$ is isomorphic

to $\mathbb{Z}_a \times \mathbb{Z}_b$.

Use the First Iso. Theorem.

Find a homomorphism

$\varphi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_a\times\mathbb{Z}_b$ that

is onto, and

$\ker(\varphi) = a\mathbb{Z}\times b\mathbb{Z}$.

Define φ

Define ψ by:

$$\psi((m,n)) = ([m]_a, [n]_b) \in \mathbb{Z}_a \times \mathbb{Z}_b$$

(Exercise: If $f_1 : G_1 \to H_1$, and $f_2 : G_2 \to H_2$, then $f : G_1 \times G_2 \to H_1 \times H_2$, with $f((g_1, g_2)) = (f_1(g_1), f_2(g_2))$ is a homomorphism.)

ψ is onto (exercise).

Let $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ with

$\psi((m,n)) = ([0]_a, [0]_b)$.

Then $[m]_a = [0]_a$ and $[n]_b = [0]_b$.

$\Leftrightarrow$ $m = ka$ and $n = \ell b$ for some $k, \ell \in \mathbb{Z}$.

$\Leftrightarrow$ $m \in a\mathbb{Z}$ and $n \in b\mathbb{Z}$.

$\Leftrightarrow$ $(m,n) \in a\mathbb{Z} \times b\mathbb{Z} = K$

$\Rightarrow$ So $\ker(\psi) = K$.

Thus $G/K \cong \psi(G)$ or

$(\mathbb{Z} \times \mathbb{Z}) / (a\mathbb{Z} \times b\mathbb{Z}) \cong \mathbb{Z}_a \times \mathbb{Z}_b$ by the First Iso. Theorem.