$G = \mathbb{Z}_4 \times \mathbb{Z}_4$

$N = \langle (3,2) \rangle$

$= \{(0,0), (3,2), (2,0), (1,2)\}$

Left cosets:

$(0,0)+N = N$

$(0,1)+N = \{(0,1), (3,3), (2,1), (1,3)\}$

$(1,0)+N = \{(1,0), (0,2), (3,0), (2,2)\}$

$(1,1)+N = \{(1,1), (0,3), (3,1), (2,3)\}$

$G/N = \{N, (0,1)+N, (1,0)+N, (1,1)+N\}$

order 4

| $+$ | $N$ | $(0,1)+N$ | $(1,0)+N$ | $(1,1)+N$ |
|---|---|---|---|---|
| $N$ | $N$ | $(0,1)+N$ | $(1,0)+N$ | $(1,1)+N$ |
| $\boxed{(0,1)+N}$ | $(0,1)+N$ | $(1,0)+N$ | $(1,1)+N$ | $N$ |
| $(1,0)+N$ | $(1,0)+N$ | $(1,1)+N$ | $N$ | $(0,1)+N$ |
| $\boxed{(1,1)+N}$ | $(1,1)+N$ | $N$ | $(0,1)+N$ | $(1,0)+N$ |

$\cong \mathbb{Z}_4$

Cayley table of $G/N$.

## Simple Groups

**Definition:** A group $G$ is simple if it has no non-trivial proper normal subgroups, i.e., the only normal subgroups of $G$ are: $\{e\}$ and $G$.

**Observe:** An abelian group is simple if and only if it has no nontrivial proper subgroups.

By the midterm:

An abelian group $G$ is simple if and only if $G \cong \mathbb{Z}_p$ for some prime $p$.

- last century, the finite simple groups were classified.
- <u>Theorem</u>: For $n \geq 5$, the group $A_n$ is simple.

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## <u>Chapter II</u>: Group Homomorphisms.

Given a group $G$ and a normal subgroup $N$, we have the factor group $G/N$.

In general, $G \not\cong G/N$, but there is a natural map from $G$ <u>onto</u> $G/N$.

$$\phi: g \longmapsto gN \in G/N.$$

$$\phi: G \to G/N$$

$$\phi(g) = gN$$

<u>onto</u>

$\phi$ satisfies

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

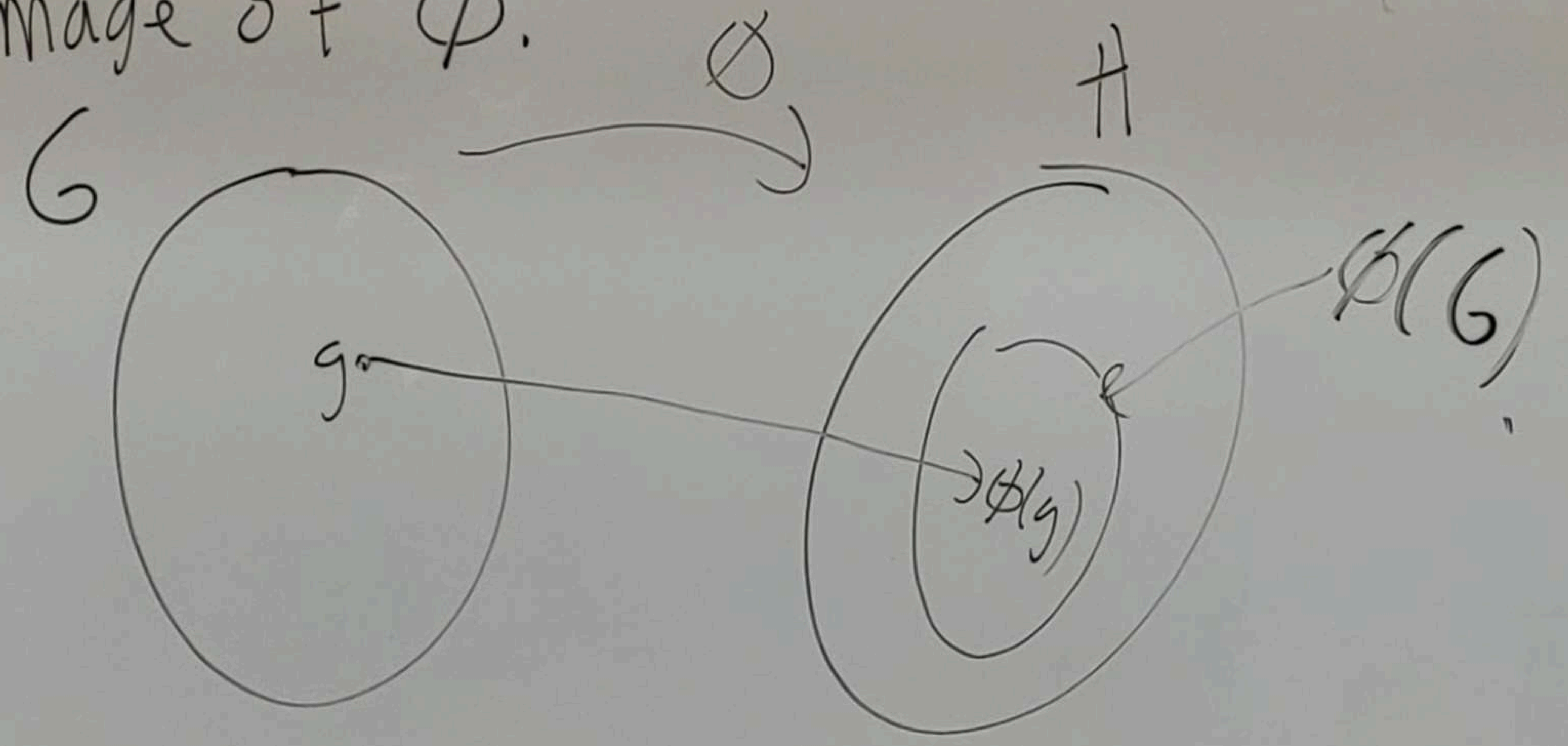
$$\left[\begin{aligned}\phi(g_1 g_2) &= (g_1 g_2)N \\ &= (g_1 N)(g_2 N) \\ &= \phi(g_1)\phi(g_2)\end{aligned}\right].$$

**Definition:** Let $(G, \cdot)$ and $(H, \circ)$ be groups.

A map $\phi : G \to H$ is a homomorphism

if for all $g_1, g_2 \in G$, $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$

- The range, or image, of $\phi$ is called the homomorphic image of $\phi$.




examples

① $\gamma : S_n \to \mathbb{Z}_2$

$$\gamma(\sigma) = \begin{cases} 0, & \text{if } \sigma \text{ is even} \\ 1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

Claim: $\gamma$ is a homomorphism.

$$\gamma(\sigma_1 \sigma_2) = \gamma(\sigma_1) + \gamma(\sigma_2)$$

Cases:

① both $\sigma_1, \sigma_2$ even

② both $\sigma_1, \sigma_2$ odd

③ one even, one odd

② $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$

Claim: this is an onto, homomorphism

Proof: Follows, since

$$\det(AB) = \det(A)\det(B)$$

③ Define $\phi: \mathbb{R} \to \mathbb{R}$ by

$$\phi(r) = r^2$$

Claim: $\phi$ isn't a homomorphism

Proof: $\phi(r_1+r_2)$ vs $\phi(r_1)+\phi(r_2)$

$$\phi(r_1+r_2) = (r_1+r_2)^2 = (r_1^2+r_2^2+2r_1r_2) \neq (r_1^2+r_2^2) = \phi(r_1)+\phi(r_2)$$

Let $r_1=1, r_2=1$.

Then $\phi(r_1+r_2) = \phi(2) = 4$

$\neq 1+1 = \phi(1)+\phi(1)$

**Proposition 11.4**: Let $\phi: G_1 \to G_2$ be a group homomorphism. Then

1. If $e$ is the identity element of $G_1$, then $\phi(e)$ is the identity element, $e'$, of $G_2$
2. if $g \in G_1$, then $\phi(g^{-1}) = \phi(g)^{-1}$
3. if $H_1$ is a subgroup of $G_1$, then $\phi(H_1)$ is a subgroup of $G_2$

$\Big\}$ proofs are identical to those for isomorphisms.

4. If $H_2$ is a subgroup of $G_2$, then $\phi^{-1}(H_2) = \{g \in G_1 \mid \phi(g) \in H_2\}$ is a subgroup of $G_1$
   - If $H_2$ is a normal subgroup of $G_2$ then $\phi^{-1}(H_2)$ is a normal subgroup of $G_1$.